What is a Scheme?

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This chapter is dedicated to, and fundamentally influenced by, Richard Skemp's pioneering work on schemes. Both of us were present in Warwick when Richard died, and we sang at his funeral. This is no maudlin sentiment but a deeply felt gratitude for the man and his work. David Tall occupies Skemp's Chair of Mathematics Education at Warwick, and both of us were influenced, as have been many mathematics educators, by Skemp's highly polished works. His contributions have an enduring quality because he tackled basic issues of mathematical intelligence.

Schemes: Psychology to Mathematics Education

The terms "schema" and "schemata" were apparently introduced into psychology by Bartlett (1932), in his study of memory. Bartlett took the term from the neurologist Henry Head who had used it to describe a person's conception of their body or the relation of their body to the world. Bartlett used the term schema in much the same way as Skemp, following him, did: as an organised structure of knowledge, into which new knowledge and experience might fit. The utilisation of Head's notion of schema in psychology was reviewed by Oldfield and Zangwill (1942a, 1942b, 1943). Bartlett's notion of schema was picked up by Skemp (1962, 1971), and then Rumelhart (1975) also resurrected Bartlett's idea and terminology, once again in the study of memory.

Minsky (1975) introduced his idea of "frames" and Schank (1975) the idea of "scripts", both of which are similar to Bartlett's schemata. Davis' (1984) influential book on cognitive science methods in mathematics learning leant heavily on the idea of schemes. It is fair to say that whilst the term scheme has been used in mathematics education (see, for example, Steffe, 1983, 1988; Davis, 1984; Dubinsky, 1992; Cottrill *et al.*, 1996) there have not been many attempts to define more precisely what might constitute a scheme.

Notable exceptions, apart from Skemp's own writings, are the articles by Dubinsky (1992) and Cottrill *et al.* (1996), the first of which at least acknowledged that some sort of definition would help

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people working in the field, and the second of which attempted a recursive definition.

Skemp (1986) made it clear that schemes play a pivotal role in relational understanding:

> *To understand something means to assimilate it into an appropriate schema.* (p. 43; author's italics).

Chapter 2 of *The Psychology of Learning Mathematics* is entitled "The Idea of a Schema". In that chapter Skemp makes it clear that he views schema as a connected collection of hierarchical relations. It is this point of view that we wish to explore.

Structuring the World: Categories

A necessary condition for higher order mental functioning is the ability of an individual to categorise things in the world. In order to count, for example, we need things to count. These things are categorised by us as instances of the same thing for the purposes of counting. Do you have enough clean shirts to take on holiday? Are there enough chairs to seat all our guests? In order to answer questions like these we necessarily have to see different things, such as different shirts or chairs, as instances of the same category for the purpose of counting them.

This ability to categorise is possessed by many animals (Edelman, 1989) and is fundamental to the ways in which human beings structure their worlds. It is critical in mathematics learning because counting, the first mathematics that most of us engage in, is so clearly predicated on an ability to categorise.

The patterned records of actions we use in mathematical activity are themselves instances of categorisation. The categorisation involved in the formation of schemes is the brain categorising its own activities. In primary categorisation, perceptual events are categorised, whereas in secondary categorisation – the type of categorisation in which schemes are formed – the brain's own responses to perceptual categorisation are themselves categorised. This reflective activity of the brain is an essential part of Piaget's theories of the development of logico-mathematical structures. Recently, it has been given a more detailed functional and structural form by Edelman (1989, 1992):

> … the theory of neuronal group selection suggests that in forming concepts, the brain constructs maps of its own activities, not just of external stimuli, as in perception. According to the theory, the brain areas

responsible for concept formation contain structures that categorize, discriminate, and recombine the various brain activities occurring in different kinds of global mappings. Such structures in the brain, instead of categorizing outside inputs from sensory modalities, categorize parts of past global mappings according to modality, the presence or absence of movement, and the presence or absence of relationships between perceptual categorizations. (1992, p. 109, author's italics)

Action Schemes and Higher Order Schemes

Apart from the ability to subitize numbers in suitably configured arrays, number involves counting, which is an action scheme – a sequence of actions performed to achieve a goal. Using counting, addition is an extended action scheme to obtain the total in two collections. Children use the initial "count-all" scheme until they recognise certain parts of the process as redundant. They then make new connections by omitting the initial counting. This new reconstructed scheme misses out the first part of the count and performs a "count-on". If they sufficiently well manage the "count-on" to allow a neuronal trace of the input to link with the output (or the results are re-corded to *see* the links), they may make a connection between input and output to give the known fact.

Essential to an understanding of schemes is the focus of attention of a learner. What it is a person focuses on in an action scheme determines the consequent structure of that scheme for them. The first attempt to use attentional systems as the basis for mathematical development appears to be von Glasersfeld's (1981) attentional model of unitizing operations. These operations play a major part in his theoretical model for an operation of the human mind that creates a unitary item from perceptual experience. They are still our only model for how it is humans form units and pluralities. So, in this sense, attention is fundamental to how we understand numerosity. von Glasersfeld stated very clearly what he meant by attention for this purpose:

> ... I want to emphasize that 'attention' in this context has a special meaning. Attention is not to be understood as a state that can be extended over longish periods. Instead, I intend a pulselike succession of moments of attention, each one of which may or may not be 'focussed' on some neural event in the organism. By 'focussed' I intend no more than that an attentional pulse is made to coincide with some other signal (from the multitude that more or less continuously pervades the organism's nervous system) and thus allows it to be registered. An 'unfocussed' pulse is one that registers no content. (p. 85)

and he cites neurophysiological experiments that support the view that:

> … attention operates above, and independently of, sensation and can, therefore, function as an organizing principle; second, if attention can, indeed, shift from one place in the experiential field to another, it must have a means of regarding ... these places and disregarding what lies in between. (p. 85).

Von Glasersfeld's model is fundamental to the Steffe-Cobb-von Glasersfeld-Richards theory of children's counting types and arithmetic operations (Steffe, von Glasersfeld, Richards & Cobb 1983; Steffe, Cobb & von Glasersfeld, 1988). von Glasersfeld provides a model of how records of experience can be obtained by a focus of attention on the perceptual stream of data. It is a basic tenet of a constructivist theory of learning, as we understand it, that these records of experience – the memories – are not veridical records of any *thing* in the world external to the recorder. Rather they are chemical traces of neuronal activity occurring as a result of perceptual interaction with the world. Importantly, for our theme, these traces are themselves capable of being taken as the primary material for experience:

> Perceptual categorization, for example, is non-conscious and can be carried out by classification couples, or even by automata. It treats *signals from the outside world* – that is, signals from sensory sheets and organs. By contrast, conceptual categorization works from within the brain, requires perceptual categorization and memory, and treats *the activities of portions of global mappings* as its substrate. (Edelman, 1984, p. 125, author's italics)

This, for us, is in essence how schemes come into being, so we make the following definition:

- An action scheme (or 0-order scheme) is a sequence of actions performed to achieve a goal.
- An n^{th}-order scheme is a categorization of lower order schemes.

Our brains take as basic data for reflection the records of our previous experience, and just as we categorize visual perceptions from cup-like things into the category of cups, so we categorize patterns of action such as directed movement with synchronized standard utterances into the category of counting.

Examples of Scheme Formation

A well-studied action scheme is children's dealing, or distributive counting, to allocate fair shares (Miller, 1984; Davis and Pitkethly, 1990; Davis and Hunting, 1990; Davis and Pepper, 1992). This is a sequence of actions that children in many, if not all, cultures can carry out to allocate fair shares , but one which, at a young age, 4– 5 years, they usually do not internalize as a 1st order scheme. By the time children are about 8 years of age, however, it is not uncommon for them to refer to dealing as "just like when you play cards". At this point, we infer, they have conceptually categorized various action schemes of dealing and so established a 1st-order scheme for dealing.

An example of higher order scheme formation appears in the thought of a student, Stephanie, described by Maher and Speiser (1997). Carolyn Maher at Rutgers University, and her colleague Robert Speiser, at Brigham Young University, reported a beautiful series of episodes involving a young high school student, Stephanie. She found the formula for calculating the binomial coefficients by relating the problem to one of counting towers of blocks – a problem she, along with many other children, had studied in elementary school.

In 1989 Maher, and her colleague Al Martino, began working with a small group of first graders to encourage the children to explore and explain their differences in thinking as they solved problems. Stephanie was one of the children in this group and when she was in grade 4 worked on the problem: how many different towers of a fixed height can we make from blocks of two colors? (Maher and Martino, 1996; 1997). In the fall of 1995 – coincidentally, just after Richard Skemp passed away – Stephanie had moved to a different school and was in grade 8. Maher and Speiser worked with Stephanie on her reasoning in mathematics problems, which now revolved around algebra. Stephanie had calculated the binomial coefficients for $(a+b)^2$ and $(a+b)^3$. One of the researchers asked her about these numbers for $(a+b)^3$ and Stephanie replied: "So there's *a* cubed. ... And there's three *a* squared *b* and there's three *ab* squared and there's *b* cubed." She then said: "Isn't that the same thing?" The researcher asked what she meant, and Stephanie replied: "As the towers."

It's fair to say that the researchers were surprised at this sudden, unannounced, appearance of towers in an algebra problem. Stephanie had not systematically counted towers of blocks in school since elementary classes. Maher and Speiser's hunch was that Stephanie was visualizing towers of height 3 in order to organize the products of powers $a^i b^j$ of a and b. They got support for this hunch when, in questioning Stephanie further about the terms in the expansion of $(a+b)^3$, she said: "I don't want to think of *a*'s. I want to think of red." In the same session Stephanie was asked: "What about *b* squared?" and she replied: "Um. Two yellow."

Stephanie seemed to interpret the numbers a and b in the expression $(a+b)^3$ as colours of blocks – red or yellow – and the index "3" as the height of a tower of red or yellow blocks. She could then bunch together those towers that had the same number of red (and so same number of yellow) blocks. This was because, in her interpretation, the order of the red and yellow in the towers didn't matter: the towers were standing in for *products* of *a*'s and *b*'s, just as the individual red and yellow blocks were standing in for individual *a*'s and *b*'s.

Imagery can act as a *generator* of mathematical thought, when the imagery acts to reduce the load on working memory. This is what happened to Stephanie when she formed images of the expansion of $(a+b)^3$ as a row of towers of red and yellow blocks. Far from being a dead weight, filling up working memory with irrelevant detail, the images acted as a powerful engine for her, allowing her to accelerate rapidly and easily through some complicated counting.

Two weeks after the episode reported above, Stephanie was asked to explain to a researcher who was not previously present, what she had said about binomial coefficients and towers. As she began to explain Stephanie described, in vivid detail, episodes from a grade 4 class in which she and two classmates had figured out how to build towers of a given height from those of height one less. Stephanie was able to write a recursive formula for the binomial coefficients by utilising her recalled images of towers of blocks.

That Stephanie had established a 1st-order scheme from conceptual categorization of the action schemes of building blocks was evident from her previous conversations about them. At the time of the first interview, above, there does not seem to be evidence that she had similarly schematized the action schemes of expanding a power of a sum of two terms. However Stephanie made the remarks: "Isn't that the same thing?" and "As the towers" in her explanation of her algebraic calculations. At this point we infer that she has, by analogy, established a 1st-order scheme for binomial expansions. When, two weeks later, she is able to write down the recurrence relation for the binomial coefficients by reference to counting towers we infer that Stephanie is able to express clearly that she sees essentially the same process occurring in these two 1st-order schemes. In other words, she has established a 2nd-order scheme through conceptual categorization of 1st-order schemes.

Schemes and Symbols

The operations of arithmetic benefit from being written in signs. These signs enable a subtle form of compression that is not immediately apparent. For many students the signs are simply indicators – for example, the sign "2 + 3" indicates to some students that one should carry out an action, whereas for other students the same sign also functions as a symbol. For these latter students there is a symbolic relation with other signs and a flexibility in interpreting a frame of reference for the signs. This is essentially what Gray and Tall (1994) refer to as a procept. Steffe (1988) has written in relation to symbols that:

> Children's operations seem to be primarily outside their awareness, and, without the use of symbols, they have little chance of becoming aware of them nor can they elaborate those operations beyond their primitive forms.

In answer to the question of whether his use of the word "symbols" referred to conventional mathematical signs, or whether there was a deeper interpretation in which the symbols were records of a process of interaction between a student and teacher, Steffe (personal communication) replied:

> I was thinking of, for example, the way in which a child might use the records of past experience that are recorded in the unit items of a sequence to regenerate something of that past experience in a current context. The figurative material that is regenerated may act as symbols of the operations of uniting or of the results of the operations in that the operations may not need to be carried out to assemble an experiential unit item. The figurative material stands in for the operations or their results. In this I assume that the records are interiorized records – that is, records of operating with re-presented figurative material.
>
> My hypothesis is that these operations will continue to be outside the awareness of the operating child until a stand-in is established in which the operations are embedded. ... awareness to me is a function of the operations of which one is capable. But those operations must become objects of awareness just as the results of operating. To become aware of the operations involves the operations becoming embedded in figurative material on which the operations operate. To the extent that this figurative material can be re-generated the operations become embedded in it.

The figurative material is also operated on again. In this way the operations are enlarged and modified.

The idea that operations are outside conscious awareness until a mental "stand-in", or symbol, is developed upon which the operations can act mentally – is of critical importance in the development of mathematical schemes. For it is these symbols that form 1st-order schemes: not only is the scope of the operations extended, as Steffe says, but what the operations operate on becomes a mental object – a 1st -order scheme.

When Bob Speiser talked about Stephanie's scheme for thinking about the binomial coefficients at the Psychology of Mathematics Education conference in Lahti, Finland, we wondered how much of the connection between the towers and the binomial coefficients Stephanie had taken on board. We asked him if Stephanie had thought about a more complicated situation, such as $(a+b+c)^3$ in terms of towers made from three colours. He affirmed that she had indeed; what's more, Stephanie could write down the recursive definition for the coefficients of the products of powers for *any number* of variables *a, b, c, d , ...* by using images of towers of blocks of the same number of colours as the number of variables.

This is a singular achievement for a grade 8 student, and it illustrates the amazing generative power of vivid images that are tightly coupled to a problem. The complicated looking signs:

$(a+b)^n =$ sum of terms $C(p,q)a^p b^q$ where $p+q=n,$ and
$C(p+1,q+1) = C(p,q) + C(p,q+1)$
with $C(n,0) = 1$ and $C(n,n) = 1$

are not abstruse to a child who has assimilated a model in terms of towers of coloured blocks. These signs simply express a recursive way of building towers. What appears to be complicated mathematics is just a way of writing this recursive relationship. The written mathematics – the marks – are what many students focus their attention on. In so doing they can, and often do, lose sight of the fundamentally simple idea that the marks, or signs, express. Worse, their only terms of reference for the signs is likely to be at an indexical level – a conditioned response. This is how many students see formulas in mathematics: as something upon which something has to be done, such as rearrangement, substitution, cancellation or similar actions. What the signs refer to for students who think this way is the actions that they themselves could carry out. Furthermore, that is all the signs refer to. A student who understands the signs symbolically can also do things like rearrangement, substitution, and cancellation. However, they can do more. They can focus on other aspects of the signs, such as Stephanie's focus on the recursive formula for the binomial coefficients in terms of a recursive procedure for building towers of blocks. This is an essential aspect of Peirce's (see Deacon, 1997) view of the icon-index-symbol relationship as a hierarchy. The ability to operate at one level in this hierarchy implies an ability to operate at a lower level, but not conversely. It is not that children who learn to operate symbolically in mathematics forget the relationship of signs to concrete objects or to remembered processes – it is just that in a particular context some of these memories are not particularly helpful. Indeed, they may well be just so much clutter, filling up the available space in working memory. The sign formulas are useful for programming a machine to calculate the binomial or multinomial coefficients, but these mathematical signs also behave as a very compact symbolic expression of the relationships they embody. So a student capable of interpreting these signs symbolically can choose to think in terms of models, such as towers of blocks, *or can withhold any such interpretation*, knowing that they could interpret it this way if they wished. Their thinking has become proceptual (Gray and Tall, 1994) and they have gained enormous flexibility and economy of thought.

This leads to a curious, and critical, chicken and egg situation. Students may not have a scheme about which they can talk, but their lack of awareness of the operations used in the procedure inhibits them from talking about the procedure in the absence of carrying it out. Talk about the procedure by a teacher would seem to be insufficient for scheme formation, because many students have no awareness of the operations of the procedure on which to hang the talk. They have no symbolic frame of reference for the teacher's words. Repeated carrying out of the procedure by itself is no guarantee that awareness will result, because there may be no necessity to reflect on the procedure in the absence of carrying it out in practice. This dilemma was summed up succinctly by von Glasersfeld (1990), who wrote:

> If it is the case that ... conceptual schemas – and indeed concepts in general – cannot be conveyed or transported from one to the other by words of the language, this raises the question of how language users acquire them. The only viable answer seems to be that they must abstract them from their own experience. (p. 35)

So if talk alone and repeated practice do not suffice for the formation of schemes, and if students must abstract them from their own experience, where does this leave a teacher? Many students, in the process of counting or carrying out elementary operations on numbers, focus their attention on what we, as observers, deem to be peripheral properties of objects, such as their colour or size (Gray, 1991; Gray & Pitta, 1996; Gray & Pitta, 1997a, b; Gray, Pitta & Tall; 1997).This focus of attention necessarily occupies a student's working memory with detail that is known to be irrelevant to the formation of higher order mathematical schemes. We don't usually care whether the chairs we are counting are red or blue, for the purposes of determining whether a given collection of people will be able to be seated. This filling of working memory with irrelevant detail has two effects. First it slows the student considerably: they are able to consider fewer examples in a given time than those students whose focus is not on irrelevant detail. Consequently, such students have considerably fewer examples to categorise. Second, with each example they encounter there is less room in working memory for those aspects that we do consider relevant for the establishment of higher order schemes. As a result these students are seeing fewer examples, and less that is relevant in each example they encounter. Consider, for example, a student at secondary level who is engaged in expansion of the algebraic square of a sum or difference of terms. Examples such as $(a+b)^2$, $(c+d)^2$, $(e-f)^2$, $(s+3t)^2$, $(x-y+z)^2$, might well be seen by a student as unrelated things to do – unrelated actions which they do not categorize as similar instances of a single phenomenon. The most likely purpose of a collection of exercises like this is to give a student practice in algebraic expansion, and to give them enough examples to enable them to categorise the exercises as examples of algebraic squaring of a sum or difference. If a student continues to focus on the actual letters, or whether there is a sum or difference, or a coefficient "3", then they are unlikely to attain the categorization a teacher intended. They remain stuck at the level of individual actions, instead of forming schemes. Consequently, algebra becomes, for such students, a hard subject, with many detailed and unrelated calculations. What a student needs to do is to learn to throw away much of the perceptual information available to them. This involves a focus of attention on a different aspect of the algebraic expressions.

Schemes as Mental Objects

Perceptual categorization gives rise to our feeling for prototypical named objects in the world. The fact that we can talk about chairs, or a chair, without referring to or pointing at a particular world-thing is a result of a process of perceptual categorization. The concept “chair” is a mental concept, and not a corporeal world-thing. Nevertheless, it is extremely convenient for us to think of “chair” as an object. We cannot even count “chairs” unless we see certain world-things as instances of “chairs”. In other words, we reify our conceptions obtained through perceptual categorization. This should give us cause to suspect that schema formation, which we have defined as categorization based on action schemes, also leads to mental objects. These mental objects are based not on world-things, as in perceptual categorization, but on world-actions.

Dörfler (1993) casts doubt on the nature of mathematical objects. He writes:

> My subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariably comes to my mind are certain patterns of dots or other units, a pentagon, the symbol 5 or V, relations like 5+5=10, 5*5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever could find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless I deem myself able to talk about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object ‘5’. (pp. 146–147)

This, however, is to miss the point of categorization. Where, in our heads do we see the object “chair”? As Dörfler intimates, we may see images of particular chairs, and even be capable of forming images of chairs we have never seen. The point is that “chair” is the name of a category to which we agree that certain world-objects belong. As such, it acquires object status: that of a mental object, a conception, resulting from perceptual categorization. Likewise, the word “dealing” refers to a category of world-actions, and as such it is a mental object resulting from conceptual categorization.

Perceptual, Social and Conceptual Categorization

Perceptual categorization is, as we have noted, common not only among different peoples, but also among animals of many sorts. Social categorization is also very common among mammals. This form of categorization is exemplified by the "them and us" syndrome: the division of a group into two on the basis of a perceived difference, such as skin color, accent, behaviour, speech, or indeed almost any perceived difference (Harris, 1998). These types of categorization are so common, occurring almost obligatorily in human society, that we often overlook them as mental constructions and take them to represent significant differences given by the phenomenal world.

Why then should the sort of categorization that we postulate as the basis of scheme formation seem to be so difficult for individuals to establish? Why should it require an elaborate apparatus of cultural transmission – classrooms, teachers and textbooks, not to mention psychologists and mathematics education researchers? The answer, one might suspect, lies in the everyday nature of perceptual and social categorization. Perceptual categorization is vital for animals that move around looking for food, shelter, and mates, in a potentially dangerous world. Social categorization is an inevitable consequence for humans who have warring social ancestors (Harris, 1998). Conceptual categorization, one might imagine, arises as a possibility, and only as a possibility, with the development of language. However, the recency, in evolutionary terms, of conceptual development does not provide a sufficient reason why conceptual categorization should be so difficult, at least in the field of mathematics.

What seems, from the empirical evidence, to be a much more compelling reason for the difficulties we see in scheme formation in mathematics is the general lack of awareness that humans have of action schemes. By "lack of awareness" we mean inability to articulate action schemes as distinct from their outcomes. Steffe, among others has remarked on this lack of awareness of action schemes, as we have noted above. Granted this lack of awareness, it is almost clear that we should have difficulty categorizing action schemes: we are not aware of them as schemes. Instead, what we are aware of is the outcome of those schemes: the results of sharing by dealing, or the results of counting, the results of an algebraic calculation. Why this should be the case is, as far as we are cognisant, not known. However, granted that it is the case, it provides a considerable obstacle for conceptual categorization of action schemes. As Steffe has remarked on other occasions, this provides a supremely important role for a teacher of mathematics in helping students to be able to articulate their action schemes regarding number, space, and arrangement – the basic elements of mathematical experience.

The question of why humans should not naturally find it straightforward to articulate an awareness of action schemes therefore assumes a great importance in the study of the acquisition of mathematical conceptual thought. A simple reason suggests itself, namely a separation between the language and motor centres of the brain. The neurologist Ramachandran (1998) highlights the problem of putting actions into words as a translation problem. He writes, a

> … fundamental problem arises when the left hemisphere tries to read and interpret messages from the right hemisphere. … crudely speaking, the right hemisphere tends to use an analogue – rather than digital– medium of representation, emphasizing body image, spatial vision and other functions of the how pathway. The left hemisphere, on the other hand, prefers a more logical style related to language, recognizing and categorizing objects, tagging objects with verbal labels and representing them in logical sequences (done mainly by the what pathway). This represents a profound translation barrier. (p. 283, author's italics)

This translation barrier is particularly evident in adults who have damage to the right brain or a disconnection of the two hemispheres. This happens, for example, when the corpus callosum, the bridge connecting the two hemispheres, is damaged or cut (as used to happen in cases of severe epilepsy). How might this explain why young children have difficulty articulating their action schemes? After all, only in rare cases will children have such sever dislocation between their two brain hemispheres. Yes, but the fact is that the corpus callosum is quite undeveloped in young children: the nerve fibres connecting the two hemispheres have not yet been fully myelinated, so nerve impulses in young children do not conduct between the left and right hemispheres as well as they do for older children and adults (Joseph, 1993, p. 353 *ff*). Whilst this might suggest a reason for young children's relative inability to articulate awareness of their action schemes, it does not explain why older children, and indeed many young adults, are equally incapable of such articulation. It is not uncommon, even in university level mathematics, for students to be able to carry out taught procedures – such as solving simultaneous linear equations by Gaussian elimination – and yet have an almost total inability to articulate how the procedure is carried out. Often the best they can do is to ask for an example, which they then proceed to calculate.

A further clue to the relative difficulty in articulating action schemes comes from work of Ullman *et al* (1997) on language difficulties in sufferers of Alzheimer's disease on the one hand and

Huntington's and Parkinson's on the other. Their work suggests that word memory relies on areas of the brain that handle declarative memory – memory of facts and events. These areas appear to be the temporal or parietal neocortex. However, rules of grammar seem to be processed by areas of the brain that manage procedural memory, the basal ganglia, which are also involved in motor actions. That there seem to be two distinct brain areas for procedural and declarative memory must make us suspicious. In mathematical settings, at least, the region devoted to declarative memory may have difficulty – that is, few mechanisms for – taking as its basic material the activities of the region responsible for procedural memory. If so, the role of teacher becomes even more evident: as an external conduit to allow declarative memories to be formed from the raw material of stored procedural memories.

Let us look again at Stephanie's categorizations in this light, because Stephanie is a child who *was* able to make higher order categorizations beyond the commonplace. First, Stephanie was motivated to seek reasons for things mathematical (Maher & Speiser, 1997). Indeed this was a prime reason for Carolyn Maher and Bob Speiser's focus on Stephanie. However, if we are right about the need for procedural memory to be externalized before it can become declarative then Stephanie must have had some external influence on the formation of her declarative memories of building towers. Did she? Indeed she did: Maher and Speiser report how Stephanie was engaged in elementary school with a group of children who built towers together, and engaged in argument and reasoning about their activities. So Stephanie's external agent in this case was her group of classmates who not only built towers with her, but also argued with her. What about Stephanie's 2^{nd}-order categorization, in which she linked in considerable detail building towers with the binomial and multinomial theorems? We have to suspect, from the records of interviews, that her external agent for utilising procedural memories to create declarative ones, was the pair of interviewers. Through the questions asked by the interviewers Stephanie was able to take her procedural memories and turn them into objects of reflection, which then created declarative memories for her.

The decisive force in the creation of higher-order schemes, therefore, may be an appropriate agent who can externalize procedural memory and utilize it, consciously or not, so that a child can form declarative memories. The reason for this, we hypothesize, is that the temporal and parietal neocortex has, in young children, or young adults, few mechanisms for taking the memory activities of the basal ganglia as raw data for the formation of new declarative memories. What has to happen, we suspect, is that an external agent needs to *externalize* those memories of motor actions from the basal ganglia and recast them in a form suitable for the temporal or parietal neocortex to process them as procedural memories.

Connections with APOS Theory

Dubinsky and colleagues (Dubinsky, 1992; Cotterill *et al*, 1996) have proposed an Action-Process-Object-Schema theory in which schemes feature as the end result of a structural organization. In the APOS setting, an *action* is a physical or mental transformation of objects to obtain other objects. A *process* arises from an action when a person is able to reflect upon and establish conscious control over the action. A process becomes an *object* when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations." (Cottrill, et al, 1996). *Schemas* enter into this theory as structural organizations of actions, process and objects.

Our proposal is that this "structural organization" is obtained through conceptual categorization in the sense of Edelman (1989). The essential point, for us, is that some mechanism must be postulated to facilitate the structural organization central to APOS theory. In line with Skemp's emphasis on brain activity and brain models of mathematical thought we believe that the process of conceptual categorization provides such a mechanism. Skemp discussed the connections between categorization and schema in *Intelligence, Learning and Action*: indeed he regarded them as practically synonymous. The only extra highlight we wish to stress is that, in line with Skemp's emphasis on intelligent, goal-driven, action (an emphasis he shared with many other seminal thinkers in mathematics learning), the focus on scheme formation in mathematics is on categorization beginning with action schemes.

Dedication

We owe a debt to Richard Skemp. Apart from pioneering work in schema, he began the process of modelling what it is that the brain is doing when it's thinking mathematically. Skemp concentrated on fundamental issues of models for brain operations in mathematical thought, and for intelligent thought more generally. Recent developments in psychology and neurology have been reinforced the gems of principles and models that he elucidated so clearly. He established a solid link between intelligent human actions and the operations of our brains, and regarded the study of mathematics learning as a way to develop models for higher-order intelligence in general. As Anna Sfard has written (this volume) " Skemp ... came to an empty field and left it with an impressive construction." Skemp, himself, wrote:

> ... it seemed that by studying the psychology of learning mathematics, the improved understanding of intelligent learning which can be gained by working in this area should be generalisable to give a better understanding of the nature of intelligence itself: with a potential for applications extending over a very wide range of activities. (1979, p. 288)

This gentle man treated human beings as intelligent creatures who have a capacity to reflect on their actions and learn from them. In so doing he was led to examine models for brain functioning that might allow us to think in this way. His excursion into brain models was an intellectually necessary part of his quest to understand what it is that allows human beings to think as they do, and to behave with the intelligence they are capable of manifesting. We are forever grateful to Richard Skemp for these pioneering efforts. They laid a clear and firm foundation for a subject whose time has now well and truly come – the nature of the mathematical brain, its relation to mathematical intelligence, and to intelligence in general.

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