CONSTRUCTION OF CONCEPTUAL KNOWLEDGE: THE CASE OF COMPUTER-AIDED EXPLORATION OF PERIOD DOUBLING

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This research focuses on students using an experimental approach with computer software to give visual meaning to symbolic ideas and to provide a basis for further generalisation. They use computer software that draws orbits of x = f(x) iteration and are encouraged to investigate the iterations of $f_\lambda(x)=\lambda x(1-x)$ as λ increases. The iterations pass through successive acts of period-doubling as $\lambda=\lambda_0, \lambda_1, \lambda_2, \ldots$; they are invited to estimate the values of λ and to compare their experimental results with the theory of geometric convergence. The supervisor acts as a mentor, using various styles of questioning to provoke links between different ideas. Data is collected in various ways to give evidence for the ways in which students develop conceptual links between symbolic theory and the visual and numeric aspects of computer experiment.

INTRODUCTION

According to Dreyfus and Eisenberg (1991), students prefer to think algebraically rather than geometrically when they are solving problems, and the authors give several reasons for this in terms of social, curricular and epistemological factors. For example, students are more likely to solve the equation $f(x) = x$ rather than drawing a graph of the function and a diagonal line when asked to find fixed points of the given function *f*. In other words, they think algebraically about a given problem using equations, symbols and logic rather than draw a diagram or pictures. On the other hand, visualisers are people who prefer visual and spatial methods and think geometrically when they are asked to solve a problem. Research concerned with the use of visual abilities reveals not only the value of visual processing in visuo-spatial problems, but also that both high visualisers and low visualisers can improve their understanding with computer-assisted learning using graphical representations (Sein, Olfman, Bostrom and Davis, 1993).

> Restoring the visual and intuitive side of mathematics opens new possibilities for mathematical work, especially now that computing has enough power and resolution to support it with accurate representations of problems and their solutions. The benefits of visualization include the ability to focus on specific components and details of very complex problems, to show the dynamics of systems and processes, and to increase intuition and understanding of mathematical problems and processes (Cunningham, 1991, p. 70).

Tall (1991a) also observes that computer graphic software can provide students with environments for intuition prior to the construction of a formal concept. With these ideas in mind, we followed a course in which a mathematics professor provided

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students with a software environment to “get their hands dirty” through an experimental approach that encouraged them to think visually and numerically rather than just symbolically. These students were first year mathematics majors in a university with a high quality student intake (all with A-grades at A-level mathematics and a minimum of AAB in three subjects). In their first two terms they had received lectures and seminars on mathematical analysis, differential equations, linear algebra, group theory. They were now attending a computer laboratory course in which they were to extend their symbolic experience with visual explorations to lay conceptual foundations for chaos theory.

Sierpinska (1987) has warned that the use of a visual representation may focus the students’ perceptions on the nature of the computer picture in an immediate, intuitive and global way, obscuring more subtle ideas in the potential infinity of the symbolic process. We investigate whether this occurs and see what kind of arguments the students use to link experimental numerical results and formal theory. Do the visuo-spatial experiences with the computer, supported by the supervisor as mentor, provide a basis for reflective activities that lead to flexible conceptual thinking relating numerical experiments, visual representations and symbolic theory?

> Experience modifies human beliefs. We learn from experience or, rather, we ought to learn from experience. To make the best possible use of experience is one of the great human tasks and to work for this task is the proper vocation of scientists. (Polya, 1954, p. 3).

PROCEDURAL AND CONCEPTUAL KNOWLEDGE

Hiebert and Lefevre (1986, p. 3) state that the crucial characteristic of conceptual knowledge lies in the rich relationships constructed between specific pieces of information. It may be considered as a well-connected web of knowledge, for flexibly accessing and selecting information. In contrast, procedural knowledge is a form of sequential knowledge, constructed in a succession of steps.

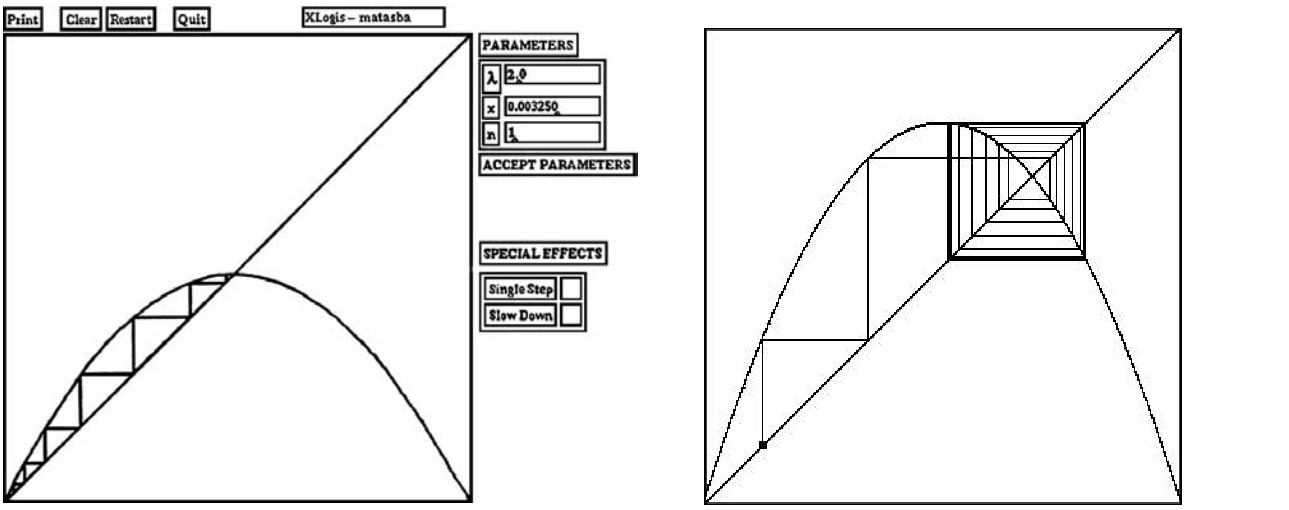
Heid (1988) showed that students in an experimental calculus class using a microcomputer as a tool for visualizing graphs and for manipulating symbols developed a broader conceptual understanding than students in a traditional class focusing mainly on symbolic procedures. She found that students gaining conceptual knowledge in this way were able to develop concepts further than those using procedural knowledge. Many other researchers (eg Tall, 1991b) contend that students using interactive dynamic computer software gain a much better insight into mathematical concepts than those following a traditional curriculum. In this study we therefore consider conceptual knowledge constructed through visualisation using interactive graphical software. To investigate this idea, the present research was conducted using the framework outlined in the next sections, focusing on the establishment of the connection between visual orbits of $x = f(x)$ iteration, the numeric information provided by the software and the underlying mathematical theory.

THE MATHEMATICAL CONTEXT

The research focuses on a mathematical activity which is part of a theoretical and experimental development leading to chaos theory. The computer class was preceded by an hour's guided symbolic investigation in which the students investigated the fixed points of $f(x) = \lambda x(1-x)$ symbolically. This involved solving the equation $x = \lambda x(1-x)$ for x in terms of λ to obtain the roots in symbolic form, namely 0 and $1–1/\lambda$. They then investigated the size of $f'(x)$ at these fixed points. If $|f'(\alpha)| < 1$ at a fixed point α, then α is an *attractor*, and iterations will home in on it. On the other hand, if $|f'(\alpha)| > 1$ then α is a *repeller* and iterations will move away. By symbolic means the student is expected to determine when the fixed point $1–1/\lambda$ is an attractor. Since $f'(x) = \lambda - 2\lambda x$, it is easy to show that $x = 1–1/\lambda$ is an attractor for $1 < \lambda < 3$.

The student is then invited to carry out similar calculations for the function $f^2(x) = f(f(x))$ which is a little more intricate but possible. (The students involved are very able and 14 out of 19 were able to complete the symbolic task.) The calculations for higher iterates $f^{n+1}(x) = f(f^n(x))$, however, become more complex and it is time to switch to the computer model.

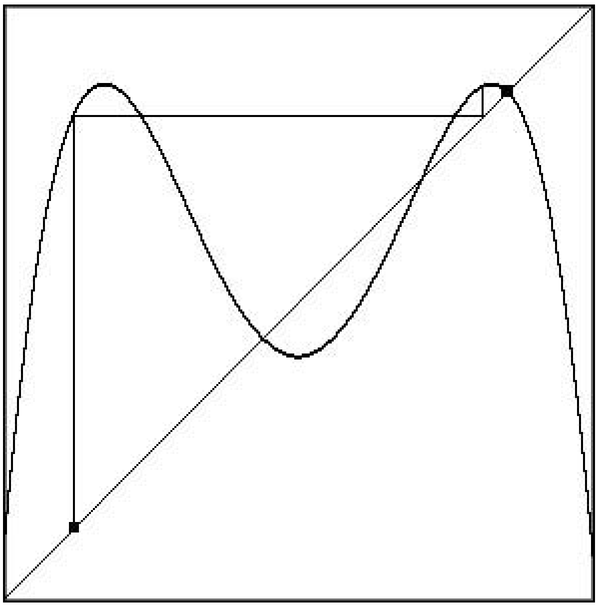
The student is invited to investigate iteration of the function $f(x) = \lambda x(1-x)$ using computer software as the parameter λ increases. For values of λ between 1 and 3, the iterations home in on the attractive root (figure 1(a)). Although this contains the seeds of the Sierpinska obstacle (that the limit may *actually* be reached visually, but not symbolically), the visual picture allows the encapsulation of the limiting process as a visual limit object, the *point x* where $x = f(x)$ iteration stabilizes. This point may then be seen to vary, changing smoothly as λ increases. When λ passes through the value 3, the attractive point becomes a repeller and the iterations begin to spiral out and settle in a period of length two (figure 1 (b)). This phenomenon is called a *period doubling bifurcation*.



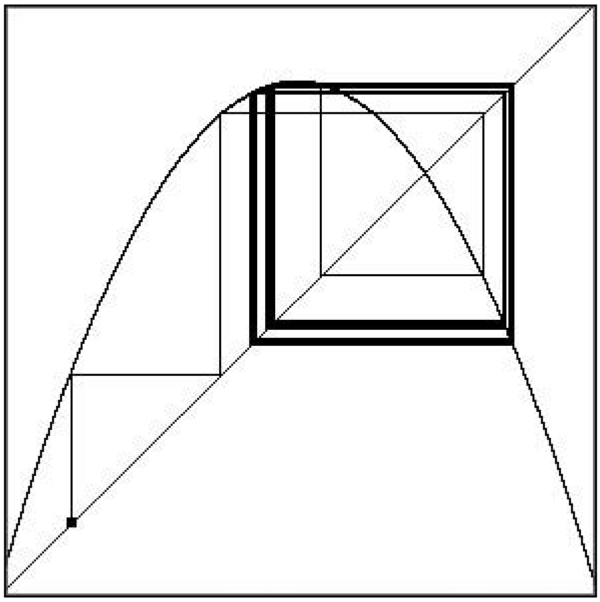
(a) convergence to one root of $f(x) = x$ (b) period doubling after λ=3 (here λ=3·2)

Figure 1. Graphic representations using the software *xlogis*

As λ continues to increase, at $\lambda = \lambda_1$, a new bifurcation occurs for *f*, from a cycle of length 2 to one of length 4. This corresponds directly to a simpler bifurcation for f^2 from a fixed point to a cycle of period 2. (Figure 2.)



(c) as λ increases, the fixed point for f^2 becomes unstable and bifurcates



(d) The corresponding cycle of period 2 for *f* opens out to a cycle of period 4

Figure 2. The second bifurcation

As λ increases further, there are a successive bifurcations of period 2, 4, 8, and so on, at values of $\lambda = \lambda_0, \lambda_1, \lambda_2$. The purpose of the computer investigation is to get numerical data on the first few values of this sequence and to get a sense that they satisfy the condition for geometric convergence. The sequence, $\lambda_0, \lambda_1, \lambda_2 \ldots$ converges to a value λ_∞ called the Feigenbaum point. The computer experience is therefore intended to give the student an experimental context offering visual pictures and numerical approximation to link to the symbolic theory. In particular, the *process* of iteration can be seen as a visual *object*—the final cycle of length 2^n—and the student can imagine the successive behaviours of this object as λ increases to give the succession of bifurcations.

THE RESEARCH FRAMEWORK

Subjects

The study involved thirty first-year students enrolled in an Experimental Mathematics course at the University of Warwick. The first-named author supervised three groups containing five, six, and seven students, respectively. The group with six students consisted entirely of students who had obtained first-class marks on previous tests. The other two groups had a broader spectrum of performance and were selected for further study reported in this paper.

Software: ***xlogis***

The software used was *xlogis* (see figure 1(a)) which is written in C and runs under the Xwindows Graphic User Interface at Sun terminals operating Sun Solaris. It is designed to enable students to control a range of parameters for iterating the function $f(x) = \lambda x(1-x)$; these include options to specify the value of λ, the starting point x and the number n of iterations in the displayed graph $f^n(x)$. In addition there are special effects to operate the iteration one step at a time or to change the pace of the iteration. The iterations drawn so far may also be cleared, enabling the user to focus on the later iterations when they have stabilized on a limit cycle.

Instruments

Various forms of data were collected in the study. The course organizer had already designed a *pre-requisite test* to test the students' understanding of "geometric convergence" (see below). The formal assessment for the course consisted of a *written assignment* handed in three days after each session, requiring students to write about their observations and inferences. In addition to these formal assessments, Soo D. Chae, who acted as supervisor and participant-observer, collected data using *audio-tapes* together with *field notes* made at the time. Finally, the students were given a *questionnaire* after the course to investigate some of their understandings relating to their visual experiences and symbolic theory.

Pre-requisite test

This was designed to investigate students' awareness of "geometric convergence" given in terms of the following definition and the accompanying question:

> A sequence (a_n) is said to *converge geometrically* if the ratio $(a_{n+2} - a_{n+1})/(a_{n+1} - a_n)$ converges to a limit r with $0 \le r < 1$ as n goes to infinity. Write down an example of a sequence that converges geometrically. Prove that a sequence which converges geometrically also converges in the usual sense.

The computer experiment

The students were given the following tasks to experiment with the logistic map $f(x) = \lambda x(1-x)$.

1. Use *xlogis* to investigate what happens when λ increases through the value 3·0.

2. Use *xlogis* to investigate the dynamics for λ between 3 and the value λ_1 for which the period 4 orbit occurs. What happens when λ goes through λ_1?

3. As you increase λ beyond λ_1, you should see a sequence of period doubling bifurcations. Use *xlogis* to obtain estimates of the parameter values λ_n for which the nth period doubling bifurcation occurs.

   What do you notice about the way the λ_n converge? The parameter value λ_∞ to which they converge is called the accumulation of period doublings. Try taking the ratios of successive differences. What does the result tell you? Can you think of a way of seeing this by drawing a graph?

In this sequence of activities, the students begin with a value of λ less than 3 to reveal the picture in figure 1(a), but as the value of λ passes through $\lambda_0 = 3$, the picture changes to the format of figure 1(b). They must then experiment with larger values of λ to estimate the values $\lambda=\lambda_1$, $\lambda=\lambda_2$, …, where successive period doublings occur. These must be performed as accurately as possible to be able to relate them to the theory of geometric convergence. In practice the accuracy is limited as it involves the student trying various values of λ and homing in on the points where the orbits change; the exact point where the change occurs can only be seen approximately on the computer screen.

The role of the supervisor

In order to improve effective experimentation, the supervisor assisted and responded to the group, providing support and explaining the phenomenon of period doubling. Sometimes the supervisor offered advice by providing directed questions to keep the students going if they were stuck. Three different types of questions were used: for *opening-up*, *structuring*, and *checking* (Ainley, 1988). For instance, an *opening-up question* responds to a student's request by asking the student to think more about it:

Student A: What is happening when the function cycles between two values? (referring to the picture in figure 1(b)).

Supervisor: How does this relate to the terms of the sequence x_1, $x_2 = f(x_1)$, $x_3 = f(x_2), \ldots$?

A *structuring question* is designed to construct concepts by linking disconnected knowledge via appropriate structured directing questions:

Student B: This equation is quite complicated to solve. How can I find the solutions? (pointing to the fourth degree equation generated by $f^2(x) = x$).

Supervisor: Do you think that fixed points of $f(x)$ also become fixed points of $f^2(x)$?

Student B: Maybe…. Umm …. Yes….

Supervisor: Why? Justify your answer. (Prompting the student to make links explicit.)

A *checking question* simply checks what the student has just done:

Student C: The function seems to be hitting four points. So, is this lambda one?

Supervisor: Will a slightly smaller value of lambda also hit four points?

Students' self-written reports

Students were asked to write up their observations and answers as they proceeded, and then to summarise their mathematical ideas and arguments clearly and hand them in within three days. According to Mason (1982), this kind of activity is valuable for helping students to reflect on what they have done and how they have done it. The supervisor graded reports using criteria that emphasised the quality of students' ideas without seeking perfect presentation. The students' reports on the mathematical questions posed during the experimentation provided a valuable source of data.

RESULTS AND DISCUSSION

Student responses to the pre-requisite test

Of the twelve students chosen, two did not give responses to the request for an example of geometric convergence, three gave incorrect responses ($a_n = 1/n$, $\sqrt{n}$) and seven gave correct examples ($a_n = 2^{-n}$, or 10^{-n}, or e^{-n}). Only five students could give a proof that geometric convergence implies convergence in the usual sense. (See the first two columns of Table 2 below.) Thus less than half of the students responded positively to what was considered a necessary pre-requisite.

Students concept images arising in the questionnaire

Case	Number (%)	Concept Image
A & C	18 (60%)	A fixed point of *f* is where the graph of *f* intersects the diagonal
A & D	1 (3%)	*[the correct response]*
A, C, & E	1 (3%)	
A, B, C & E	1 (3%)	D is not on the graph *y=f(x)*
C	2 (6%)	
E	1 (3%)	
No response	6 (20%)	

Table 1. Concept images for fixed points

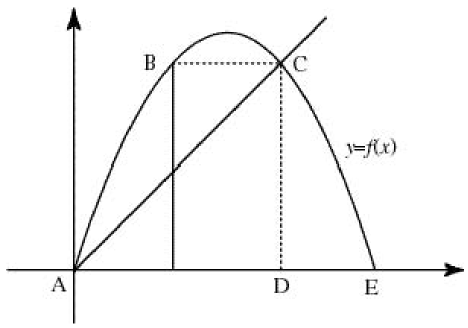


Figure 3. Which points are fixed points of the given function?

The post-course questionnaire included a question (figure 3) which gave the definition of a fixed point and asked the student to identify the fixed points of the iteration $x = f(x)$ in a picture. Despite the definition stating that it is a point *x* such that $y = f(x)$, the visual representation focuses instead on the point (x, y) where the line $y = x$ meets the curve $y = f(x)$. As we see from Table 1, only one student answered correctly while most students chose the intersection points *A* and *C* on the diagonal line. This is one example of a geometric obstacle. In this case it proved to be easy to reconcile the definition with the picture through discussion.

The major Sierpinska obstacle—the literal interpretation of the approximate picture on the screen as representing the actual underlying mathematics—is more subtle. We have already seen students using language that focuses on the role of the picture onscreen—student A earlier is quoted saying 'the function cycles between two values' and student C refers to 'the function seems to be hitting four points.' However, this does not mean that the student is not aware of what is going on with the convergence. For example, student C says *'seems* to be hitting', suggesting a distinction between what is seen onscreen and the underlying behaviour. This language allows the student to refer to the end result of the continuing process as a mental object. Certainly, when they come to discuss the convergence of the values λ_0, λ_1, λ_2, λ_3, …, nine of the twelve students use the concept of geometric convergence, which carries with it that, in this case, the convergence of the sequence of points of period-doubling in theory continues without actually reaching the limit. This suggests that some students can evoke the idea that the picture on the screen is only a limited accuracy model of the underlying theory.

Students' formulations of period doubling

The students' written reports were analysed to see how they responded to the tasks given them. One student, typical of those who were successful, estimated the numerical sequence of period doubling bifurcations as approximately λ_0=3, λ_1=3·449, λ_2=3·54, λ_3=3·559, λ_4=3·563, λ_5=3·564, λ_6=3·5642. He observed:

> The values for λ_n appear to be converging to a value between 3·5 and 3·6. The parameter value λ_∞ to which the λ_n's would converge is called the accumulation of period doublings. Taking the ratio $(\lambda_n - \lambda_{n-1})/(\lambda_{n-1} - \lambda_{n-2})$ of successive differences, we find 0·2027, 0·2088, 0·2105, 0·25, 0·21.

He noted that—despite the poor accuracy afforded by the limitations of the experiment—the values of the ratio were initially around 0·2, which is far less than the critical value 1 for geometric convergence.

Our main concern is the manner in which each student coped with the convergence of the sequence in the written assignment (table 2). Student S1 attempted to give a symbolic proof of the convergence, in addition to drawing a graph of his numeric computations, plotting values of $\lambda_n - \lambda_{n-1}$ against $\lambda_{n-1} - \lambda_{n-2}$ to reveal a line of

	Pre-requisite Test		Written Assignment			
	Geometric convergence	Proof	Numeric	Graphic	Symbolic	Geometric convergence to λ_∞
S1	yes	yes	yes	yes	yes	yes
S2	yes	no	yes	yes	yes	yes
S3	no	no	yes	yes	yes	yes
S4	no	no	yes	yes	yes	yes
S5	no	no	yes	no	yes	no
S6	yes	no	no	no	no	no
S7	no	no	no	no	no	no
S8	no	yes	yes	yes	yes	yes
S9	yes	yes	yes	yes	yes	yes
S10	yes	yes	yes	yes	yes	yes
S11	yes	no	yes	yes	yes	yes
S12	yes	yes	yes	no	no	yes

Table 2. Responses of observed groups in the pre-requisite test and written assignment

gradient approximately 0·2. Student S2 drew a similar graph without any supporting symbolic argument. Student S3 and S4 plotted the values of $(\lambda_n - \lambda_{n-1})/(\lambda_{n-1} - \lambda_{n-2})$ for λ against n to obtain a sequence of points approximating to the horizontal line λ=0·2. Student S5 obtained the numerical values and simply observed that they were approximately 0·2. Students S6 and S7 were not able to obtain satisfactory numerical values to be able to attempt the task.

Nine students (S1 to S4, S8 to S12) were able to give some kind of explanation relating the experimental results to the theory of geometric convergence. This included S3, S4 and S8, who had not responded satisfactorily to the pre-requisite test. This simply means that these three students were able to *use* the test for geometric convergence without being able to give examples or to give a formal proof that geometric convergence implies convergence.

Nine of the twelve students (S1 to S5, S8 to S11) succeeded in relating their experiences to their earlier use of symbolism in the preliminary work. Interestingly, all of these students also succeeded in using numeric representations in the written part of the assignment. More importantly, eight students out of the nine who were able to provide a graphical representation of their numerical data were also able to link their numeric data to the geometric convergence of the sequence to the Feigenbaum point. (Table 2.) These findings underline the fact that two thirds of the students involved were successful in connecting their visual and numeric observations to theoretical aspects of the situation, intimating that the experiment was successful in aiding their construction of wider conceptual knowledge.

SUMMARY

This investigation into students using computer software to gain visual insight and to obtain numerical approximations to link with theory proved to be successful for eight out of twelve students, including three who did not have the desired pre-requisite knowledge of the notion of geometric convergence. These three were able to operate by simply substituting into the given formula $(\lambda_n - \lambda_{n-1})/(\lambda_{n-1} - \lambda_{n-2})$. One minor cognitive obstacle encountered by the majority of students is that the picture gives the impression that the fixed points are where the curve $y = f(x)$ meets the line $y = x$ rather than the value of x for which $f(x) = x$. This was easily resolved.

Several students used language that intimated that the limiting cycles of length 2, 4, etc., represented the end result of the process of iteration, which allowed them to speak of the cycles as mental objects yielding a sequence of bifurcations as λ increased. This seems to be a perfectly natural process of *encapsulating the process of iteration* to give *visual objects that could be mentally manipulated*, without necessarily falling into the Sierpinska obstacle of equating what was onscreen precisely with the underlying potentially infinite mathematical processes.

Overall, two thirds of these students were able to use flexible links between numeric, graphic and symbolic representations of geometric convergence to construct their own ideas of the convergence of the points of bifurcation to the Feigenbaum point. Thus the claim by Dreyfus and Eisenberg that students prefer to think algebraically rather than geometrically should not be interpreted to mean that students *never* think geometrically. By giving students environments in which flexible thinking is encouraged, flexible thinking relating numeric, visual and symbolic representations can—and does—occur.

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Biography of Soo Duck Chae

Soo Duck Chae is a PhD student in the Mathematics Education Research Centre at the Institute of Education, University of Warwick, where she is working on the construction of conceptual knowledge using experimental mathematics under the supervision of David Tall. She worked as a researcher in the field of graphic programming at the Korea Institute of Defence Analysis and as a lecturer at the Korea Military Academy and Seowon University. She enjoys fine art, especially oriental painting and computer graphics.

Biography of David Tall

David Tall is Professor in Mathematical Thinking at the Mathematics Education Research Centre, University of Warwick. He has previously held posts in primary mathematics at Woodloes Middle School, secondary science at Wellingborough Grammar School, university mathematics at Sussex and Warwick Universities, and mathematics research at The Princeton Institute for Advanced Study. He has authored or co-authored over 30 books and manuals, and over 200 papers on mathematics education, with special interests in computers, visualisation, symbolisation and cognitive growth in mathematics from child to adult. He enjoys the music of Percy Grainger, *The Bill*, and Ardbeg single malt.