Developing Formal Mathematical Concepts over Time

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This paper investigates the development of university students' understanding on 'equivalence relations & partitions' over a period of time. Although these ideas are taught in the same topic, they have quite different cognitive properties. We find that, although the concept of 'relation' can be visualised, an 'equivalence relation' is more subtle. A partition, however, is more easily visualised than remembered formally. Our focus is on if and how these different properties influence students' concept development.

Introduction

Chin & Tall (2000) focused on a theory in which informal mathematics becomes formalised by introducing definitions, proving theorems and compressing formal concepts into cognitive units appropriate for powerful formal thinking. The theory was tested by a questionnaire filled in by 36 students after 6 weeks studying the formal theory of equivalence relations and partitions. It was found that:

> Less than half gave *formal* responses in terms of *definitions* or *theorems*. […] This confirms a picture in which the majority of students following a formal course at a highly rated university responded at an informal level after several weeks' experience of formalism. At the same time, two able students worked in a different way using the *compressed concept* that encompassed both equivalence relation and partition. Chin & Tall, 2000, p. 183

In this paper we follow the development over a longer time period to gain further insight into the students' constructions. We focus on fifteen students, of whom ten were tutored by the first author for an hour per week during the first two terms and on into the second year. Data was collected through audio-taping tutorials and in-depth interviews, with a second application of the questionnaire to determine long-term changes in conceptions.

An evaluation of the Foundations course by the students at its conclusion in the first year revealed that the students considered 'relations' to be the most difficult topic—a comment that had been repeated for several previous years of assessment. Summarising the perceptions of the students, the annual report commented that '*Euclid's algorithm* and *symbolic logic* were well understood, *basic set theory* and *functions* generally required extra work, but the topic on *relations* was often poorly understood.' On an average, only about 20% of students declared that they understood relations well with nearly a third of students claiming that, even after extra study, they only understood the topic poorly. It was this observation that drew us to study the topic of 'equivalence relations and partitions'. We considered that an understanding of students' difficulties in this topic that they found most problematic might shed light into wider difficulties in the understanding of formal mathematics. In particular, *why* do the students claim to have such difficulty with 'relations'? We now focus on the longer-term development from the first to second year of the course.

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Analysis and comparison

The subjects are 15 second year mathematics students following a course in the highest ranked pure mathematics department in the whole of the UK. Their marks for the first year are widely distributed—three are over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, three between 40 to 49. They answered the same questionnaire on the topic of 'equivalence relations & partitions' that they have already learnt for about a whole year and were interviewed during the first term in their second year.

The formal definition of equivalence relation

The formal definition of equivalence relation in terms of being 'symmetric, reflexive, transitive' proves to be relatively easy for students to learn and reproduce, though the precise use of quantifiers in each part of the definition is a little more subtle. Table 1 shows that 14 out of the 15 students reproduced a definition although only 5 of these gave the full quantified definition, 4 gave the formal definition without quantifiers and 5 gave an informal response in terms of the three words 'reflexive, symmetric, transitive'.

	First Year (N=15)	Second Year (N=15)
Formal/detailed	5	9
Formal/partial	4	5
Informal/outline	5	1
Total definition	**14**	**15**
Example	0	0
Picture	0	0
Other	1	0
No response	0	0

Table 1: Responses to 'equivalence relations'

Only one student—whom we shall call 'Arthur'— did not give the formal definition in the first year. He explained later that he could not remember the definition at the time; instead he attempted to explain the notion of equivalence classes in terms of a partition:

An equivalence relation generates a subset of elements
that are all related to each other, and divides
the set into ~~partions~~ partitions.

Note the imprecision of the language here, for example 'generates *a subset*' and 'divides the set into *partitions*'. Arthur obtained 50% in the end of year examination and had to resit one of his courses. Nevertheless, even he was able to give a formal detailed response in the second year. This gives us our first major piece of evidence: *all* of the students could reproduce the definition of equivalence relation; 9 out of 15 gave a complete version, the other 6 at least remembered 'reflexive, symmetric, and transitive'.

This was reflected in responses to an informal question asking if the relation 'has the same surname as' is an equivalence relation on the set of students in the class (Table 2).

		First Year (N=15)	Second Year (N=15)
Informal	Informal Definition	2	1
	Other	0	1
	No response	0	0
Formal perhaps with some informal language	Definition	12	13
	Theorem	0	0
	Partition	1	0

Table 2: Responses to the informal 'surnames' question

Both the students giving informal responses in the second year were able to give a full formal response in interview. Both thought the question too trivial to merit a detailed response in writing. For instance, John (whose response was classified as 'other') wrote:

Since the relation is to do with "equality" of the surname, it must be an equivalence relation.

He was a talented student with a mark of 68% in the first year examination who showed his understanding to be definition-based on all his assignments and in tutorials. Thus all fifteen students were capable of a formal definition response by the second year.

Table 3 shows the responses to the following question:

> A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection $f: X \to Y$. Is this relation an equivalence relation?

		First Year (N=15)	Second Year (N=15)
Informal	Informal Definition	3	0
	Other	1	1
	No response	0	0
Formal perhaps with some informal language	Definition	7	2
	Theorem	3	12
	Partition	1	0

Table 3: Responses to the formal 'bijection' question

This data shows that after being given a period of time to digest what they had been taught, whilst only 3 were theorem-based in the first year test, 12 are able to upgrade their understanding to the theorem-based level in the second. This is consistent with the successive move from definition-based conceptions to theorem-based conceptions over a time in which the ideas are being used formally.

The definition of equivalence relation on a set *S* as a subset of *S*×*S*

When responding to the notion of equivalence relation, none of the selected fifteen students used the general notion of relation as a set of ordered pairs in their definition. Only one student (Nathan) in the first year alluded to the idea as follows:

when r relates to itself, ie $r \sim r$, $r \sim s$, $s \sim r$ $\forall r,s$ and if $r \sim s$ and $s \sim t$ then $r \sim t$. (~~Relates~~ 2 items relate if they're both in the chosen part of a cartesian product set $\{m,n\}$ $m \in M$ $n \in N$

Even here the notion is an afterthought following the definition in terms of the notation $a\rho b$, for the relation ρ rather than the notation $(a,b)\in\rho$ which was given initially in the lectures. Notice that even here Nathan used the notation $\{a,b\}$ (used in the course for *un*ordered pairs) rather than the correct notation (a,b).

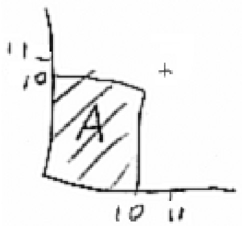
In the second year, only one student (Simon, the most successful with a mark of 85%) referred to a relation as 'a subset of $A \times A$' in his response to the meaning of 'equivalence relation'. He also was the only student to give a satisfactory answer to the following:

$A = \left\{(x,y) \in \boldsymbol{R}^2 \mid 0 \le x \le 10,\ 0 \le y \le 10\right\}$. Is A an ***equivalence relation*** on $\boldsymbol{R}$?

In the first year *no* student responded positively to this question. Several wrote explicitly that they did not understand what the question meant:

$A=\{(x,y) \in \boldsymbol{R}^2 \mid 0 \le x \le 10,\ 0 \le y \le 10\}$. Is A an ***equivalence relation*** in $\boldsymbol{R}$?
Answer (yes or no or don't know): Don't know.
Full Explanation: A defines points in the plane x-y where $0 \le x \le 10$ and $0 \le y \le 10$. But don't understand the relation.

In the second year, Simon responded as follows:



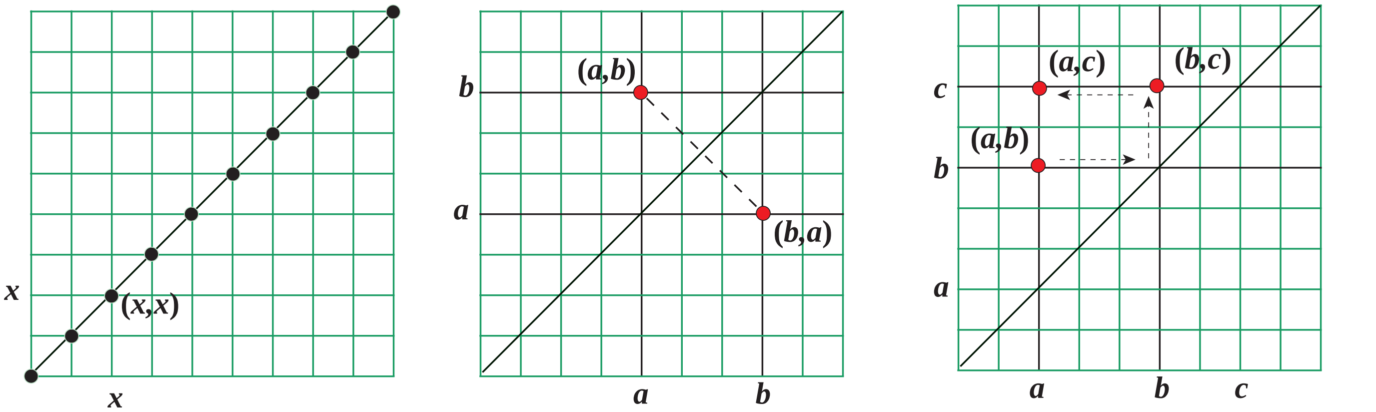
Consider $11 \in R$, as $(11,11) \notin A$, not reflexive.

He described an equivalence relation as 'a subset of $A \times A$' with reflexive, symmetric and transitive properties that can divide a set into a partition. He also offered the formal definition with all the detail. He therefore had a conception of equivalence relation and partition as a rich cognitive unit.

We therefore obtain our second major piece of evidence: *all but one* of the students did not relate the notion of relation as a set of ordered pairs with the notion of equivalence relation.

The gap between relations and equivalence relations

We see that *all* fifteen students could work with the notion of equivalence relation using the notation $a \sim b$, but only *one* evoked the notion of relation on a set S as a subset R of $S \times S$. On reflection, one can see that the notion of 'equivalence relation' on a set S does not have an easy visual image. Seen as a subset R of $S \times S$, the reflexive law can be pictured by saying that the diagonal elements (x,x) are all in R, the symmetric law can be seen in terms of reflection of the element $(a,b) \in R$ in the diagonal to also give $(b,a) \in R$, but the transitive law (a,b), $(b,c) \in R$ *implies* $(a,c) \in R$ is a little more sophisticated. (The transitive law moves horizontally from (a,b)—maintaining the second coordinate b—to the diagonal then vertically to the point (b,c), completing the rectangle to give the third point (a,c).)(Figure 1).



reflexive: **$\forall x \in S$, $(x,x) \in R$**

symmetric: **$(a,b) \in R \Rightarrow (b,a) \in R$**

transitive: **$(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$**

Figure 1: Visual representations of the three axioms for an equivalence relation R on a set S.

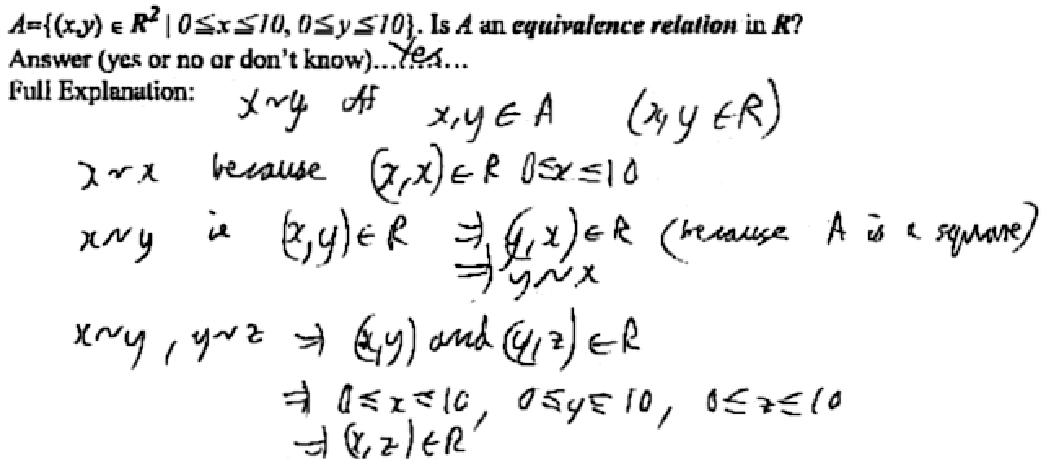
The complexity of the visual representation is such that it was not taught in the course. Thus, although the notion of relation on a set S is given in terms of a subset of $S \times S$, *it is never represented as a visual picture*. In this way there is a complete dichotomy between the notion of relation (interpreted as a subset of $S \times S$) represented by pictures and the notion of *equivalence relation* which is not.

Furthermore, the topic of ‘relations’ also includes *order relations*. We hypothesise that the typical student will find it difficult to give a coherent overall meaning to the notion of ‘relation’ that encompasses both order relations and equivalence relations. Partial support for this hypothesis is the students almost total failure to respond to the equivalence relation defined as a set of ordered pairs compared with almost total success with questions using the form $a \sim b$.

After interviewing 10 of the 15 students, the authors find that these students learnt the definition of relation on a set formally as: ‘a subset of the cartesian product of the set itself’. But they learnt the definition of ‘equivalence relations’ focused on the three properties of reflexive, symmetric and transitive. The following conversation recorded in an interview with two students (whom we name Jack and Nathan, respectively) offers some evidence. Jack and Nathan were being asked about the question in which the relation A is defined as the subset $A = \left\{(x,y) \in \boldsymbol{R}^2 \mid 0 \le x \le 10, 0 \le y \le 10\right\}$ of $\boldsymbol{R}^2$

Jack: Sorry! I can’t understand what this question means?
Interviewer: O. K. Nathan, can you understand it?
Nathan: Well, umm…(*Pondering for a while.*) No, I don’t think so.
I: Can you think of the formal definition of ‘relations’ first?

They started trying to recall their memory of ‘relations’.
N: I think it’s a sort of ordered pairs, isn’t it?
I: Yes. You are right. Can you say it more formally?
J: Let me think. It’s ages ago, I don’t think I can remember it.
I: How about you, Nathan?
N: *(Shaking his head.)*
I: O.K. Let me write it down on the board.
The interviewer wrote the definition (Stewart & Tall, p.69) on the board and explained it to them.
J: Yes. I see. That should be what we learnt in the lecture a long time ago.
I: O.K. Now, can you try to answer this question again?
Nathan immediately made the whole deduction, answering ‘yes’ after checking the three conditions although he did not include the quantifier in ‘reflexivity’. Jack still seemed confused.


$A=\{(x,y)\in \mathbf{R}^2 \mid 0\le x\le 10,\ 0\le y\le 10\}$. Is A an ***equivalence relation*** in $\mathbf{R}$?
Answer (yes or no or don’t know)…Yes…
Full Explanation: $x\sim y$ iff $x,y\in A$ $(x,y\in R)$
$x\sim x$ because $(x,x)\in R$ $0\le x\le 10$
$x\sim y$ ie $(x,y)\in R \Rightarrow (y,x)\in R$ (because A is a square)
$\Rightarrow y\sim x$
$x\sim y,\ y\sim z \Rightarrow (x,y)$ and $(y,z)\in R$
$\Rightarrow 0\le x\le 10,\ 0\le y\le 10,\ 0\le z\le 10$
$\Rightarrow (x,z)\in R$

J: I still can’t see how to check *A* is an equivalence relation in *R*.
I: You can understand the definition of ‘relations’ we just reviewed, can’t you?
J: Yes. I think so.
I: ‘Equivalence relation’ is just a kind of ‘relation’ but with some more properties, isn’t it?
J: Yes.
I: Just add the three properties to the definition of ‘relation’, then try to answer this question again.
Jack was stuck checking ‘reflexivity’.
J: I’m getting confused. What’s the point of checking ‘reflexive’?
I: Nathan has finished his deduction. Let’s have a look at his answer then I’ll answer you, Jack. Do you think Nathan’s answer is correct?
J: mmm…*(Pondering for a while.)* Yes, I think so.
I: O. K. Let’s have a careful look at ‘reflexivity’. What is the quantifier for it?
N: For all the elements in *A*?
I: What do you think, Jack?
J: Should be ‘for all the elements in *R*’.
I: Do you agree with Jack, Nathan?
N: mmm…Yes. Yes. I think he’s right.
I: So, do you think *A* is ‘reflexive’ now?
J: I see. No. Because *A* doesn’t cover the whole plane, so it won’t be ‘reflexive’.
N: mmm… So the answer should be ‘no’.
I: I think you both get the point. Now can you check if *A* is ‘symmetric’ or ‘transitive’?
N: Yes, both of them. I think I used the wrong notations. They should be *A*, not *R*.
I: Well done! Jack, could you make a conclusion of this question?
J: You mean *A* is not an equivalence relation because it is symmetric and transitive but not reflexive.

This data seems to suggest that even students who do not have a formal concept image of a former idea (like the idea of 'relations') can still build up their understanding of the next relevant idea without having much particular difficulty (like the idea of 'equivalence relations') if the two definitions are not directly related.

Partitions

The development of the notion of partition also improved over the year (table 4).

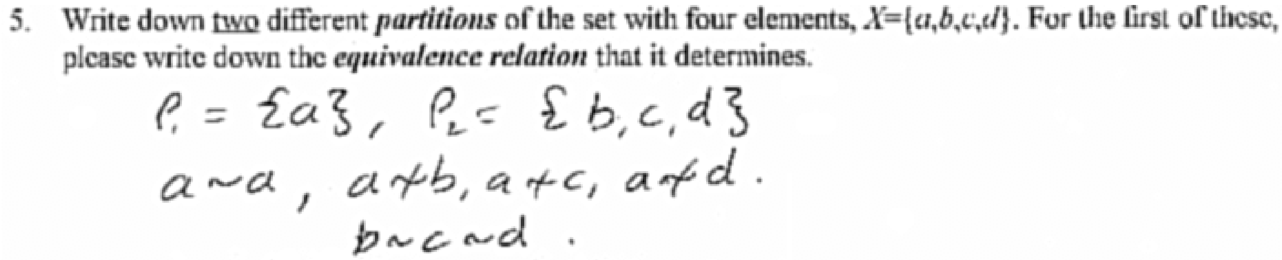
	First Year (N=15)	Second Year (N=15)
Formal/detailed	2	8
Informal/outline	6	3
Total definition	8	11
Example	0	0
Picture	1	1
Other	4	3
No response	2	0

Table 4: Responses to 'partitions'

The number of detailed formal definitions increases from 2 to 8 and the overall definitions increase from 8 to 11. However, 4 students fail to give a definition for partition when all can give a definition for 'equivalence relation'. Looking closely at the responses reveals that the majority of students tried to use *their own language* to interpret the definition of 'partitions' so that their answers were highly varied.

Interestingly, all ten students interviewed said they had a mental picture of a partition. Nine of them thought they understood 'partitions' better than 'equivalence relations'. The exception, Jack, explained that although he could picture partitions, he still did not know the formal definition and was happier handling the formal definition of equivalence relation. Of the other nine, Arthur was typical in saying that he felt he understood 'partitions' better than 'equivalence relations' because he could visualise 'partitions' but not 'equivalence relations'.

When asked to give examples of partitions, twelve out of fifteen gave satisfactory answers. The other three revealed an interesting misconception. Jack wrote:


5. Write down two different *partitions* of the set with four elements, $X=\{a,b,c,d\}$. For the first of these, please write down the *equivalence relation* that it determines.

$P_1 = \{a\}$, $P_2 = \{b, c, d\}$

$a \sim a$, $a \nsim b$, $a \nsim c$, $a \nsim d$.

$b \sim c \sim d$.

At first sight this may seem as if Jack has written down one correct partition. However, in interview, he explained that he thought that his two partitions were P_1 and P_2. All three of the students giving unsatisfactory responses shared the same misconception: that the term 'partition' referred to each individual subset, not to the collection of all subsets.

In this way we see that the class as a whole retain their understanding of 'equivalence relation' at the definition level and apparently shift their perception of partition to the theorem-level, whilst some are still having difficulty with the definition of partition.

Conclusion

In this paper we have been considering the development of 'equivalence relations' and 'partitions' a year after the students first met the concepts in the Foundations course. During this time they would have met the ideas in other courses and revised for the end of year examinations. We questioned why the students claimed that the notion of 'relation' was the most difficult in the whole of the Foundations course. We found that, after a year, although all 15 students could give the definition of equivalence relation using the notation $a{\sim}b$, only *one* could respond to a question where an equivalence relation was given in terms of a subset of the cartesian product. We showed that, although the notion of relation is easily visualised, the notion of equivalence relation is difficult to visualise but easy to remember as a verbal definition. We also hypothesised that the introduction of the very different notion of order relations at the same time gives little common ground amongst the examples of relation to allow a coherent link to be made between the examples and the general concept.

We note that nine out of ten students interviewed claimed that they felt they understood partition better than equivalence relation, whereas in fact their performance on the test showed that they were able to handle equivalence relation better than partition. This is accompanied by the observation that they say they can visualise a partition, but not an equivalence relation. We consider this to be consistent with the notion of 'embodied mathematics' (Lakoff & Johnson, 1999; Lakoff & Nunez, 2000) giving a deeper human sense of meaning. Thus the development of the formal thinking characteristic of the 'rigour prefix' (Alcock & Simpson, 1999) is here underpinned by the embodied concept-image and formal concept-use in the sense of Moore (1994).

Over the year there is a general shift from 'definition-based' deduction referring specifically to the formal definition to 'theorem-based' deduction, using already proven theorems. One student clearly had the composite notion of equivalence relation and partition as a rich cognitive unit. The investigation of whether others have such a cognitive structure is more likely to arise in interview rather than standard written questions. This remains a topic of our current research.

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