# UNIVERSITY STUDENTS' EMBODIMENT OF QUANTIFIERS

David Tall & Erh-Tsung Chin
Mathematics Education Research Centre
University of Warwick
Coventry, CV4 7AL, UK
<david.tall@warwick.ac.uk; e-t.chin@warwick.ac.uk>

*This paper investigates the novice university students' understanding of the formal definition of "equivalence relations", especially their understanding of the quantifiers in the definition. Even though the definition is relatively simple and only involves the universal quantifier, we find that half of a class of highly qualified university students are unable to test whether an explicit relation on a set with three elements is an equivalence relation. Analysis of the data, from a questionnaire answered by 277 students and interviews with 36, reveals subtle influences of language and of conceptual embodiments. In particular, the transitive law, which is shared with the notion of order relation, may evoke an embodied image of order that is highly misleading.*

## INTRODUCTION

Chin & Tall (2000) proposed a theory of development of formal thinking, moving from informal concepts, to the introduction of definitions, to 'definition-based' deduction and on to 'theorem-based' deduction, in which we hypothesised that successful students would compress formal concepts into cognitive units appropriate for powerful formal thinking. In particular, we focused on 'equivalence relations & partitions' which students claimed to be the most difficult topic in the course. Our purpose was to determine *why* the students found the topic to be problematic.

In our first study above, there was evidence of responses in the categories of informal responses, definition-based deduction, theorem-based deduction, with some evidence of a few students having cognitive units that linked the notions of equivalence relation and partition. Our second study (Chin & Tall, 2001) followed the development over a period of time and revealed a general shift from 'definition-based' deduction using the formal definition to 'theorem-based' deduction referring to already proven theorems. However, it also revealed quite different developments for 'equivalence relation' and 'partition'. An equivalence relation has an apparently simple definition that almost all students are able to reproduce but it does not seem to have a natural embodiment. A partition has a subtler definition but has a simple prototypical embodiment as a set broken into subsets. Even though equivalence relations and partitions are mathematically equivalent, they develop as qualitatively different cognitive units linked by a formal theorem.

Our focus here is on the way that students employ the definition of 'equivalence relation'. Is the definition *operable* in the sense that it can be used as the basis for formal deduction (Bills & Tall, 1998)? In particular, do they have a full understanding of the use of the quantifiers, as, for instance, in '*$a \sim a$ for all $a \in S$*', or are there other aspects of the definition that the students use intuitively or implicitly?

## THEORETICAL FRAMEWORK

Formal definitions and deductions are the essential ingredient in advanced mathematical thinking (Tall, 1995), but they are known to cause great difficulties (Vinner, 1991). Some formal thinkers attempt to use logical deductions, others use a natural approach that relies more on their concept image (Pinto, 1998). The latter often involves thought experiments using embodied images in the sense of Lakoff and Johnson (1999, pp16-44). Our earlier papers showed that the notion of equivalence relation did not have a strong embodied image in the same way as the notion of partition. So, precisely how do students reason when they use the definition? For instance, how do they use the universal quantifier which occurs in the reflexive law 'for all $a \in S$, $a \sim a$'? The use of quantifiers in definitions has been considered "one of the least often acquired and most rarely understood concepts at all levels, from secondary school on up — even, in many cases, into graduate school" (Dubinsky, et al., 1988, p.44).

The definition of equivalence relation, uses only the universal quantifier:

> An *equivalence relation* on a set $S$ is a binary relation $\sim$ on $S$ that is
> *reflexive*: $a \sim a$ for all $a \in S$
> *symmetric*: if $a \sim b$ then $b \sim a$ for all $a, b \in S$
> and *transitive*: if $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in S$. (Stewart & Tall, 1977)

However, the quantifier here plays a very subtle role, as is shown by the following question, (which the first author well remembers being unable to answer for several days whilst studying Birkhoff & Maclane (1953) as an undergraduate):

> Given an equivalence relation "$\sim$" on $A$. Let $a \sim b$, then $b \sim a$ (by symmetry) and $a \sim b$, $b \sim a$, implies $a \sim a$ (transitivity). So symmetry and transitivity imply reflexivity.

The subtlety is two-fold. The first axiom asserts that the relation $a \sim a$ must hold *for all $a \in S$*. The second and third axioms use implication in a more subtle way, that *if* the premises holds, *then* the consequence follows. We decided to use a specific instance of this idea to test if the students' definitions are truly operable.

## EMPIRICAL STUDY

The study was performed on 277 (first year mathematics) students taking a course on 'foundations' in one of the top five ranked mathematics departments in the UK, 151 from a class of pure mathematics majors and 126 in a class consisting of those following courses such as statistics, economics, or physics. Both classes covered the same material over a ten-week period with three-hour lectures per week supported by weekly examples classes in groups of up to four students. The second author acted as supervisor for 12 pure mathematics students selected to cover a range of abilities. The topic of 'equivalence relations' and 'partitions' was formulated in the third week and developed in subsequent weeks. The students conceptions were studied by a questionnaire given to all students in the ninth week, (six weeks after the definition of equivalence relation was given and subsequently developed) and the responses were triangulated with interviews with the other 24 selected students, fourteen in pure mathematics and ten in other mathematically-linked subjects, together with field notes made in tutorials with the 12 pure mathematics students.

In this study we focus on the following question to investigate how the students understand the definition of "equivalence relations":

> Let $X$={*a, b, c*} and the relation ~ be defined where *a*~*b*, *b*~*a*, *a*~*a*, *b*~*b*, but no other relations hold. Is this an **equivalence relation**? If not, say why?

**Student Responses & Analysis**

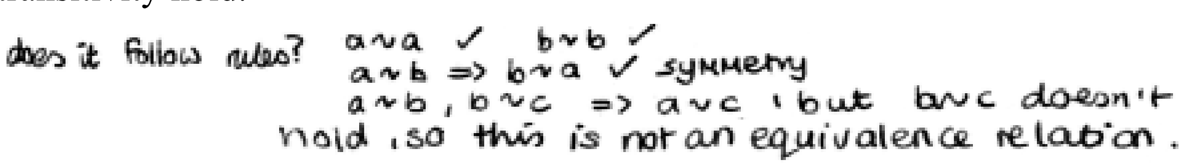
The student responses to the main question were categorised as follows:

- Correct deduction & answer,
- Incorrect deduction with correct answer,
- Incorrect deduction & answer,
- Don't know/ no response.

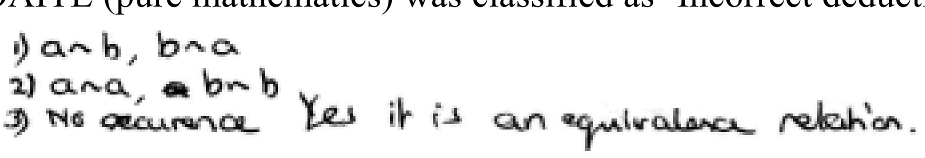
The following examples illustrate the categories.

To be classified as 'Correct', an answer must either have a general statement that the reflexive property does not hold for all elements, or specifically note that *c*~*c* does not hold. For instance, ANNWAN (pure mathematics) wrote:


GILWIN (other mathematics) was classified 'Incorrect deduction with correct answer.' Not only did he miss the universal quantifier in the reflexive property, he tested the symmetric property only with the distinct elements *a*, *b*, then denied transitivity because "*b*~*c* doesn't hold" for making *a*~*b*, *b*~*c* $\Rightarrow$ *a*~*c*. However, he even did not notice that *a*~*c* does not hold either when making the deduction. In the interview, he explained what he means is it needs three *different* elements to make transitivity hold.




JOAITE (pure mathematics) was classified as 'Incorrect deduction and answer'.



He used only the four related pairs and assigned each of them to his own versions of the three rules, (presumably in the order 1: symmetry, 2: reflexivity, 3: transitivity) without exhibiting any conception of the subtlety of the universal quantifier.

| | *Correct deduction & answer* | *Incorrect deduction* | | *Don't know/ No response* | *Total* |
|---|---|---|---|---|---|
| | | *"correct" answer* | *incorrect answer* | | |
| Pure mathematics | 94 | 24 | 30 | 3 | 151 |
| Other mathematics | 45 | 44 | 18 | 19 | 126 |
| Total | 139 | 68 | 48 | 22 | 277 |

**Table 1: Responses to the "use of quantifiers" question**

The distribution of the categorisation is shown in table 1. Visibly, the pure mathematics students give more correct responses (in terms of both deduction and answer) than the other mathematics students ($\chi^2$=19.34, p<0.0001). These 'correct' responses show a satisfactory use of the universal quantifier in the reflexive property at either a general or specific level. (This is not to say that a student who gives a correct response to the reflexive property necessarily understands the whole definition, for there are seven students who correctly assert the falsehood of the reflexive property who are categorised as having an 'incorrect deduction' because of difficulties with the symmetric or transitive properties (usually the latter)).

We investigated whether the nature of these responses correlated with the quality of the definition that the students offered for an equivalence relation. To determine the latter we combined the results of the following question:

> Say what "***equivalence relation***" means to you,

and a later question focusing on their concept definition:

> Look back at what you wrote about the meaning of "***equivalence relation***", do you consider it to be a *formal definition*? If you consider it is not a proper formal definition, please write down the *formal definition*.

The combined response to these questions were placed into four categories:

- formal-detailed (including full use of quantifiers),
- formal-partial (giving the three properties in symbolic form without quantifiers),
- informal outline (mentioning 'reflexive, symmetric, transitive' only)
- other, or no response

The student responses are cross-tabulated in tables 2, 3. There is no statistical correlation between the responses of the pure mathematicians and the quality of their definitions ($\chi^2$=10.94, p=0.28). But for the other mathematicians, the correlation is highly significant ($\chi^2$=29.39, p<0.001). However, closer examination shows that the significance arises from the correlation between those students responding 'other/no response' in both categories. If these are removed, then the correlation on the remaining data is $\chi^2$=3.11, p=0.54 for the pure mathematicians and $\chi^2$=5.91, p=0.21

| Definition \ Deduction | *Correct deduction & answer* | *Incorrect deduction* | | *Other/ No response* | *Total* |
|---|---|---|---|---|---|
| | | *correct answer* | *incorrect answer* | | |
| Formal-detailed | 31 | 7 | 5 | 1 | 44 |
| Formal-partial | 25 | 6 | 7 | 1 | 39 |
| Informal-outline | 35 | 10 | 15 | 0 | 60 |
| Other/no response | 3 | 1 | 3 | 1 | 8 |
| Total | 94 | 24 | 30 | 3 | 151 |

**Table 2. Pure mathematics students: giving definitions versus making deductions**

| Definition \ Deduction | *Correct deduction & answer* | *Incorrect deduction* | | *Other/ No response* | *Total* |
|---|---|---|---|---|---|
| | | *correct answer* | *incorrect answer* | | |
| Formal-detailed | 9 | 5 | 1 | 0 | 15 |
| Formal-partial | 21 | 23 | 6 | 4 | 53 |
| Informal-outline | 12 | 12 | 9 | 11 | 41 |
| Other/no response | 3 | 4 | 2 | 8 | 17 |
| Total | 45 | 44 | 18 | 19 | 126 |

**Table 3. Other mathematics students: giving definitions versus making deduction**

for the others. Thus the statistical difference is due solely to the fact that a significant minority of the other students cannot handle the definition at all.

Our attention focuses on those students who gave unsatisfactory responses.

***Unsatisfactory responses declaring the relation to be an equivalence relation***

Forty eight students asserted incorrectly that the relation is an equivalence relation. Thirty gave no explanation. Of these, three were interviewed and in each case the response indicated that the students concerned did not think deeply about the problem. For instance, one was capable of giving a 'formal/detailed' response with full quantifiers in the interview, but was unable to explain any further.

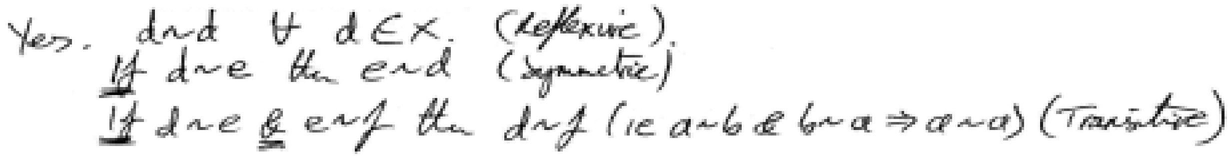
Of the 18 students offering a written explanation, there was a tendency to simply overlook the role of the quantifier in the reflexive property. SANSON (pure mathematics), for example examined the reflexive rule for all four given relational pairs, then indicated one clear example of the transitive law, but did not consider symmetry in detail.

This is reflexive because if $a \sim b$ then $b \sim a$
if $a \sim a$ then $a \sim a$
& if $b \sim b$ then $b \sim b$

It is transitive because $a \sim b$ $b \sim b$ & $a \sim b$ etc.

& I think it will also be symmetric ∴ is an equivalence relation

MAROOD (pure mathematics) wrote out the definition in full, but then did not apply the definition to the specific example.

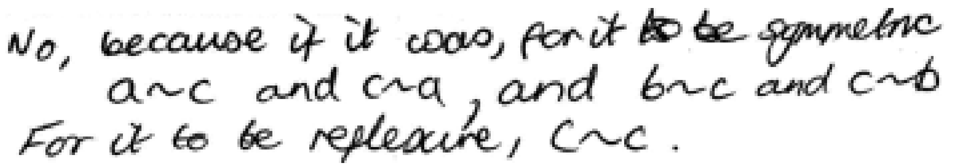



In general, even though many of these students could give the definition in detail, they failed to implement it in the given case.

***Saying the relation is not an equivalence relation***

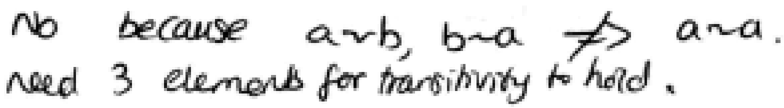
The case of students who asserted the relation was *not* an equivalence relation, but made errors in explanation, is the most revealing category of all. *Eighty two percent of them (19 of 24 pure mathematicians and 37 of 44 other mathematicians) offered the same reason: "the relation is not transitive so it's not an equivalence relation".*

Their reasons varied. SUSDLE (pure mathematics), who could give a formal/ detailed definition, correctly dealt with the reflexive property, but noted incorrectly, that the symmetry law needed to hold for more elements:

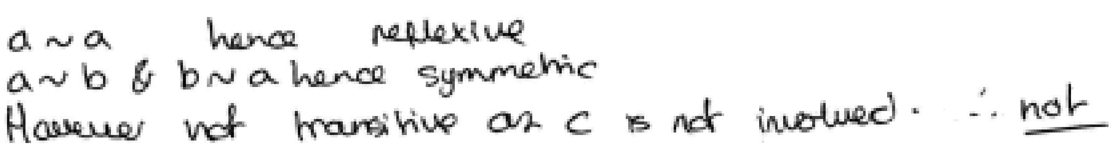



In the interview, he was asked how about transitive property. He replied that it must include all the relations between each two out of any three *different* successive elements. He also explained that he thought in this way because the universal quantifier is used in all three axioms.

LUCCCO (other mathematics), classified as 'formal/detailed', misunderstood transitive property because he did not think $a \sim b$, $b \sim a \Rightarrow a \sim a$. He clearly pointed out there should be three elements for transitivity to hold.




SUSURT (pure mathematics), classified as 'formal/ partial' because he omitted quantifiers in his definition, missed the universal quantifier in reflexivity, only examined symmetry with the distinct elements *a* and *b*, then disagreed with transitivity "as *c* is not involved".




In the interview SUSURT was asked why he thought *c* should be involved for transitivity to hold. He echoed the thinking of LUCCCO and GILWIN (quoted before) by replying "because it needs *three* elements to make transitivity hold". In the interviews, we found that six of the seven interviewees who asserted transitivity failed also shared this view. Furthermore, when asked, 'If the two relations $a\sim b$, $b\sim a$ are removed from this question, what will happen?', five out of these seven replied, 'symmetry will not hold either'. Three students said explicitly that they used the concept of an order relation as an embodiment of transitivity and two of these explicitly said they used '$a<b$, $b<c$ imply $a<c$' as a special example for transitivity in their concept image.

## DISCUSSION

In our data, approximately half the students are unable to handle the definition in a simple example using only three elements. The reasons are diverse, but 82% of those giving an incorrect reason for the example not being an equivalence relation focus on the transitivity law where there is a sense that 'the transitivity law must involve three elements' and even that the transitive law is interpreted using an embodiment that is the same as the axiom in an order relation. This has been a fundamental underlying conception in mathematics even amongst those who insist on formal thinking. In his famous address to the International Congress of Mathematicians in 1900, David Hilbert said:

> Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea "between"? (Hilbert, 1900)

The students involved in this study have been given an exposition of the theory in which the concept of relation is defined first, then, in quick succession, the special cases of function, order relation and equivalence relations. We therefore hypothesise that what is happening is that the students fail to get a workable mental image of each of these three conceptually different kinds of relation. In the case of an order relation, it is a natural thought process to imagine the elements ordered in a line, and, in the absence of an embodied image of the notion of equivalence relation, in using the transitive law, it is natural to link to the self-same image.

This phenomenon is an essential element in the transition from elementary to advanced mathematical thinking. In moving from a way of thought that uses related imagery at will to a formal way of thinking that is intended to be formal, the student can not have conscious control over all the mental connections that are made. Lakoff and his colleagues (Lakoff & Johnson, 1999; Lakoff & Nunez, 2000) argue powerfully that all thought is embodied. However, it is tautological to claim that human thought occurs in a physical brain. A more useful distinction needs to be made. Mathematicians are acutely aware that their formal thought occurs in an embodied mind, but they struggle to make their deductions as free from embodied influences as is humanly possible. Tall (2002) theorises how embodied thought experiments can suggest formal theorems and how formal deduction can prove them, sometimes giving structure theorems that have new, more sophisticated, embodiments.

What is happening with these students is that their introduction to formal thought concerning equivalence relations, order relations, and functions as examples of relations occur in a brain in which the concepts are intimately linked together. Some make connections in a manner familiar to the mathematical community, but most, if not all, have a variety of other mental linkages which need to be addressed for serious progress to be made.

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