Attainment and Potential: Procedures, Cognitive Kit-Bags and Cognitive Units

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This paper considers the case of students who attain the same level of performance at the end of one course and yet reveal very different levels of success on the course which follows. A comparison is made between two students attaining a grade B in college algebra who perform differently in the succeeding Pre-Calculus Course. Interviews reveal quite different cognitive structures. The successful student had a variety of approaches to problems, checking mechanisms, and an overall grasp of equations in one variable to build up links as if it were a ***cognitive unit****. The student who struggled had* ***a cognitive kit-bag of procedural techniques*** *with no flexibility or checking mechanisms. She relied on her calculator to help her over difficulties with negative numbers and fractions. She had the same attainment but very different potential to cope with the ensuing course.*

Introduction

Examinations may fulfil a variety of purposes, including a summative assessment of performance on the material studied, a diagnostic assessment of strengths and weaknesses, and a predictive assessment of future potential. In practice, final examinations in a course are often summative in nature and may have little diagnostic or predictive value. In this paper we consider the case of two students who attain the same level in a summative examination at the end of a college algebra course and yet perform very differently on the succeeding Pre-Calculus course, showing the weakness of predictive value of the results. We seek an understanding of the phenomenon by using diagnostic interviews to reveal the reasons for the gap between attainment and potential.

Crowley & Tall (1999) studied the diffuse cognitive structure of a student who was less successful in a course studying straight-line equations. This paper will compare and contrast the work of a similar student—who struggled with very straightforward algebra concepts—with that of a student who attained the same grade and yet proved to be more successful in the succeeding course. As the students attained the same grade, the difference between them will be a qualitative rather than quantitative.

Our thesis is that a successful student organizes his or her mathematics into a network of connected cognitive units (Barnard & Tall, 1997) with flexible links. She or he is able to move among the cognitive units almost subconsciously as needed to solve a problem—the network structure contains (usually multiple) procedures toward a solution, checking mechanisms, and links to a larger mathematical structure. Although the two students have the same attainment, we seek qualitative factors in their responses which may explain the difference in future potential.

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Students who attain the same level but have different potential

Nancy and Kathy both earned a "B" in college algebra, and both enrolled in pre-calculus the following semester. Nancy had little difficulty in pre-calculus, but Kathy had a great deal. Because it is a degree requirement, Kathy needed to pass the course; she dropped it once, re-enrolled and eventually passed it, but with much hard work and a great deal of anguish. Why do "so many of the population fail to understand what a small minority regard as being almost trivially simple?" (Gray & Tall, 1994). Why, as in this case, do students with apparently similar attainment go on to perform so differently?

To seek insight into these questions, we will explore the cognitive structure demonstrated by the two students working problems involving graphs of linear functions from the first college algebra course.

The interview took place when the students were already studying pre-calculus, so it would already show longer-term understanding rather than the knowledge the students may have learnt at the time of the exam.

In separate interviews the first-named author asked each student to find the slope of the line from the graph in Figure 1.

Nancy demonstrated a flexible, efficient solution process. She began by noting the direction of slope:

> **Nancy: What is the slope on the graph here? It goes down.**

The conversation focused on the two displayed points:

> **Nancy: Do you want me to solve it using the points?**
>
> Researcher: I don't care how you solve it.
>
> **Because you can just really look at it and tell…It's a negative 2 slope.**
>
> So you did it by counting squares.
>
> **Yeah, just by looking at it. I mean, if the squares weren't there, I could do it by taking the two points and finding the slope like I've done.**

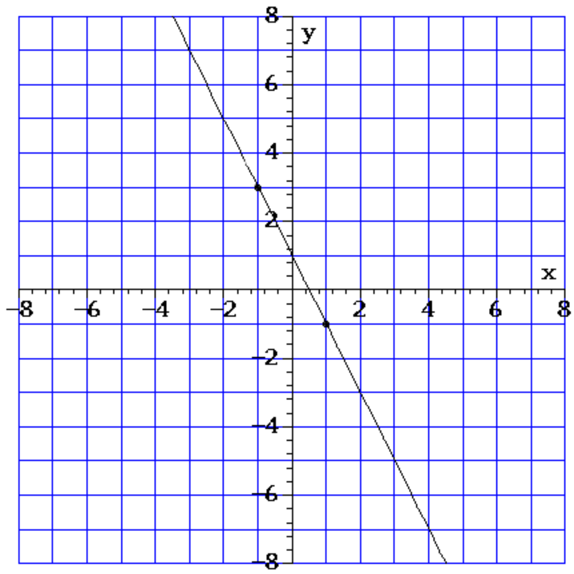

Figure 1: Find the slope of the line

Nancy's strategies are flexible (figure 2, overleaf). She seeks to know if the teacher wants a certain type of solution, and when encouraged to use a method of her choice, she uses the direction of slope and counts along the squares to "see" the slope. also explained that she could also use two points and the formula for the gradient if required to do so.

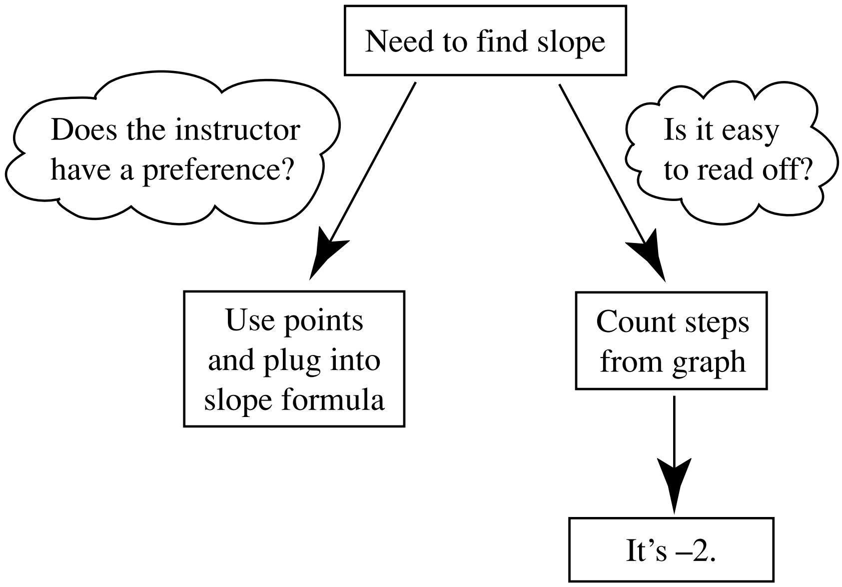

Figure 2: Nancy's solution to slope of the line

Kathy's solution is not so efficient. She fixes on the two marked points, reads them and writes them down.

$(1,-1)$
$(-1,3)$

Researcher: Can you find the slope of that line?

Kathy: One, negative one, and negative one, three.

She then uses the formula, writing it down correctly, but then works out the numerator by subtracting one from three, and silently writing –1 in the denominator.

$$\frac{3-(-1)}{-1-1} = \frac{2}{-1}$$

Kathy: Shoot.

Researcher: *(pointing at the left and right sides of the equation.)* So how did you get from here to here?

By subtracting…you've got me all bamboozled. My handy-dandy calculator, I rely on that. That would be four. [pointing at the numerator].

Kathy turns to her calculator and calculates $3-(-1)$ as 4 and changes her answer:

Four over two? Does that look right?

$$\frac{3-(-1)}{-1-1} = \frac{4}{2}$$

I think it does, but I could be wrong.

Well, you made a mistake, not a very big one…

One I can't catch, though. It'll make me feel really stupid later.

Researcher: *(Encouraging.)* I'm just trying to protect you from making a dumb mistake on your exam! … The denominator, what's minus one minus one?

Zero.

No, it's not.

Two, is that it?

It's negative two, isn't it?

Yeah.

She completes the solution: $\frac{3-(-1)}{-1-1} = \frac{4}{2} = -2.$

Kathy has great difficulty with negatives and fractions; she freezes when asked to manipulate them without a calculator. When she has a calculator she uses it to do her arithmetic and can then cope with simple calculations. Her solution process is diagrammed in figure 3.

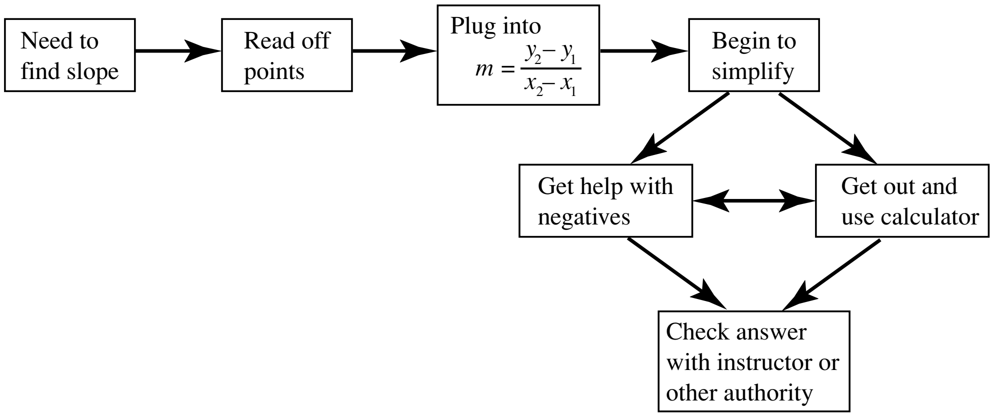

Figure 3: Kathy's solution to slope of line from graph

Nancy had flexibility; she could either read the slope off the graph, if that was easy, or she could find the slope by identifying two points and using the slope formula. This flexibility saved her a lot of effort on this particular problem. Kathy only evoked a formula for the slope, which required two points. She was unable to link the slope to the change in y over the change in x from the graph. This problem was, for her, consequently much more complicated, ultimately involving computations with negatives, which caused her much difficulty. She was also insecure when she finished the problem; she always checked with an authority figure—the instructor, the interviewer, or the answers in the back of the book—for assurance that she had found the correct answer.

In a second problem (figure 4) to write down the equation of a straight-line graph, Nancy's solution was not the most efficient. She was able to read off the slope, but then used the point-slope formula to write the equation, whereas it would have been more efficient to simply use the y-intercept from the graph. Nevertheless, she demonstrated flexibility in finding the slope, and again in checking to ensure that the point she was using in the formula was correct. An outline of her strategy is given in figure 5.

Meanwhile, Kathy responds to the problem by reading off two points (1,2), (4,5), using the formula for the gradient to find it is 1, and then the formula for a line through (1,2) with gradient 1. She too makes an error, but makes no effort to correct it until she is prompted by the researcher.

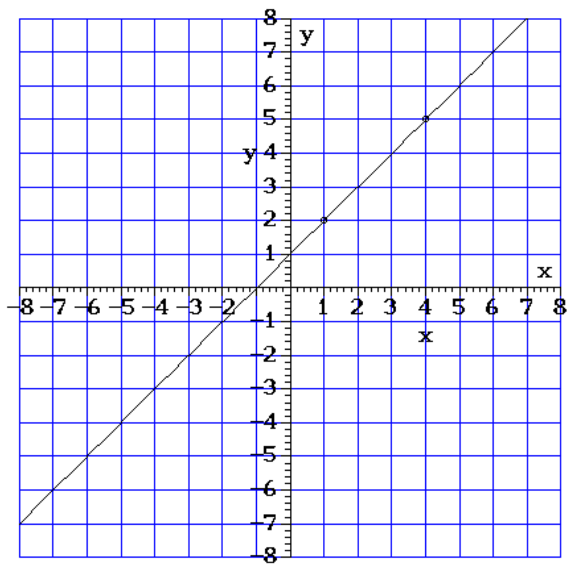

Figure 4: Write the equation of the line.

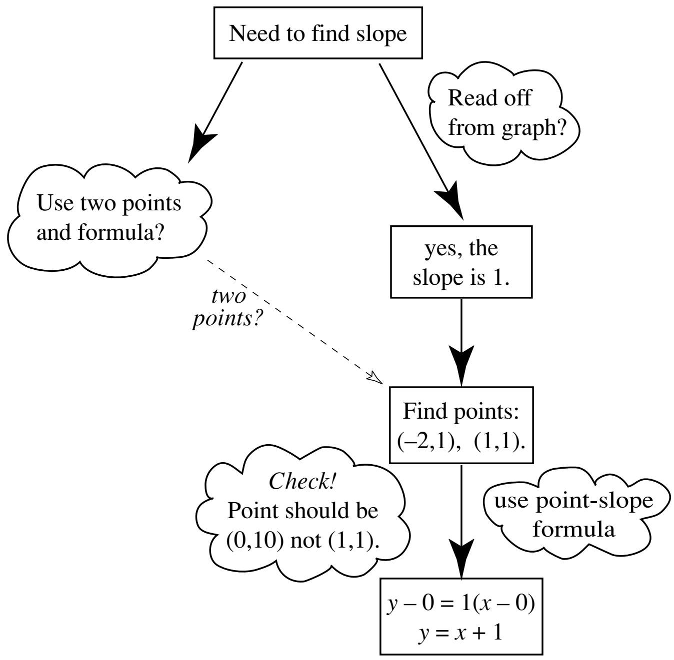

Figure 5: Nancy's solution to the problem in figure 4

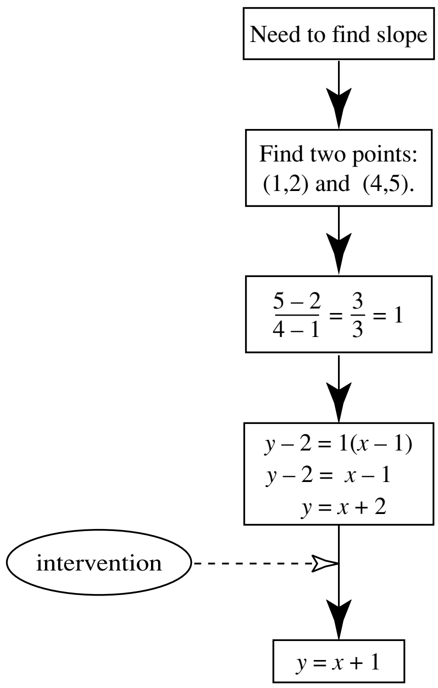

Figure 6: Kathy's solution

A third problem (figure 7) to find the equation of a straight-line graph has the same format of problem 2 but the negative gradient from problem 1. Nancy again shows flexibility, reading the slope m=–1 by inspection and then reading off the y-intercept b numerically to give the equation in the form $y = -1x + 1$.

Kathy did not exhibit the same flexibility or checking mechanisms:

Three, negative two…I need to find another point…
That's a good plan.
Four, negative three…
What are you going to do now?
Find the slope first. Negative three…

She uses the formula incorrectly …

$(3,-2)(4,-3)$

$\frac{-3-(2)}{4-3} = \frac{-1}{1} = -1$

$y-(-3) = -1(x-4)$
$y+3 = -x+4$
$y = -x+1$

Now wait a minute. Why did you…this is right.
Why did you tell me that $-3-2$ is –1?
Wait, is it negative 5?
Yes, however, you made a mistake earlier. Where you had $-3-2$, it's $-3-(-2)$, isn't it?
Oh, yes.
So then what do you want up here?

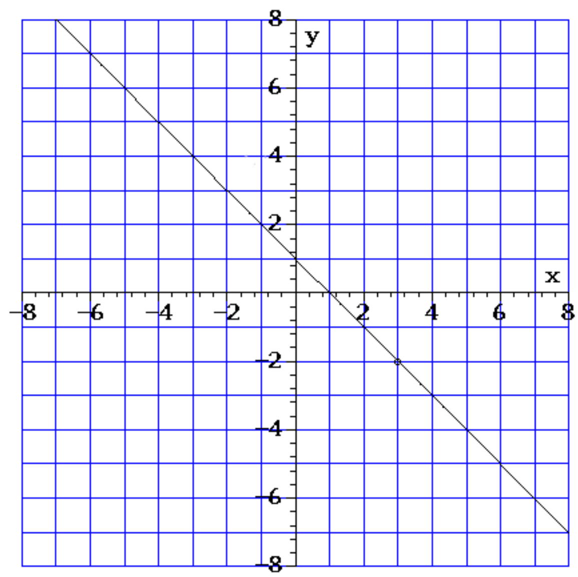

Figure 7: write the equation of this line

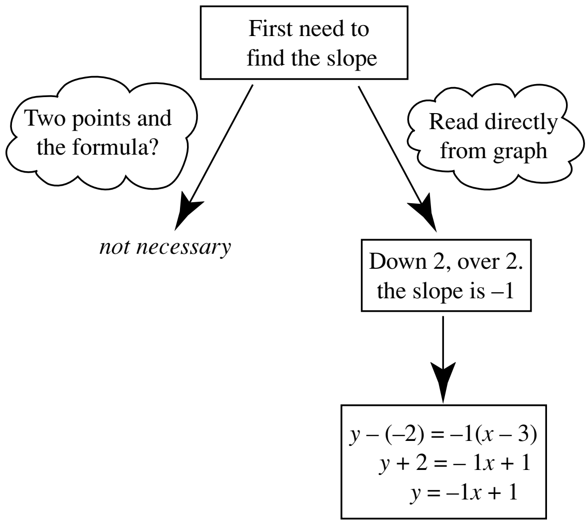

Figure 8: Nancy's solution to figure 7

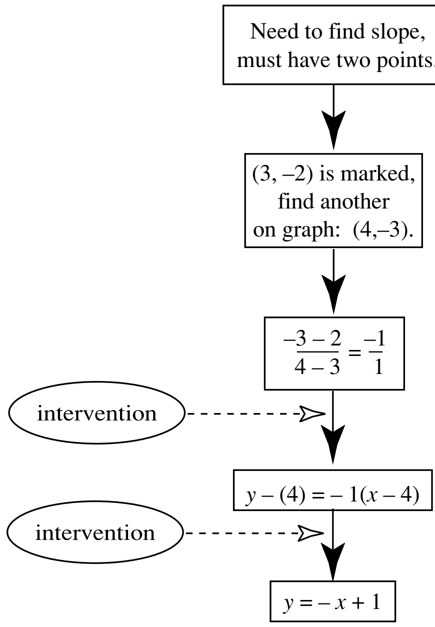

Figure 9: Kathy's solution

That would be positive 5?
Isn't minus a minus a plus?
Yes.
Minus three plus two is minus one.
Which is what I got, I just didn't have the negative sign.
(note: she had made canceling errors)
So you've got the slope, now what are we going to do?
Use the…
What's the y-intercept on that line?
The y-intercept?
Where does it cross the y-axis?
One, er, zero, one.
So you should get something x plus one, shouldn't you?
Yeah.

In all three problems so far, Nancy demonstrated flexibility in choosing a route to a solution, thus showing evidence of links between graphs, formulas, and other aspects of a problem. She also routinely checked her work using alternative methods, another indication of useful links. She found her own error in the second problem. Kathy on the other hand had at most a single procedure in each case and was prone to make mistakes.

There were similar features elsewhere. For instance, two successive problems were as follows:

Write the equation of the line through the points $(1,-4)$ and $(3,8)$

and:

Is the point $(5,14)$ on the same line as $(1,-4)$ and $(3,8)$?

Both students solved the first problem by finding the slope through the two points and then substituted (3,8) to get $y-3=6(x-8)$ and simplified to obtain the equation $y=6x-10$. Once again, without her calculator, Kathy made a mistake and needed to be corrected. Both used the point (3,8) rather than $(1,-4)$. When asked why, Nancy replied :

Because I don't like negatives… Less chance for messing up if you don't use the negatives.

Kathy replied:

Because that had a negative. If it has a zero in it, I choose it.

In this way we see that both students avoided the use of negatives where possible. However, Nancy coped with them and Kathy, without her calculator, was liable to error.

When shown the second question, neither initially saw the link to the previous problem. The conversation with Nancy went as follows:

Nancy: So what you want to do is you want to find out the equation for this and then put that [(5,14)] into the equation …

Researcher: You've already got this one (*pointing to the previous question.*)

Oh, it's the same one?

She wrote $y = 6x - 10$ *and continued …*

That's the equation for that…So then maybe I could just find out the slope for this and that…

$y=6x-10 \qquad (5,14)\ (3,8)$

$$\frac{8-14}{3-5} = \frac{-6}{-2} = 3$$

The slope is 3. My slope is different, so it can't be on the same line.

With Kathy the conversation was more one-sided:

If you wanted to know if the point (5,14) is on the same line as $(1,-4)$ and (3,8).

…silence…

This is the same as those (*points to the previous problem.*)

…silence…

How would you figure out if a point is on the same line?

…silence…

I don't remember…

Okay. That's okay.

In this problem we see Nancy does not always make the necessary links between information. However, with a little prompt, she followed the problem straight through, not by substituting the coordinates into the equation to see if the point lay on the line, but insightfully by comparing the gradients of two lines to show they are not the same. Kathy was unable to cope.

Summary

In all the questions considered there is a broad common thread. Nancy demonstrated links between graphical and symbolic representations, as well as links to and between procedures. Although she made mistakes, she had methods of checking and self-correcting. She did not always make the necessary connections and had some fears about negative numbers, but was broadly successful. Kathy obtained the same grade on her examination but merely learned a set of procedures and had difficulties with negative numbers and fractions that she coped with in routine questions by using her calculator. The procedures she has learned have allowed some success, but she must work very hard, and the procedures are not organized in a useful way that will allow her to build on them in the subsequent pre-calculus course.

This data revealed fits with a range of theories in mathematics education. For instance, the work of Krutetskii (1976) specifies clearly that the more successful students work with 'curtailed solutions', and are able to 'switch easily to another solution method', while less successful students 'tend to remember only specific details'.

The performances of Nancy and Kathy may also be considered in terms of the SOLO Taxonomy, in particular the concrete symbolic mode with its cycle of unistructural, multi-structural, relational, extended abstract. Kathy reveals a uni-structural response in tmany of these problems. Taken together with her other knowledge of equations, she has, at most, a multi-structural level consisting of her cognitive kit-bag of little related procedures. Nancy, on the other hand is more relational in her treatment of solution processes, moving towards an extended abstract level where she is beginning to treat the whole straight-line/linear relationship schema as a coherent cognitive unit. However, when Kathy uses a calculator to carry out her arithmetic, she can cope better with negatives and fractions. As a result she can pull up her level of performance so that, in the end, both she and Nancy attain the same grade B. But in the subsequent course, Nancy succeeds first time and Kathy has great difficulty in passing at the second attempt.

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