Continuities and Discontinuities in Long-Term Learning Schemas

(Reflecting on how relational understanding may be instrumental in creating learning problems)

David Tall
University of Warwick

> Master. … I wil propound here ii examples to you whiche if you often doo practice, you shall be rype and perfect to subtract any other summe lightly …
>
> Scholar. Sir, I thanke you, but I thynke I might the better doo it if you did show me the working of it.
>
> Master. Yea but you muste prove yourselfe to do som thynges that you were never taught, or els you shall not be able to doo any more than you were taught, and were rather to learne by rote (as they cal it) than by reason.
>
> (Robert Recorde, *The Ground of Artes*, 1543)

The notion of "learning by reason" rather than "learning by rote" has long been a focus of creative teaching. In writing the oft-quoted paper on "instrumental understanding" and "relational understanding", Richard Skemp (1976) is a significant link in the chain of those who developed notions of "meaningful learning". Skemp, however, had wider goals in life. For him, relational learning was part of a broader plan of developing a "long-term learning schema" for life-long learning (Skemp, 1962).

In his publications *Understanding Mathematics* (1964) for secondary school and *Structured Activities in Learning Mathematics* (1993–4) for younger children he developed learning schemas for children's learning over long time periods. He also developed a rich theory of human learning that has proved significant in shedding light on how the cognitive processes of human thinking can lead to the logic and aesthetic beauty of formal mathematics.

Published in *Intelligence, Learning and Understanding – A Tribute to Richard Skemp* (eds David Tall and Michael Thomas), 2002, pp. 151–177. The book is available for sale from PostPressed, http://www.postpressed.com.au

It is the purpose of this paper to consider the development of long-term learning. The logic and rigour of mathematics is such that it seems that a curriculum can be built in successive stages where each stage builds on preceding learning. Indeed, the British National Curriculum for children aged 5 to 16 is formulated in ten levels where each level is seen as developing a coherent set of ideas building on earlier levels. This would suggest that a well-designed curriculum could be continuous in the sense that each stage builds smoothly on those already experienced. Such a "self-evident truth" is, in fact, false. This paper will show that there are many points in learning in which cognitive discontinuities occur. Thus the building of a long-term curriculum is likely to need to face various situations in which difficulties arise and need to be conquered.

Skemp was aware that appropriate schemas need to be developed that are appropriate to a given learning task:

> Since new experience which fits into an existing schema is so much better remembered, a schema has a highly selective effect on our experience. What does not fit into it is largely not learnt at all, and what is learnt temporarily is soon forgotten. (Skemp, 1986, p.41)

A major problem occurs when what may be highly appropriate at one stage may be unsuitable later. Skemp formulated this in his overall theoretical position saying:

> … not only are unsuitable schemas a major handicap to our future learning, but *even schemas which have been of real value may cease to become so* if new experience is encountered, new ideas need to be acquired, which cannot be fitted in to an existing schema. A schema can be as powerful a hindrance as help if it happens to be an unsuitable one. (*ibid.* (my italics))

An example of a schema of short-term value is the use of so-called "fruit salad" algebra, in which meaning is given to expressions such as $4a+3b$ by thinking of the letters as standing for actual objects such as "4 apples plus 3 bananas". This will give support in manipulation of expressions such as $4a+3b+2a$ to give $6a+3b$ by simply thinking in terms of manipulating the numbers of each fruit. However, this short-term gain soon leads to difficulties in interpreting expressions such as $7ab$. Does it mean 7 apples and bananas? Is 3 apples plus 4 bananas equal to 7 apples and bananas? Is $3a+4b = 7ab$? Initial simplistic approaches to subjects that subsequently lead to inappropriate links may harm long-term development. My contention takes this observation a step further: even well-designed learning tasks can—at a later stage—harm future understanding.

In the remainder of this paper I begin by considering the general problem of discontinuities in a long-term curriculum, followed by a range of examples of discontinuities occurring at various stages of school mathematics. Since every journey begins with a single step, I start with what I term a *cognitive root*; this is a starting point having meaning for the learner at the beginning of a learning sequence, yet containing the possibility of long-term meaning in the later theoretical development. I then question whether a successful beginning will necessarily lead to long-term success, considering various aspects of the calculus, where visuospatial ideas can act as a long-term foundation in a range of different possible approaches.

But I also reveal that the different representations—symbolic, algebraic and numeric—do not always have obvious links between them. The consequence is that different parts of the subject may benefit from different kinds of representations and pose different kinds of problems for different students. A powerful tool to address these problems is the willingness of the student to reflect carefully on new ideas, to see how they are similar and how they *differ* from earlier meaningful ideas. Learning by rote may allow the student to cope with similar problems, but reflecting on the nature of the mathematics is more likely to support flexible long-term learning.

Continuities and Discontinuities in Long-Term Learning Schemas

On the assumption that the long-term curriculum designer should be more concerned with ultimate coherence and successful learning than settling for a short-term gain, a good solution would seem to be a long-term curriculum that builds steadily and continuously on previous experience. This seems to be the underlying aim of much curriculum design, after all, mathematics is a coherent and logical subject, so its teaching should be amenable to coherence and logic.

A quarter of a century ago I remember, as an earnest young mathematically oriented educator, suggesting long-term learning schemas based on my interpretation of Skemp's ideas. For me at this stage the quest entailed

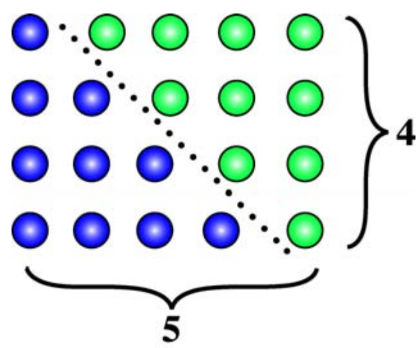


Figure 1. 1+2+3+4 is half of 4 times 5.

looking for ways in which I could present subtle mathematical ideas in simple ways at appropriate times and revisit them later in successively more sophisticated ways—for instance, seeing the sum of an arithmetic sequence 1+2+3+4 as a triangular array of dots which could be fitted to an identical array to give a rectangle of four rows and five columns. Thus the sum 1+2+3+4 is half of 4 times 5. This could be written as an arithmetic sum by adding 1+2+3+4 to 4+3+2+1 in pairs to get 4 lots of 5:

add columns ↓

1 + 2 + 3 + 4
4 + 3 + 2 + 1

---

5 + 5 + 5 + 5 = 4 x 5

Figure 2. Two lots of 1+2+3+4 give 4 lots of 5.

At a later stage this could be generalised algebraically to give

$$1+2+3+\ldots+n = \tfrac{1}{2}n(n+1),$$

and finally proved by induction when, and if, this may be considered appropriate.

The logical development from physical objects, through arithmetic, algebra and formal proof seemed to me an ideal part of a continuously developing long-term learning schema that can be revisited at successive times in more subtle ways. However, such an "ideal" coherent sequence of activities proves to have an Achilles' heel. Although the sequence of development may be apparent to an experienced teacher, at the time of learning the links may not be so readily apparent to the growing child.

The reality is that one must build on the ever-changing cognitive structure of the individual. Achieving a continuous and coherent development proves far more elusive than one might expect. The problem is essentially that the brain works by building connections between neuronal circuits, evoking many internal links, most of which are unconscious:

> Conscious thought is the tip of an enormous iceberg. It is the rule of thumb among cognitive scientists that unconscious thought is 95 percent of all thought—and that may be a serious underestimate. Moreover, the 95 percent below the surface of conscious awareness shapes and structures all conscious thought. If the cognitive unconscious were not there doing this shaping, there could be no conscious thought.
>
> (Lakoff & Johnson, 1999, p. 13)

These links, deeply embedded in our biological brain, produce a complex mental image of concepts (termed the *concept image* by Tall & Vinner, 1981), which operates in many subtle ways. Vinner (1997) speaks of *pseudo-conceptual thinking* in which students learn to do the things that give satisfactory immediate success but may not plumb deeper conceptual issues. These deeper mathematical structures may have no meaning or purpose for the learner at the time so that—even if the curriculum is presented in a manner that is (to the teacher) coherent and logical—it can fail to be understood by the learner.

Skemp was a master at formulating theories to cover such phenomena. He referred to the difference between learning which is an *expansion* of current knowledge—where new ideas fit easily with current schemas—and *reconstruction* of knowledge, where the old schemas must be reflected upon and modified to fit with the new ideas. In a given context, conceptually well-structured learning may be based upon mental images that "fit" the context. Yet these self-same images may later fail in new, as yet unknown, situations.

In designing long-term learning schema, it becomes important to consider if and where reconstruction is likely to be necessary, even in instances where previous learning has been relational. It is my contention that shifts of context causing cognitive conflict occur far more widely in the mathematics curriculum than might at first be apparent.

Cognitive Discontinuities Throughout the Curriculum

To get a sense of the kind of difficulties that occur, I begin with a number of examples encountered personally at different points in the mathematics curriculum.

John, a "slow learner" aged 12, could not imagine negative numbers. In primary school his early number experience began with counting objects and the number track consisting of distinct unit blocks. He continued to think of number in terms of counting and found it impossible to imagine a "minus number" of objects. How on earth can anyone imagine *minus two cows*? To him, it didn't make sense.

My son Christopher, then aged 8, was easily able to conceive of "minus numbers". The temperature in the winter measured in Centigrade often went below zero, and he enjoyed playing in the snow. But he could not perform arithmetic with minus numbers. When his younger brother tried to explain to him that 5 take away minus 2 is 7, Christopher stamped around in a temper. He *refused* to believe that when something is taken away the result could be *bigger*.

Jane, aged 12, could not make sense of multiplication of fractions. In particular she could not believe that the product of two fractions could give a *smaller* answer. All her previous experience of whole numbers had intimated that multiplication always leads to a (much) larger number.

Rachel, aged 13, did well at arithmetic and could calculate accurately and efficiently. But her first encounter with algebra was a disaster. The teacher explained that a letter such as x could be used to stand for a number, so that if x is a number, then $x+3$ is "the number plus 3". For instance, she explained, if x is 2, then $x+3$ is 5, if x is 3, then $x+3$ is 6, and so on. But Rachel didn't understand why 'x' had been introduced or what it meant. If she didn't know x, she couldn't calculate $x+3$ and if she *did* know x, she didn't need algebra, it was far easier just doing arithmetic. Why complicate things by using letters? At a later stage when asked to simplify $3+2x+1$, she wrote $6x$, by adding together all the numbers (after all, they *do* have an addition sign between them) and "leaving the x" because she didn't know what to do with it. Algebra, for her, became a meaningless manipulation of symbols using arbitrary rules.

Robert, aged 14, did well with arithmetic and algebra. He understood that 3×3 was written as 3^2, $3\times3\times3$ as 3^3, $3\times3\times3\times3$ as 3^4 and so on. He could even see that 3^n means n lots of 3 multiplied together and x^n means

$$\underbrace{x\times x\times\ldots\times x}_{n\text{ times}}$$

From this it was a short step to see that

$$\underbrace{x\times x\times\ldots\times x}_{m\text{ times}}\times\underbrace{x\times x\times\ldots\times x}_{n\text{ times}}=\underbrace{x\times x\times\ldots\times x}_{m+n\text{ times}}$$

or that

$$x^m\times x^n=x^{m+n}.$$

He had a clear relational idea about the meaning of the power notation in the context, but when he was shown that $x^{1/2}$ must be $\sqrt{x}$ because

$$x^{1/2}\times x^{1/2}=x^1=x,$$

he suddenly became confused because he could not make sense of the notation $x^{1/2}$. If x^n means "n lots of x multiplied together", what does "half a lot of x multiplied together" mean? For the expert, the familiar formula is just being "generalised" to apply to fractional exponents. For Robert, such a strategy had no meaning whatsoever.

James was a successful fifteen year old being taught the rudiments of calculus. First the teacher explained ideas in terms of a picture, with a secant approaching the tangential position, then he went through some numerical examples for $y = x^2$, focusing first on the secant through $x = 1$ and $x = 2$, then successively calculating slopes from 1 to 2, from 1 to 1.1 and from 1 to 1.01. He showed that these slopes get closer and closer to 2, and then moved on to consider the ideas algebraically using $(x+\delta x)^2 - x^2$ divided by δx. After calculating the slope for $y = x^2$ and $y = x^3$, he revealed the general pattern of the derivative for x^n as nx^{n-1} and showed how the rule worked for general polynomials. "That's typical," said James to the teacher after the class, "you always show us the hard ways first before getting down to the simple way to do it." In James's class *all* of the pupils learned to differentiate polynomials but *none* could give any relational explanation of the process. They knew from experience that their teacher would always help them by teaching them a simple rule. The rest of the performance was to please him, not them.

Alec, aged 16, never did understand all that stuff about limits, but he knew that the derivative of $3x^4+5x^2$ is $12x^3+10x$. The derivative of a sum is just worked out by adding the resulting derivatives. He used the "same" rule for a product such as $x^2(x^3+x)$, to give the "derivative of the product" as the "product of the derivatives", namely $2x(3x^2+1)$.

The incidents discussed in this section show children of various ages finding difficulty with a variety of aspects of mathematics in new situations. Even if children have previously been successful through several sequences of learning they may eventually meet a situation that makes no sense. Although these examples arise in a range of different contexts, an underlying story can be uncovered.

Sources of Cognitive Discontinuities

Several of the examples just considered clearly involve a new context in which previous ideas cause a conflict—for instance, John's inability to conceive of negative numbers, or Christopher's ability to give them a meaning but inability to perform arithmetic with them, or Jane's difficulty with a product of fractions giving an unexpected smaller result. The reality of the growth of neuronal connections reveals a vast complex of growing connections that operate in a range of ways to support our human activities. The coordination required for counting involves seeing, pointing to the things seen, performing it in a sequence that includes each object once and once only, saying the number sequence at the same time. This involves a vast number of neural connections, the majority of which are subconscious. Initially the act of counting involves physical objects in physical situations, linking numbers inextricably to real world referents. When a new situation occurs which causes a dissonance with these connections, the individual may feel confused yet be unable to pinpoint the reason for the confusion. Negative numbers "feel wrong" because "you can't have less than nothing." Even if one can envision positive and negatives as credits and debts, or temperatures above and below freezing point, these particular embodiments carry no sense of a full array of arithmetic operations. One may "start at a positive temperature +2°C and *go down* 3°C to end up at the negative temperature –1°C." This may give a sense of difference between temperatures, perhaps even linking to a conception of subtraction. By talking about "taking away a debt" one may even give a sense that "taking away a minus" is the same as "adding a plus". But these are straining the meaning of the original neuronal links involving combining and removing physical objects. The idea of *multiplying* negative numbers is, for most, a bridge too far.

My son Christopher was able to conceive of negative numbers in terms of temperature at the age of eight but had no sense of how to do arithmetic with them. His younger brother Nic, then only five years old had, without any prompting, actually *asked* how to multiply minus numbers. He too knew about the concept of minus number in terms of temperature. He reasoned that "ordinary numbers" could be added and multiplied so why couldn't you do the same with "minus numbers". On discussing the concept of lending him pocket money and putting pieces of paper with minus numbers in his purse to record the operation (in denominations of –10 pence), I asked him how I could give him 50 pence if I had no money at the time. He said "you could take away five of the 'minus tens'." He then thought for a moment, smiled, and said "… oh, so *two minuses make a plus*." He could then do *any* sum, difference or product involving plus or minus numbers. His one generative idea enabled him to do them all in a consistent manner *without any further teaching*.

How can two children brought up in the same environment be so different? My interpretation of this situation is that Nic happened to think about arithmetic by focusing his attention on the essential detail of the symbols and operations. He probably did not link operatively to physical reality at the time. For him, therefore, it was natural to work in his cognitive context and seek how to do the same operations on minus numbers. I conjecture that such a conception was not possible for his brother Chris because of his neuronal links to physical situations where multiplying negatives clashed with his own meanings. Nic was faced only with a (pleasurable) task of cognitive expansion, building on his existing manipulation of numbers. Chris was faced with a (difficult) task of cognitive reconstruction that challenged his very relationship with the world as he perceived it. For him new ideas did not make sense. They did not fit.

The primitive brain has a way of reacting to perceptions that appear strange or threatening. The lower limbic system unconsciously produces neuro-transmitters that affect the operation of the brain, encouraging some activities and suppressing others. The child who finds certain concepts "do not make sense" is therefore likely to be at a disadvantage in attempting to process the information. I conjecture that, not only are the ideas harder because they are more diffuse and more difficult to co-ordinate, the emotional activity of fear generated by the limbic system suffuses the brain with neurotransmitters that makes the contemplation of these ideas even harder.

The case of Robert reveals an individual with relational understand of the power law for whole number powers, who is then confused when he he is faced with the need for cognitive reconstruction in the new context where the powers are fractional. We may hypothesise that his relational understanding was neuronally connected to the idea that the power represents repeated multiplication. Raising x to the power $\frac{1}{2}$ makes no sense because he has no idea what it means to compute "half an x multiplied together." An idea, which an experienced mathematician sees simply as a"generalisation" of a formula to apply to a larger range of examples, is not meaningful for a learner who has meaningful understanding of the power notation in its whole number manifestation.

It is salutary to realise that children may develop a *relational* understanding in a given context and yet encounter new contexts where the old links no longer work and new links must be forged. This occurs widely when a child meets an extension of the number system, say from counting numbers to negative integers or from whole numbers to fractions, or from real numbers to complex. Old intuitive rules that are part of the essential being of the individual are often deeply embedded in the psyche. The deep belief that "you cannot have *less* than nothing" in an everyday sense may prove too strong even in a world of bank accounts and credit cards where negative quantities become readily available. The practical solution in this case has been to continue to use ordinary numbers but widen the system to allow the same numbers to act separately as debts. In cognitive terms, this practical solution permits a cognitive expansion, rather than requiring a cognitive reconstruction.

In a long-term learning schema, such a focus on expansion rather than reconstruction may work in the short-term, it may even work for the life-time of a bank teller, (or at least until that teller is made redundant by technology). But in the long-term mathematical development towards the full deployment of real numbers, the need for some kind of cognitive reconstruction is inevitable. Faced with a new context which causes internal cognitive conflicts, the child must either make a serious cognitive reconstruction or take the line of least resistance by learning a new rule by rote to "get the right answer." The latter strategy leads to (temporary) survival in the mathematics class. This is seen in the example of the student James, who was perplexed by the teacher's attempt to introduce the limit concept in the calculus but realised that the teacher would later give "the simple way" to differentiate a polynomial using the formula.

Regrettably it does not take many such set-backs before the standard response is to learn "rules without reason" in Skemp's memorable phrase. When Alec learnt to "use the rules" in calculus, he extended them in an inappropriate manner, thinking that the same rule would work for multiplication that worked for addition.

The fall-back to "learning the rules" is widespread.

> I was interviewing a number of students about how they worked through their mathematics. What became very clear was the desire of the students to 'know the rule' or 'the way to do it'. Any attempt on my part to provide some background development or some context was greeted with polite indifference – 'Don't worry about that stuff; just tell me how it goes.' (Pegg, J. 1991, p. 70.)

I conjecture that the *vast majority* of learners reach this point at some stage. I further suggest that the "polite indifference" not to wish to "worry about that stuff" is not just an attitude of mind, but a sign that the student may not be *able* to talk or even think about "it", because there is no "it". A cognitive structure built on procedures to "do" mathematics may not have the mental concepts to "think" about mathematics and therefore may not be amenable to relational understanding.

As an example, consider a straight-line graph through (1,5) and (–3,–3). Its gradient can be found by the formula

$$\frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{-3-5}{-3-1} = \frac{-8}{-4} = 2$$

which, for many students, involves using meaningless "rules" such as "a minus over a minus is a plus". The equation is now $y = 2x+c$ where c can be found by substituting one of the points. But which point is substituted? One has minus numbers in it, the other has positive numbers, and for students who are struggling, the latter may seem more attractive. This gives

$y = 2x+c$ where $x = 1, y = 5$,

so

$5 = 2+c$
$c = 3$

and the equation is

$y = 2x+3$.

Having followed such a sequence of procedures, does the student genuinely identify this equation with "the line through (1,5), (–3,–3)"? Are the two notions different ways of viewing the same conceptual entity, or are they different entities, loosely connected in the brain? Many students appear not to link them at crucial points of an argument. When asked whether a third point, say (2,4), is on the same line, instead of substituting this into the equation, some students, who are struggling, go back to calculating the equation of the line through (2,4), (1,5) to see if it gives the same equation (Crowley & Tall, 1999).

We suggest that these students do not readily see all the different forms of a linear equation as being different versions of one mental entity and so they have no mental entity in their minds to manipulate. They cannot talk about "it" because, for them, there is no "it". As long as they remain fixated on the detail of the procedures, they may work very hard but with little reward. For example, in their case we cannot speak of the various 'representations' of the function concept for there is no function concept to represent. All they can do in this state of mind is to learn an increasing collection of procedures to do specific, but limited, tasks that may grow increasingly difficult to relate in any coherent overall structure.

Long-term Learning Schemas and Cognitive Roots

In devising initial activities for long-term learning schemas that encourage conceptual ideas rather than just procedural competence, I had the privilege of being Richard Skemp's last PhD student (Tall, 1986). Given the possibility (even probability) of discontinuities in the learning process, a long-term learning schema needs to take into account that cognitive reconstruction is likely to occur at various times. At the outset, I decided that the journey should begin by building an inner cognitive sense of the concept that carried the potential of long-term development. I formulated the notion of a *cognitive root* as a concept that a student meets at the beginning of a period of study that is familiar to the student at this stage, yet contains the seeds of long-term learning of the formal theory (Tall, 1989).

To start from "where the students are" to build to what you wish them to learn is quite different from building a curriculum which focuses on "where the students are desired to get". Many curricula in the calculus build towards a logical meaning of differentiation and integration and therefore decide that they must begin with the limit notion as this is the logical foundation of the formal theory. But it is not a good starting point for the learning of students.

I found that few students would naturally invent the limit concept for themselves. For instance, Tall (1986) presented a graph of the parabola $y=x^2$ with a line drawn through the points (1,1) and (k,k^2) and asked first to write down the gradient of the line and then to explain how they might calculate the tangent at (1,1). Only *one* out of over a hundred students produced a limiting argument as $k\to 1$,and he was part of a minority who had already been *taught* the limiting notion.

On the contrary, every piece of research I have ever seen underlines how difficult the limit concept is to the beginning student (Cornu, 1991). Discussed in a dynamic sense, in terms of "getting close" or "getting small", it builds up intuitive notions of variable quantities that are "arbitrarily small", giving a number line of constants and quantities which act like infinitesimals. This gives a system different from the real number line that is desired by the mathematical community.

For the first step in my own long-term learning schema for the calculus, therefore, I decided to build on human foundations that were widely available to all individuals, the sensori-motor facilities of vision and action. I alighted on the notion of *magnifying* a small portion of the graph to *see* the gradient. Specifically I considered those graphs which, when drawn on a computer screen and magnified highly (maintaining the relative scales on the axes), eventually "look straight".

I emphasise that this is at a more fundamental cognitive level than that adopted by many curriculum builders in the calculus reform movement who start with "local linearity". Local linearity already demands a facility to handle the equation of a straight line that is the "best" approximation to the curve at a point. Local straightness is a direct appeal to perception, in which the limiting notion is implicit in the action of magnification.

When the first group of students who ever used the software were asked to draw any graph they fancied and to see what happens when it is magnified, the first comment was that "it looks less curved as you magnify it". The notion of magnification to "look straight" proved an appealing idea to everyone in the class.

At first I followed the mathematical route of "fixing x" to find the gradient at a point, but then realised that the students were able to do something more intuitive—to look along the graph to "see" its changing slope.

Software was designed the student to move a "magnifying glass" along a curve to see the locally straight gradient change (Tall et al, 1990). This aided students to get a gestalt idea of the gradient function using a sensori-motor feeling of "tracing along the graph" of a "locally straight" function.

The visual computer approach drew a numerical chord from x to $x+h$ for fixed h and to plot the gradient of the curve as x moves along the curve. For small h the gradient graph can be seen to stabilise. This can be linked directly to the symbolism. For instance, in the case of $f(x) = x^2$, the gradient of the segment between x and $x+h$ is

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h} = 2x+h \text{ (since } h \neq 0).$$

The graph drawn for the gradient is $y = 2x+h$. For small h, this is indistinguishable from the graph $y = 2x$. Thus *at a sensori-motor level*, the student has the opportunity to sense that the gradient of the parabola $f(x) = x^2$ is indistinguishable from $2x$. The subtleties of mathematics are linked to deep primitive modules in the brain that are part of the essential being of the human psyche.

At the time I remember being very pleased with myself, but was surprised that the students simply took up the idea as if it were “obvious”. In one school I set up the computer software on a single computer intending to demonstrate it to the students studying calculus. I ended up sitting alone in the corner of the room, completely neglected by a large group of students standing round the computer to “see” how to do their homework. I found to my surprise that many students found it to seem so *simple* as to be “obvious”. It was as though my great discovery (which was largely unused by the mathematical community) was evident to the student mind. The explanation, to me, is that it *is* obvious, because it links with deep cognitive structure that is part of the essential being of the individual.

To compare the mental imagery generated with that of students following a more traditional course, both kinds of student were asked to sketch the derivatives of various graphs, including those in Figure 3.

There are at least two possible routes to a solution. One is to proceed symbolically by guessing the formula, differentiating it, and drawing the derivative. This is easier for graph (a) (which looks like the familiar $y=x^2$) than for graph (b) whose formula is not familiar to the student. A second approach is to proceed visually, looking along the graph to see its changing gradient. For instance, graph (b) starts with a positive gradient decreasing to zero and then stays zero.

It is reasonable to hypothesise that students with less visual insight would find problem (b) significantly more difficult and this is supported by the statistics (table 1). Three experimental classes in school (A, B, C) using computer software to visualise the gradient succeed in both problems significantly better than control classes D, E, F, G) following a standard text. The difference in problem (b) was even greater. The experimental classes also perform better than control group H of highly able students studying double mathematics, and on a par with first year university students, group I.

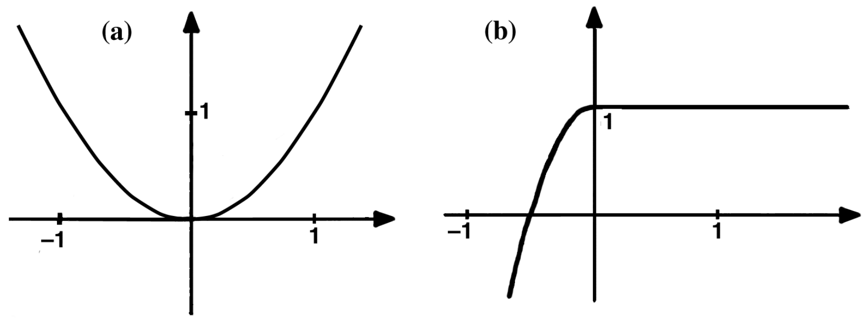


Figure 3. Sketching the gradient graphs for given graphs.

Table 1

Student Responses to Sketching Derivative Graphs

Experimental Groups	(a)	(b)
School 1, group A	100%	70%
School 2, group B	99%	87%
School 2, group C	92%	89%

Control Groups	(a)	(b)
School 1, group D	82%	31%
School 2, group E	73%	16%
School 2, group F	47%	3%
School 2, group G	39%	0%

Others (non-experimental)	(a)	(b)
School 2 (double math), group H	91%	56%
University year 1, group I	91%	89%

Not only did the idea of "local straightness" help many students to build a richer concept image of the idea of derivative, it can also be used later on in differential equations—"knowing the steepness of the function at each point, and requiring to build a function with that slope." Again, computer software was designed to allow the student to imagine physically building a solution of a (first order) differential equation by placing short line segments of the appropriate gradient end to end (Figure 4).

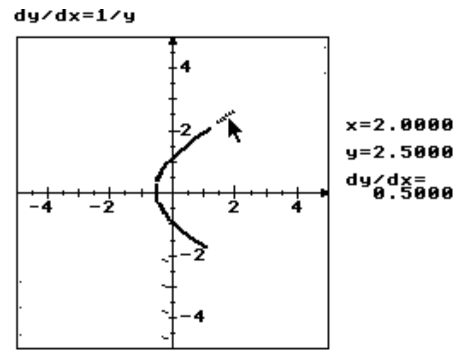


Figure 4. Enactively building a solution of a differential equation.

These ideas again provide a visual conception of the solution process, based on sensori-motor activity. They also provide a sense of existence and uniqueness of solutions. Everywhere that the gradient *dy*/*dx* is defined, the direction of the solution is given as a small segment. This supports the sense that, through every point where there is a defined direction there is one, and only one solution. Thus the existence and uniqueness of solutions can be conceptualised before any techniques of solution have been introduced. It allows conceptual links to be formed in an intuitive sensori-motor sense which may later motivate the formal aspects of the theory when and if it is studied.

Does a Cognitive Root Guarantee Long-Term Success?

Having found a starting point that seems to be able to provide some motivation for later theory, the question arises as to whether this will necessarily lead to later success. The answer is that although a cognitive root has a *potential* for development into a long-term formal theory, it does not *guarantee* that such a foundation will lead to the later formalism for all those who attempt to study it. The cognitive root operates in a sensori-motor manner that can be visualised, verbalised and discussed. But this does not provide a basis for *all* possible formal links.

As an example, consider the symbolic rules for differentiation and integration. The derivative of a product $f(x)g(x)$ is not easily seen by looking at the visual notions of the derivatives of $f(x)$ and $g(x)$. For instance, if one can visualise the derivative of $\sin x$ by looking along its graph to see its changing gradient as $\cos x$, and can similarly "see" the gradient of the graph of e^x how does this allow one to "see" the gradient of their product $e^x \sin x$? It is certainly not obvious by looking at the graphs of the two functions and "seeing" an easy relationship with the graph of the product function.

The usual visual method is to look at the product uv of two quantities as an *area* and to see the increment in the area caused by increas-ing u to $u+\delta u$ and v to $v+\delta v$ as being two strips area $v\delta u+u\delta v$ and a rectangle in the corner of area $\delta u\delta v$ (Figure 5).

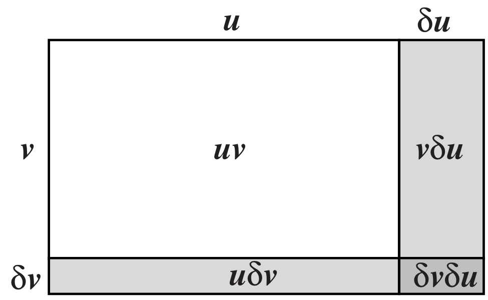


Figure 5. The change in uv is $v\delta u+u\delta v+\delta u\delta v$.

This visual representation corresponds precisely to the following symbolic manipulation:

$$\frac{\delta(uv)}{\delta x} = \frac{(u+\delta u)(v+\delta v)-uv}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta x} + \delta u\frac{\delta v}{\delta x}$$
$$\to u\frac{dv}{dx} + v\frac{dv}{dx} + 0\frac{dv}{dx}.$$

There is therefore a direct link between this rectangular representation and the symbolism, but no simple visual link between the locally straight graphs and the gradient of their product function.

The School Mathematics Project asked me to provide such a graphical link for the product function. I could not see one. I *did* provide a graphical link for the composite of two functions, by seeing $y = f(g(x))$ in terms of the functions $u = f(x)$, $y = g(u)$ and plotting the three graphs in three dimensions with axes x, u, y. Figure 6 shows the software I developed to show this idea (Tall, 1991) in the case using

$$u = x^2,\ \ y = \sin u\ (= \sin(x^2)).$$

There are four different windows: the x-u plane in the bottom right corner, with $u = x^2$, the u-y plane in the bottom left corner, with $y = \sin u$, and the x-y plane in the top right, with $y = \sin(x^2)$. Each of these is a projection of the curve in three dimensions in which $u = x^2$ and $y = \sin(x^2)$.

Seen as a static picture in a book this may be quite difficult to visualize, but an option to turn the three dimensional graph around gives a sensation of space, making it easier to visualise it as a three dimensional object.

The tangent to the curve has components dx, du, dy in directions x, u, y and the chain rule for differentiation becomes:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

as a relationship between the lengths of the components of the tangent (Figure 7).

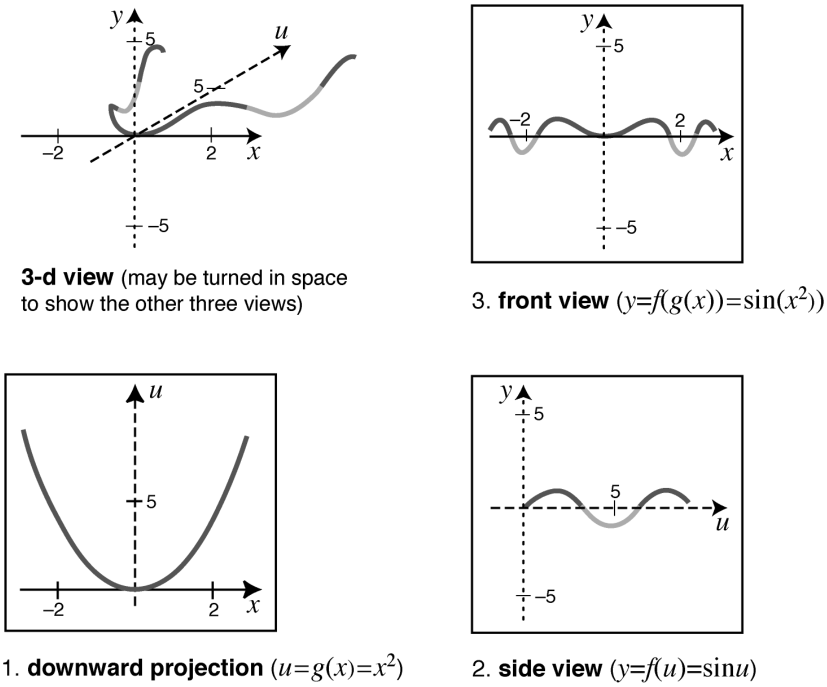


Figure 6. A composite function represented in three dimensions.

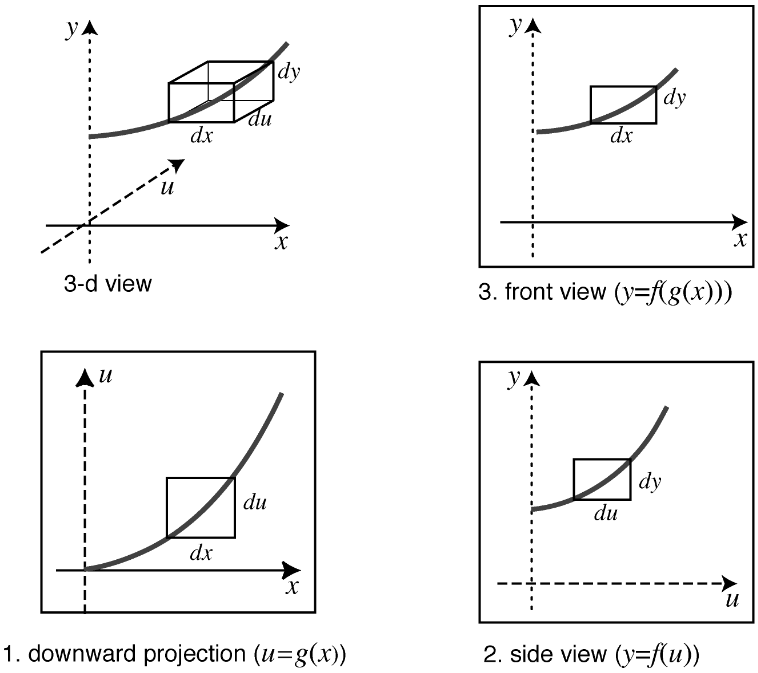


Figure 7. *The gradient box for the composite of two functions.*

This idea is best seen as a perspex model of a cuboid sides *dx*, *du*, *dy*. Such a model could be turned by hand to see the projection of the diagonal onto each of the sides where the latter have gradients *dy*/*dx*, *dy*/*du* and *du*/*dx* respectively.

These examples reveal that, although the various "rules of the calculus" can all be handled in a similar *symbolic* manner, each one has various visual images providing support which do not have simple, direct links between them.

The symbolic differentiation procedures themselves operate on the symbolism seen as a sequence of steps of evaluation (such as sums, products, quotients, composites etc). This calculation is not represented in the gestalt visual picture of the graph. *Thus the processes of "seeing" the relationships between the differentials and the procedures of symbolic use of rules focus on totally different aspects of the activity.*

This difference between the use of symbols and visualisation and the failure of one representation to link easily with the other is widespread and goes back to the earliest operations on functions. For instance, to express the line through (5,3) with gradient 5 in the form $y=mx+c$ begins with

$$y - 3 = 5(x - 4)$$

then the brackets is multiplied out and 3 added to both sides to get:

$$y - 3 = 5x - 20$$
$$y = 5x - 17.$$

We found that students remarked that, when manipulating symbols, they could not "see" what is going on (Crowley & Tall, 1999). Reflection on what is happening reveals that, as the symbols change, the graph *remains the same*. In a very real sense, a student would not be able to "see what is going on" because *nothing is happening visually to the graph*. Of course, to an expert, the fact that nothing is happening to the graph has real meaning because the line remains invariant under the symbolic manipulations. But for the student who lacks the conceptual links between disparate cognitive structures, the situation may be meaningless.

In each of these examples we see that the desired coherence of different ways of looking at a particular mathematical activity do not necessarily fit together in a precise correspondence. Some representations represent some aspects better than others, and some processes in one representation may not be mirrored in a one-to-one way to processes in the others.

The corollary of this discussion is that the cognitive development of the calculus cannot ever hope to lead to a totally coherent theory without encountering the need for various cognitive reconstructions on the way. To cope with the meaning of the calculus requires the student to reflect on the various situations to understand them in their own terms and to fit them together in a way which "makes sense". If this does not happen, then the default position is liable to involve instrumental learning of procedures without supporting conceptual links.

However, the cognitive root of local straightness can be used for a wide range of students to provide a platform with the potential to lead to various different approaches to the calculus. It can provide sensori-motor support for applications in physics, biology, economics and so on, including a mental image of notions such as the solution of a differential equation. It can be used as a based for the theory of formal analysis, either in terms of the standard "epsilon-delta" definition or in terms of an alternative "non-standard" theory using infinitesimals (Tall, 1981).

It may not be possible for all students to come to terms with all of the mathematics, but it may certainly prove possible for students who have great difficulty with some parts of the theory (eg symbol manipulation) to have conceptual insight into the nature of the theory. For instance I have had experiences with students who are not strong in mathematics being able to imagine functions that are continuous everywhere but differentiable nowhere, or to visualise functions that were differentiable everywhere once but nowhere twice, based on the notion of "local straightness" (Tall, 1995). But these self-same students were not necessarily able to cope with complicated symbol manipulation or to make the transition to fluent use of formal proof.

In this sense a cognitive root cannot guarantee that all students will understand all the theory which is to come. However, an idea such as local straightness is a "cognitive root" that appears to make sense to most learners, and also has the *potential* to be used as an introduction to various highly subtle aspects of formal theory encountered later on.

One feature of the use of local straightness as a cognitive root showed a profound difference between the conceptual structures of those that used this approach and those that followed a traditional course.

The idea of a tangent to a curve has long been known to contain cognitive conflicts requiring reconstruction. For instance, in the geometry of a circle, it is a line which touches a circle just once and it is often envisaged as a line which "touches" a curve but does not cross it. When a curve with an inflection point is encountered, students often misconceive the notion of tangent. For instance, the tangent to $y=x^2$ at the origin is seen by many, not as a horizontal line through the origin, but as a line that "touches but does not cross". This gives what is sometimes termed a "generic tangent" as reported by Vinner (1991), (Figure 8).

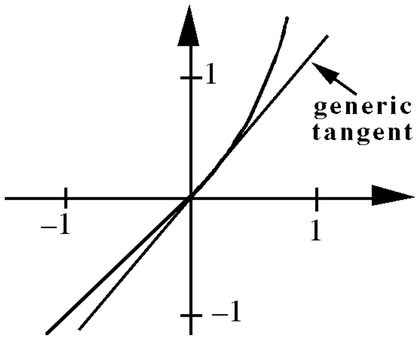


Figure 9. A generic tangent.

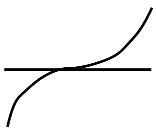
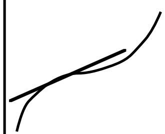
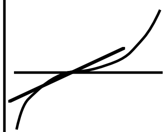
A	B	C	D	E
				
The right answer	A generic tangent	two tangents	Another drawing	No drawing
18%	38%	6%	10%	28%

Figure 8. Drawing the tangent to $y = x^2$ at the origin (Vinner 1991).

The "locally straight" visual approach did not completely eliminate the belief in a generic tangent, for instance, the graph:

$$y = \begin{cases} x & x \leq 0 \\ x + x^2 & x \geq 0 \end{cases}$$

presents a cognitive problem at the origin. A tangent exists but it coincides with the graph to the left and therefore does not intuitively "touch" the graph. A generic tangent is drawn a little to the right to show it as "touching" the curve. (Figure 9.)

The students responded as in Table 2. This reveals that the experimental students are significantly more likely to link the nature of the tangent to its standard form with less influence from the earlier concept image of "touching" the graph.

Table 2
Responses to the Tangent in Figure 9

	Yes			No	
	Standard	Generic	Other	2	0
Experimental (N=41)	31	8	0	2	0
Control (N=65)	22	30	2	15	1
University (N=47)	29	14	0	4	0

In terms of developing a long-term learning schema, it is helpful for the imagery to be developed in a way that encourages manageable reconstruction at a later stage. An example of this is to look forward to the existence of left and right derivatives, which is usually considered far too technical to discuss in a first course. However, zooming in to see a "corner" with different left and right gradients is available using the graphical software. This allowed discussion of different left and right gradients to occur simply and naturally. To test the effect of this introduction, experimental and control students were asked to consider the graph of the function

$$y = \left|x^2 - x\right| + x + 1$$

given also in the alternative form:

$$y = \begin{cases} (x^2 - x) + x - 1 = x^2 - 1 & \text{(for } x \le -1 \text{ or } x \ge 1) \\ -(x^2 - x) + x - 1 = x^2 - 2x - 1 & \text{(for } -1 \le x \le 1) \end{cases}$$

Formally it has no tangent and the slope and derivative are undefined at x=1. The students were asked to say (with reasons) how many tangents, derivatives, slopes the graph has at $x = 1$ from the following possibilities: 0, 1, 2, more than two, other (e.g. "infinity"). The response deemed formally correct was zero in each case. In addition there were many other reasons that were possible to give, say "two" (to refer to the different cases on left and right) or even "infinity" tangents which are capable of touching and not crossing through the point (1,1).

In Table 3 we see that, although only 6 out of 14 students in experimental group A gave the accepted answer (none), in the other two groups there were 9 out of 12 and 13 out of 16. However, in the control groups, only 6 out of a total of 52 gave the accepted answer.

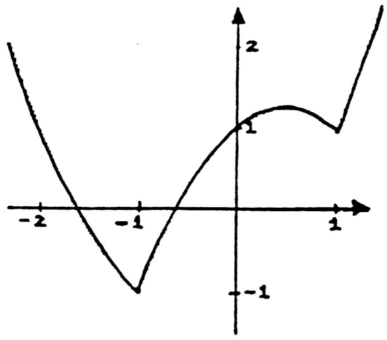


Figure 10. The graph of y = abs(x(x–1))+x+1.

The amazing statistic is that, in three out of four control groups, there were almost as many *different* responses as there were students. Pandora's box opens and reveals all the variety of possibilities in the students' minds. In experimental group A the teacher (myself) had encouraged discussion and participated actively, but had not given explicit instruction.

Table 3

Student Responses to Tangents, Derivatives, Slopes at a "Corner" Point

Experimental Groups	none	two	Other solutions	Total number of *different* solutions
School 1, group A (N=14)	6	1	7	9
School 2, group B (N=12)	9	0	3	3
School 2, group C (N=16)	13	0	3	4
Control Groups				
School 1, group D (N=9)	0	0	9	8
School 2, group E (N=13)	2	2	9	9
School 2, group F (N=17)	3	4	10	10
School 2, group G (N=13)	1	1	11	13
Others (non-experimental)				
School 2, group H (N=12)	5	0	7	8
University year 1, (N=57)	14	8	35	14

There was a combination of discussion and more explicit instruction in Group C leading to a greater uniformity in (correct) response. We thus see the power of the software to encourage discussion and the value of the directed participation of the teacher. By teaching in the traditional way the students had been left with a wide variety of possible connections as yet unassigned. The response of the university students shows the consequences of this. Some students had discussed left and right derivatives in class and 8 gave a response consistent with this, in addition to the 14 giving the response "none" to all three parts. But there were still 14 different possibilities given in total, leaving a wide range of beliefs to be brought together in teaching the notion of differentiability and non-differentiability at university.

Conclusions

In this paper we have looked at the long-term building of learning schemas. Many examples have been given of discontinuities in the curriculum in which a change in context requires considerable cognitive reconstruction from many children which may prove too difficult for many of these. These include many contexts which occur in the English National Curriculum which provoke difficulties for children and cause many of them to take the line of least resistance and learn ideas which are meaningless to them by rote.

Even when long-term curricula are carefully designed using a cognitive approach, some of the conflicts persist. If they are directly addressed by the teacher as mentor in a meaningful context then it is possible to give a wider spectrum of students more coherent insights into the nature of the mathematics. A standard curriculum which takes things in a steady order, avoiding difficulties till later can create a Pandora's box of different ideas that need to be rationalised (but are more probably, ignored) later on.

In all this development of long-term learning schemas, it becomes eminently clear that individual children make cognitive links in a wide variety of ways. To address this requires mutual activity on the part of the teacher and learner. The learner must become increasingly aware of his or her own part in reflective thinking about the concepts and helped to grow in confidence in dealing with the mismatches that occur in new contexts. The teacher, as mentor, must be aware of several different facets that may not be so clear to the pupil. One is the pupil's need in cognitive growth, the second is an awareness of those things that are mutually agreed in the mathematical community and the third is the balance between pupil need and the focus on those aspects of mathematics that give long-term power.

I do not believe it is sensible or practical to attempt to design a smooth long-term curriculum in which each idea builds easily on previous ones. New contexts will always demand new ways of looking at things and often require significant cognitive reconstruction.

Awareness of the essential nature of cognitive reconstruction is vital in long-term curriculum design. The National Curriculum for England and Wales, for instance, consists of a sequence of ten levels which are to be taught successively over the years of schooling from the age of five to the age of sixteen. It is expected that some children will move quicker and more effectively through the levels whilst others may take it at a slower pace. The aim of the British government is to "raise standards" by getting a greater percentage of children to a given level. There is even a concept of "value added" namely the average improvement in grade level of the children in a given school. This metaphor of development presupposes a sequence of successively more subtle ideas and a naïve assumption that less successful children can move through the same sequence of ideas but at a slower pace.

However, the notion of discontinuity suggests an impediment to such progress. If a child meets a point requiring cognitive reconstruction and this reconstruction does not occur, then any later developments are hampered by subsequent misconceptions. These misconceptions arise not only from mistakes or misunderstandings in earlier mathematics, but also in the failure to adapt to new contexts where the old ideas are no longer completely appropriate. Far from being an occasional problem in learning, I claim that discontinuities in development are widespread in the curriculum and must be taken into account so that more children are able to succeed in personal reconstruction of the powerful mathematical ideas that may bear fruit in later life.

References

Cornu, B., (1991). Limits. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Kluwer: Dordrecht.

Crowley, L., & Tall, D. O., (1999). The roles of cognitive units, connections and procedures in achieving goals in college algebra. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME* (pp. 225–232). *Haifa, Israel, 2*.

Lakoff, G., & Johnson, M., (1999). *Philosophy in the flesh: The embodied mind and its challenge to western thought*, New York: Basic Books.

Pegg, J. (1991). Editorial, *Australian Senior Mathematics Journal*, 5 (2), 70.

Skemp, R. R., (1962). The need for a schematic learning theory, *Brit. J. Educ. Psych., xxxii*, 133–142.

Skemp, R. R., (1964) *Understanding Mathematics*. London: University of London Press.

Skemp, R. R., (1976). Relational understanding and instrumental understanding, *Mathematics Teaching* 77, 20–26.

Skemp, R. R., (1986). *The Psychology of Learning Mathematics*, (2nd edition). Penguin Books Ltd.

Skemp, R. R., (1993–4). *SAIL through Mathematics*, Volumes 1 & 2. Calgary: EEC Ltd.

Tall, D. O. (1981). Comments on the difficulty and validity of various approaches to the calculus, *For the Learning of Mathematics*, *2*, 2 16–21.

Tall, D. O. (1986). *Building and testing a cognitive approach to the calculus using interactive computer graphics,* Ph.D. thesis, University of Warwick.

Tall, D. O. (1989). Concept images, generic organizers, computers & curriculum change, *For the Learning of Mathematics*, *9*,3, 37–42.

Tall, D. O. (1991). *Real Functions & Graphs: SMP 16-19* (computer programs and 64 page booklet), Cambridge: Cambridge University Press.

Tall, D. O., (1995). Visual organizers for formal mathematics. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with* *computers in mathematics education* (pp. 52–70). Springer-Verlag: Berlin.

Tall, D. O. (1997). Functions and calculus. In A. J. Bishop et al. (Eds.), *International handbook of mathematics education*, 289–325, Dordrecht: Kluwer.

Tall D. O., & Vinner S., (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity, *Educational Studies in Mathematics* 12, 151–169.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 65–81). Dordrecht: Kluwer.

Vinner, S., (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning, *Educational Studies in Mathematics*, 34, 97–125.