Schemas and processes for sketching the gradient of a graph

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This study investigates the manner in which students build up meaning to draw the gradient of a given graph. Some researchers claim that students tend to use algebra to solve calculus questions. This research suggests this may happen when students are encouraged to use algebra (a common occurrence in teaching and examining). However, one teacher emphasised a visual embodied approach following the shape of the curve with her hand in the air and encouraged the students to do the same, linking visual and symbolic ideas; her colleagues did not. Her students were more likely to use a visual process looking at the graph from left to right. They scored significantly better than her colleagues' students who needed to coordinate a schema of ideas.

Introduction

It has often been remarked that students are reluctant to visualize (Dreyfus and Eisenberg 1991) and avoid visual considerations in calculus (Vinner, 1989). On the other hand, Lakoff and his colleagues (Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) theorize that mathematics arises from embodied metaphors which suggest that the biological basis of perception and action provide s an important foundation for mathematical ideas. Our own belief is that the concept of derivative is best approached initially by basing the idea of 'rate of change' on 'local straightness' of the graph, actually *seeing* the gradient of the graph change as one moves one's eye along the graph from left to right. This is an *embodied* activity giving a *sequential process* to sketch the gradient of the graph.

Experience of plotting a graph itself varies depending on the information given. From a table of values a number of points may plotted by hand and joined in a left-to-right process. Drawing a graph from a formula using a software package is also usually computed from left-to-right with the software calculating and plotting successive points (perhaps with additional subroutines to check locally for asymptotes and other features). However, sketching a graph from a formula often involves assembling a collection of pieces of information that need to be sorted in order to plot the curve. These might include the behaviour of the graph for large x (positive and negative), perhaps the behaviour for x small, a few known points (eg where x=0), perhaps solving to find where the graph meets the x-axis, and differentiating to find maxima, minima and points of inflection. Experience plotting a function by hand therefore involves what we term a *schema of activities*—producing non-sequential information—as opposed to the possible left-to-right sequential process possible for sketching the gradient of a given graph.

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Our research focuses on the teaching of the notion of gradient in a school where one teacher—whom we shall call 'Teacher V'—privileges a visual-enactive approach, representing the graph by the inclination of her hand moving through space; she encouraged her students to do the same, building a physical sense of the changing gradient. She also links this approach to the symbolism, which gives the student the opportunity of a choice of methods. The other teachers use a more traditional approach, sketching graphs and also their gradient by developing an appropriate schema of activities. This follows the SMP (1997) curriculum which begins by studying the gradient of a straight line and then the gradient of a curved graph by marking it with "+" or "–" signs to show where the gradient is positive or negative. The teachers are encouraged to use computer graph plotters and graphic calculators to discover the formula for the changing gradient of a function. This is followed by numerical calculation of the gradient at a point and on to algebraic methods of calculating the gradient. However, only the one teacher privileged the use of physical enaction and sensing the change of gradient.

The students involved are all at one Comprehensive school in which 57 candidates took Mathematics, with 23 gaining a grade A, and the school's average A-level point score of 20.2 was above the national average of 18.5.

Methodology

All students responded to a test taken in class time that invited them to sketch the gradient functions of six different functions. All were given as graphs without formulae except for the second, given as a cubic formula $f(x) = x^3 + 2x^2 - x - 2$ with no graph. This was included to seek data about the manner in which these students sketch a graph from a formula. It was preceded by a straightforward example of sketching the gradient of a given graph to set the scene. The last four graphs posed a succession of different problems in sketching the gradient function. All the graphs had numerical values marked on the axes.

The research questions that interested us were as follows:

- Do students use a left-to-right sequential process or a more general schema of activities to sketch the graph or its gradient?
- If they build up information in a schema of activities, in what order is the information determined?
- Do they use the information they may be able to read from the graph, such as the numerical values on the axis?

Data collection : Sketching gradients of functions given graphically or symbolically

The first question (figure 1) caused few problems. Forty out of forty-one answered this question correctly. Twelve students approached the problem by initially marking the graph of the function with + and – signs along whole curve. Ten of these students also marked the stationary points. Eleven other students marked only stationary points.

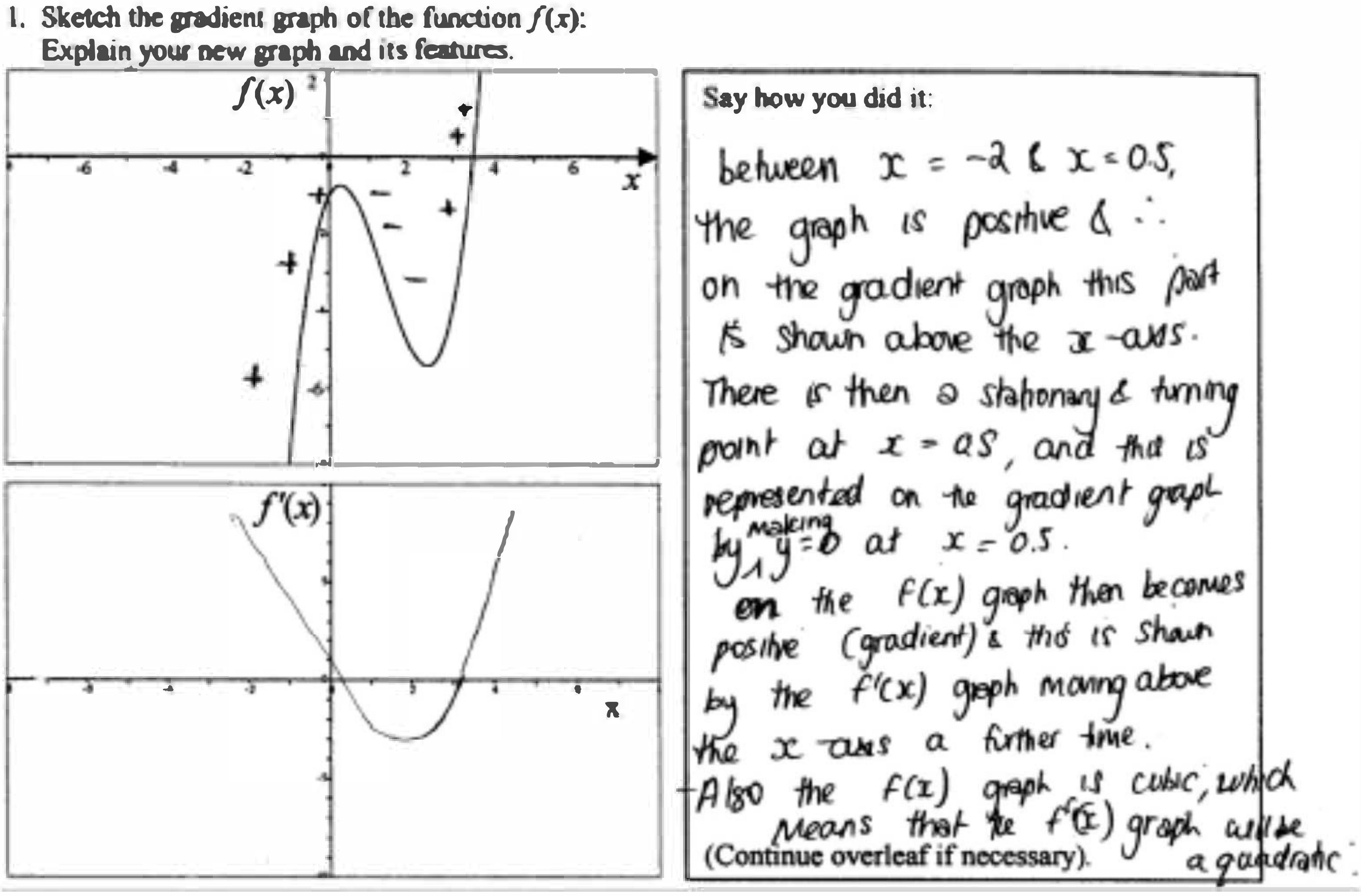


Figure 1: Question 1: sketching the gradient of a given graph from left to right by following the slope

When asked, "Say how you did it", 29 out of 41 students wrote that they read off the x-coordinates of the stationary points of the graph and then proceeded to look more carefully at the rest of the graph. Six students mentioned that the gradient graph should have a shape of a quadratic. After marking of their stationary points, 25 students indicated that they looked at the sign of the gradient and defined its change from left to right. An example is shown in figure 1. Other examples are as follows:

> "then I saw where the $f'(x)$ was positive and it started in the +ve …"

> "at –1 steep +ve gradient gradually decreases until around 0.5 where it=0. Then graph goes 'down' ∴ -ve gradient, until again it levels out, this time at 3.5 approx. Gradient then becomes ever more positive"

> "The part of the $f(x)$ graph up to the first stationary point is a steep gradient getting smaller (eventually to 0 smaller……"

The second question asked students to draw a cubic graph, given the equation

$$y = x^3 + 2x^2 - x - 2$$

and then to draw a gradient graph of this function.

Thirty six students out of 41 drew a cubic graph, four did not answer this question, only one drew a negative cubic graph. All but three who drew a cubic graph realised that the y-intercept has to be at $y = -2$. Only 16 students looked further at other features and found the 3 correct roots by factorising the equation. (Many students in the group find factorising a cubic difficult and may only attempt it using a graphing calculator to find a root.) Thirteen students were aware that the graph should cross the y-axis with negative gradient, eleven students differentiated the cubic equation to show that the gradient graph is in quadratic form, but did not use the quadratic equation to improve the precision of their graphs. In this question, thirty six students who drew the gradient graph used features they found for the cubic graph. Only *three* (including two from Teacher V's class) used a visual process for the gradient graph (as in figure 2). Using an algebraic starting point coincided with more students using algebra and just a few using the visual process of looking along the gradient of the given curve.

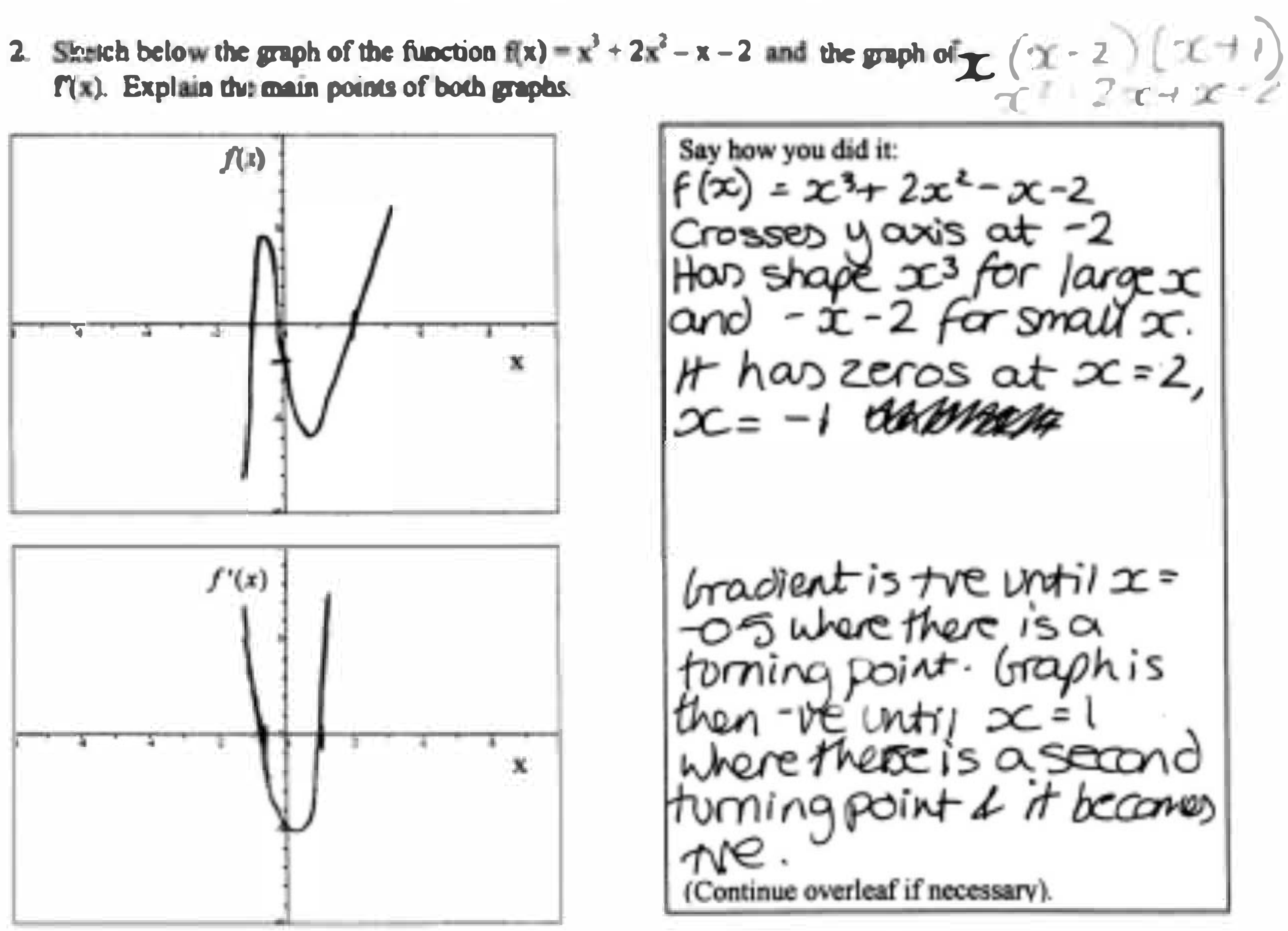


Figure 2: Sketching the graph using a schema of activities and the gradient as a visual process

Analysis of the second part of the investigation

The next part of the test was intended to introduce students to increasingly difficult problems. There are four questions in this section, requesting the student to draw the graph of the gradient of the following graphs:

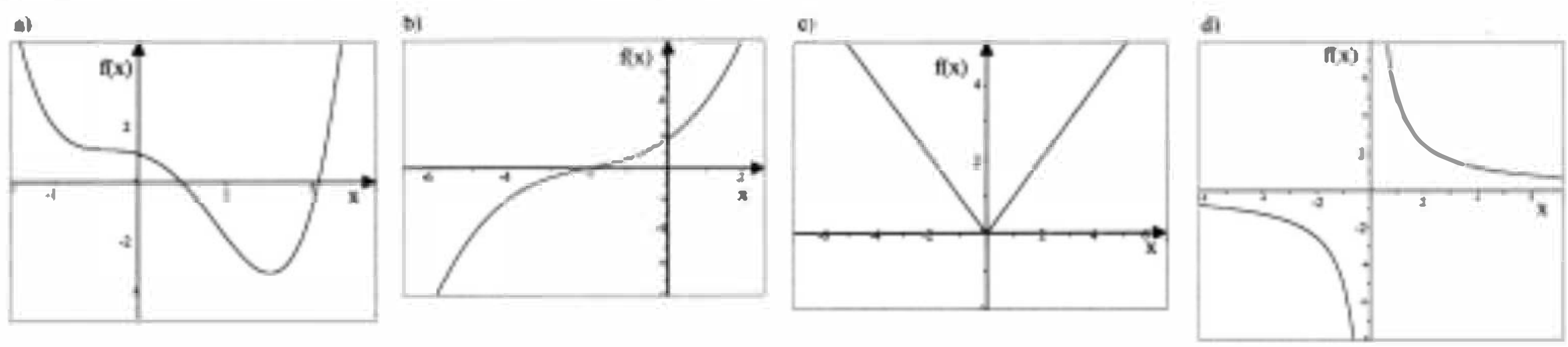


Figure 3 : Q. 3(a), (b), (c), (d) – draw the gradient of these graphs

Graph 3(a) has a point of inflection and a local minimum. Students who did the first two questions correctly should have no problem with the first point but the point of inflection tested the students' ability to recognise what a zero gradient really means.

Graph 3(b) is of a cubic function without any stationary point. What will students do when they cannot use their first method when they are so used to of marking the stationary points on the x-axis of the gradient graph?

Graph 3(c) represents the function $y = |x|$. Students start their study of the graphs with straight-line graphs but will they recognise the function as two straight-lines connected at one point? And what will they do at the origin?

Graph 3(d) is of the function $y = 4/x$. Will they recognise it as a $y = a/x$ graph and try to differentiate the formula or will they follow the graph in order to draw a gradient graph? It should be obvious from this question whether the students are using a sequential process.

The research questions are:

- How many students who answered the first question correctly will also answer most of the second section correctly and therefore prove that they have the intuitive conceptual knowledge of gradient?
- How many students will reach for their schema of algebraic activities because they have no such knowledge?

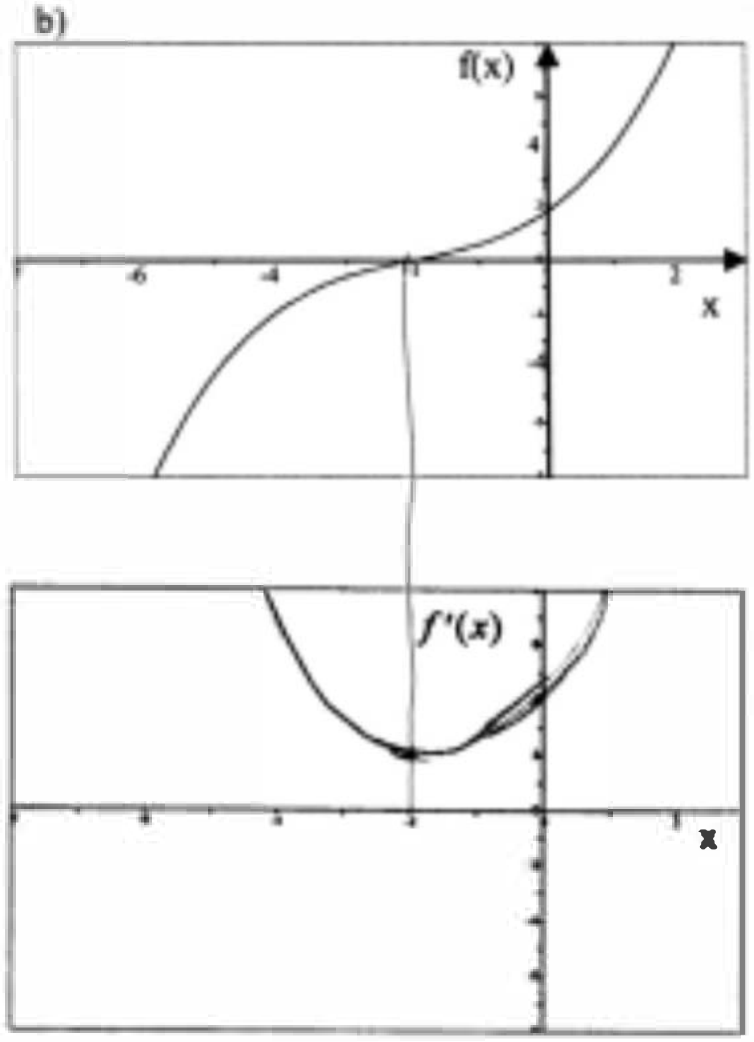
In this section of the test 33 out of 41 students answered question 3(a) correctly. Three students could cope with the local minimum but did not know what to do with the point of inflection. One student marked the point of inflection as zero on the x-axis of the gradient graph. One student marked both zeros but did not know how to proceed from there, two students drew a quadratic curve and another drew a cubic graph but with three roots.

On question 3(b), without any stationary points, the number of correct responses fell to 26 out of 41 students. Of these, two commented that the cubic function will have a quadratic gradient and 15 others either drew a line pointing from the point of inflection on the given graph to the lowest positive value of the gradient, or commented about the value in their written description. When asked, " Say how you did it," 23 students responded in a sequential way typified by the following:

> "The gradient graph doesn't ever go through the x-axis because there are no stationary points, the gradient is always positive but never reaches 0."

One of these responses is given in Figure 4.

Five students observed that the original graph had an inflection point and drew a positive quadratic gradient graph touching the x-axis. Three students thought that because the gradient was positive all the time they should draw a straight line with a positive gradient. Two students drew the graph correctly over the part of the domain where the graph was in the picture, but when the graph left the picture, they started turning the edges of the graph as if the gradient was getting smaller. Two others reflected the graph in the x-axis (one making a further error, because she explained she was trying to reflect to obtain $f^{-1}(x)$). Two students did not answer.



Say how you did it:

The gradient is smallest when $x = -2$ but it isn't stationary so the gradient graph doesn't cross the x-axis.
The gradient graph is a quadratic because the graph is a cubic graph.

(Continue overleaf if necessary).

Figure 4 : A solution to a gradient of a cubic using a conceptual schema of activities

Question 3(c) with the graph $y = |x|$ was answered satifactorily by 18 out of 41 students. 13 of these looked at the values on the axis and drew two horizontal lines at $y = 1$ and $y = -1$ for the correct domains. Eighteen other students drew a single straight line graph the whole width of the window, 12 of these with a gradient equal to 1. A remark by one student in interview revealed a possible reason for this. He explained that on the left of the y-axis the gradient is negative, on the right it is positive so the solution is drawn as a straight line from negative to positive. Five did not answer the question.

Question 3(d) was answered correctly by only 14 students. Four of these recognized the shape being similar to $1/x$ and used algebraic differentiation to answer this question correctly (figure 5). These responses involve a schema of activities switching from the visual graph to its algebraic equation, differentiating algebraically, then sketching the graph of the algebraic derivative. One of these was a student from Teacher V's class who showed a flexibility to use visualisation on some occasions and algebra where it was helpful.

The written explanation of (at least) six revealed the use of a sequential process looking along the graph. Five of these were from teacher V's class. (Figure 6 overleaf.)

Say how you did it:

The equation of the $f(x)$ graph is of the form $\frac{1}{x}$.
By differenciation using the formula $f'(x) = ax^{n-1}$ I can obtain the equation for the $f'(x)$ graph:

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$= -\frac{1}{x^2}$

(Continue overleaf if necessary).

Figure 5 : using algebra

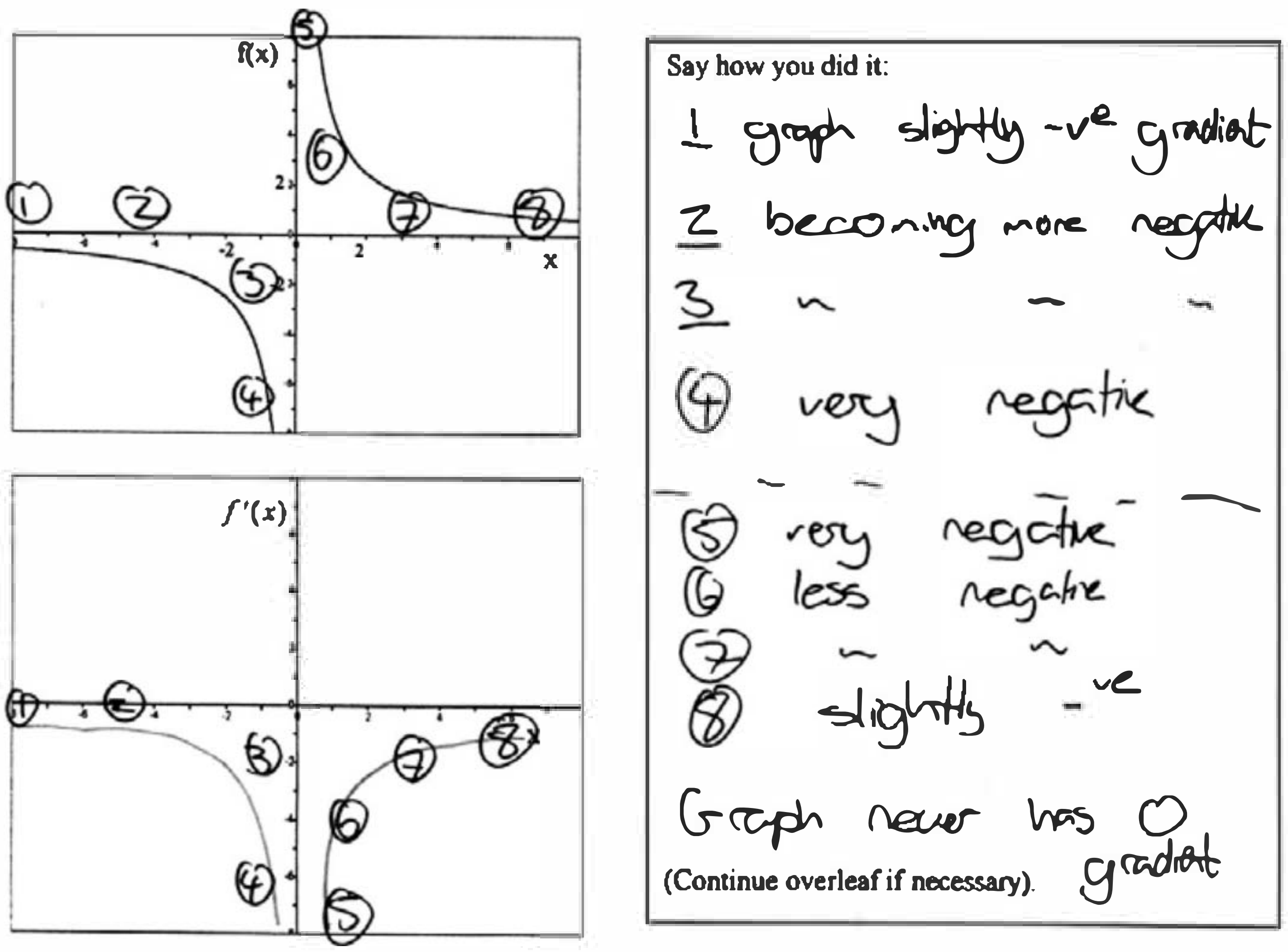


Figure 6 : Drawing the gradient of a graph as a left-to-right process

Quantitative analysis

Table 1 reveals the various levels of success. Forty out of forty one obtain at least one correct gradient graph. This is the cubic graph in question 1 which has clear turning points. The level of success falls with increasing difficulty.

	All 6 questions	5 or more questions	4 or more questions	3 or more questions	2 or more questions	1 or more questions
Correct gradient graph	7	17	26	35	38	40
The sign of the gradient marked on the curve	0	1	4	7	11	16
Described in writing how they did it	0	6	12	20	25	36
Used the information on the axis	0	0	6	12	23	32

Table 1: Cumulative scores for various aspects of students' responses

One might assume there is a correlation between drawing the correct gradient graph (row 1) and students' marking information on the graph (row 2) but this does not happen—the most able students do not need to do it. The correlation is –0.25.

A simple categorisation of verbal skills was made into those who gave clear explanation of their solution and those not so classified (row 3). There was a high positive correlation (0.68) between this measure of students describing in writing what they did and their level of success. Students who answered questions well have good communication skills. Even when Krutetskii (1976, p. 318) categorises children into analytic, geometric and harmonic, he notes that among the gifted children, even those thinking more geometrically have an above average verbal-logical component.

The final line of the table yields a small correlation (0.37) between the use of numerical information on the axis and success in drawing the gradient.

The group of students taught by teacher V, with greater emphasis on visual explanation of the gradient method, showed the best results. Their answers strongly indicate that the method of looking at the changing gradient along the graph has been used in answering the questions. On an algebraic test (not reported here), these students showed their capabilities, obtaining an average of 79% compared with 69% of the rest. Apart from question 2, where the format starting from algebra encouraged an algebraic response, these students—who were highly capable in algebra—*chose to visually follow along the graph to sketch the gradient function.*

This sample of students is not consistent with the study of Vinner (1989) which suggested that students' preferred understanding is typically algebraic and not visual. It would indicates that the results depend on the way the students are taught and allowed to construct their own knowledge.

Correct gradient graph	All 6 questions	5 questions	4 questions	3 questions	2 questions	1 question
Group taught visually (N=13)	3	7	2	1	0	0
All other students (N=27)	4	3	7	8	3	2

Table 2: Comparison of responses from Teacher V's class and the other classes

The χ^2 test comparing the subdivision into categories with 5 or more against 4 or less reveals a significant difference at the 0.01 level. This is consistent with the hypothesis that teaching with an emphasis on visual method students' increases the likelihood that students will be able to draw the gradient function of a given graph.

Examination questions encourage students to use algebra rather than the graphic or visual mode. The students in the group with emphasis on the visual construction of knowledge, preferred a visual approach except in question 2 which privileged the algebra representation by starting from an algebraic formula.

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