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Cognitive Development In Advanced Mathematics Using Technology

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This paper considers cognitive development in mathematics and its relationship with computer technology, with special emphasis on the use of visual imagery and symbols and the later shift to formal axiomatic theories. At each stage, empirical evidence is presented to show how these forms of thinking are enhanced, changed, or impeded by the use of technology.

Professional mathematicians vary enormously in their thinking processes, from those who are logically based—building from axioms in a coherent and steady way—to those who use a wide range of intuitions to suggest entirely new mathematical theories, with others who combine the two in various ways. There are also those who are theoretically based, developing mathematics for its intrinsic value and those who see it as a tool for use in a wide range of applications.

In the twentieth century pure mathematicians sought to base their work on a logical formal basis, the most famous being the work of the group of French mathematicians, corporately called *Bourbaki*:

> Logical analysis was central. A mathematician had to begin with solid first principles and deduce all the rest from them. The group stressed the primacy of mathematics among sciences, and also insisted upon a detachment from other sciences. Mathematics was mathematics – it could not be valued in terms of its applications to real physical phenomena. And above all, Bourbaki rejected the use of pictures. A mathematician could always be fooled by his visual apparatus. Geometry was untrustworthy. Mathematics should be pure, formal, austere. (Gleick, 1987, p. 89)

This attitude remains widely prevalent, but it has a serious weakness. It is one thing to produce the final version of mathematics in a form that is axiomatic and logical, however, it is another to invent that mathematics in the first place. In creating new mathematical ideas, the human brain operates in subtly different ways that need to be nurtured to work to their fullest extent.

The arrival of the computer has provided just such an environment in which experimental creative thinking may occur:

> When I came into this game, there was a total absence of intuition. One had to create an intuition from scratch. Intuition as it was trained by the usual tools—the hand, the pencil and the ruler—found these shapes quite monstrous and pathological. The old intuition was misleading. … I"ve trained my intuition to accept as obvious shapes which were initially rejected as absurd, and I find everyone else can do the same. (Mandelbrot, quoted in Gleick, 1987, p. 102)

We therefore see that the computer provides a quite different environment in which mathematicians and students may work. However, we must question what the environment can do for some individuals, if this can be done for all individuals, and whether different individuals are better off with different environments.

Recent work of Dehaene (1997) and subsequent announcements in the press reveal his evidence from brain scans that individuals doing arithmetic cover a spectrum from those who use only the language centre (including mathematical symbols) in the left brain whilst others combine this with the visual cortex to support their thinking. This counsels us that there are genuine differences between individuals, which need to be considered in using technology for providing an environment for learning. We cannot even be sure to what extent these differences are hard-wired and to what extent they may be altered by being carefully nurtured. Gaining insight into this phenomenon continues to be a major challenge for mathematics education in the future.

Sensori-motor and Visual Aspects of Mathematical Thinking

Some activities in mathematical thinking are explicitly seen as part of mathematics, such as using numeric, symbolic, and graphical methods to carry out computations or represent mathematical ideas and using axiomatic definitions and deductions to build up formal theories. There are other, deeper, human activities that act as a basis for all thought. The most primitive involve sensori-motor activity (physical sensations and bodily movement) and visual imagery.

These activities play an important part in the computer interface. For example, the sensori-motor system involves stored actions that allow decisions to be made intuitively using mouse and keyboard. These low-level cognitive actions also provide support for high-level theoretical concepts. Consider, for example, Figure 1 which shows software to build graphical solutions to (first order) differential equations by using the mouse to move a small line segment whose slope is determined by the differential equation. A click of the mouse deposits the segment and the user may fit line segments together to give an approximate solution.

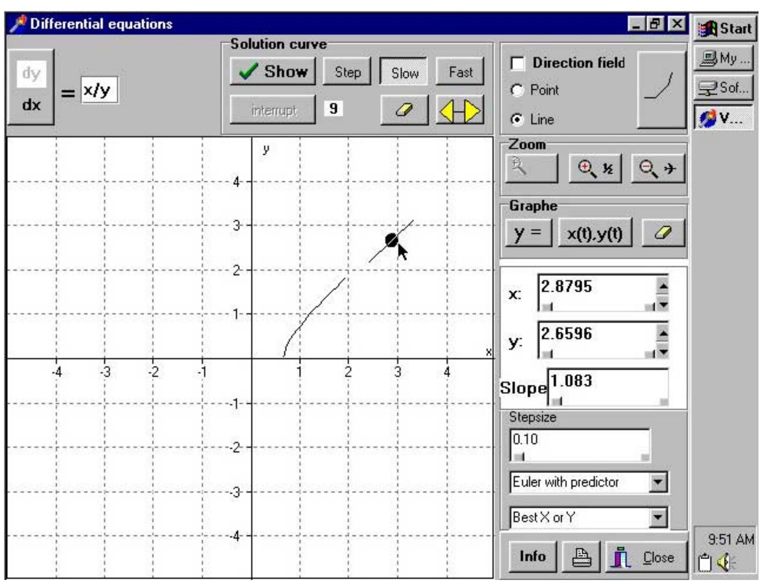

Figure 1. Enactive software to explore the solutions of differential equations (Blokland, Giessen, and Tall, 2000).

Such an activity can be performed intuitively with little knowledge of the theory of differential equations. Yet it already carries in it the seeds of powerful ideas about possible existence theorems (that a typical first order differential equation will have a unique solution through each point), and following the changing direction will build up a global solution curve. By considering selected examples, it is be possible to look at the wider view of what happens to a whole range of solution curves and to see their behaviour. In this way, an intuitive interface can provide advance organisers for formal theory, especially to those individuals who naturally build on visual imagery. This is essentially what happened in the modern study of non-linear differential equations and chaos theory. The drawings of fractals inspired and generated new mathematical theories.

Such examples show that our fundamental human senses of visuo-spatial perception can provide intuitions for more sophisticated theory. However, before can turn our attention to the formal theories of mathematics we need to consider what I term the "technical" aspects of advanced mathematical thinking, involving calculations and manipulations with numbers and symbols.

Numeric and Symbolic Aspects of Mathematical Thinking

In the nineteen-eighties, numeric calculation on computers became enhanced by symbolic manipulation. There was widespread belief that the computer could do away with all the unnecessary clutter of calculation and manipulation, allowing the individual to concentrate more on the essential ideas. In business, computers and calculators remove much of the tedium of calculation. An individual with no arithmetic skills beyond typing in numbers can enter the cost of items in a shop and the machine will give the total and even issue the correct change. Furthermore, entry of information can be simplified by scanning bar codes, and stock control can be handled by referring the items sold to the stock database to allow replacements to be ordered automatically. Soon even more of the economy will be taken over by such technology.

Yet the use of calculators and computers in mathematics has not always been as successful as it might be. In England, the use of calculators with young children has been discouraged in the hope that their absence will enable children to build mental arithmetic relationships. Perhaps this attitude has more to do with the misuse of the calculator (for performing calculations without having to think) than with any inherent defect in the apparatus itself. Used well—to encourage reflection on mathematical ideas—the calculator can be very beneficial, as Gray and Pitta (1997) showed in their work with a slow-learner having difficulties with arithmetic. By focusing on relationships, they were able to give their subject Emily a sense of number relationships where, before, she could not get beyond cumbersome counting procedures that filled the whole of her working mind.

On the other hand, Hunter, Monaghan, and Roper (1993) found that a third of their students could answer the following question before a course in which a computer algebra system was used, but not after:

What can you say about u if $u=v+3$, and $v=1$?

As they had no practice in substituting values into expressions during the course, the skill seems to have atrophied. As the old adage has it, "If you don't use it, you lose it".

Other evidence suggests that the use of symbol manipulators to reduce the burden of manipulation may just replace one routine paper-and-pencil algorithm with another even more meaningless sequence of keystrokes. Sun (1993), reported in Monaghan, Sun and Tall (1994), describes an experiment in which nine highly able 16 to 17 year old students taking a further mathematics course had unlimited access to the software *Derive*. When asked to find a limit such as $\lim_{x\to\infty}\frac{2x+3}{x+2}$, eight out of nine Derive students used the software procedure to produce the answer 2. They claimed they knew no other method, even though they had been shown the technique of dividing numerator and denominator by x:

$$\lim_{x\to\infty}\frac{2x+3}{x+2}=\lim_{x\to\infty}\frac{2+(3/x)}{1+(2/x)}=\frac{2+0}{1+0}=2.$$

In a comparable group of 19 students using only paper-and-pencil methods, 12 students used the above method, 3 substituted various numbers and 4 left it blank. This slender evidence indicates two things. Firstly, almost all the Derive students obtained the correct result, whereas a considerable minority of the others failed—showing the power of the system. Secondly, the Derive students had no alternative method available and appeared simply to be carrying out a sequence of button presses—showing the distinct possibility of using technology with a lack of conceptual insight.

The same phenomenon occurred when the students were asked to explain the meaning of $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Here, both the Derive group and one of the paper-and-pencil groups had participated in a discussion of the meaning of the notation. All of the non-Derive group gave a satisfactory theoretical explanation of the concept, but none of the Derive group could give any theoretical explanation. Four of the Derive group gave examples by replacing $f(x)$ with a polynomial and going through the sequence of key strokes to calculate the limit. Although the curriculum may include initial theoretical activities (and most introductions to calculus begin with some discussion of the limit concept), what students actually learn is dependent on the way they sense and interpret those activities. In the case of the students who used symbol manipulators, they had learned the button-pressing activities and carried them out as automatic routines.

Why is it—even though we feel we give students introductions to the idea of limit—that this concept seems so difficult for them to comprehend? To set this in context, we need to focus on the cognitive structure developed by students calculating with numbers and manipulating symbols. It is my contention that mathematical thinking uses one of the most powerful and natural constructions of the human mind—the ability to use symbols to switch between concepts and processes.

Procept Theory

The notion of *procept* was introduced by Gray & Tall (1991, 1994) to denote the use of symbol both as a *pro*cess (such as addition) and as a con*cept* (the sum). Suddenly the whole of arithmetic, algebra, and symbolic calculus was seen to be populated by procepts. The processes often begin as step-by-step procedures that are slowly routinised into processes that can be thought of as a whole without needing to carry them out. Symbols are used to allow the mind to pivot between the procedure or process on the one hand and the mental concept on the other. This spectrum of symbol usage can be described more precisely as follows:

- A *procedure* consists of a finite succession of actions and decisions built into a coherent sequence. It is seen essentially as a step-by-step activity, with each step triggering the next.
- The term *process* is used when the procedure is conceived as a whole and the focus is on input and output rather than the particular procedure used to carry out the process. A process may be achieved by n procedures ($n \geq 0$) and affords the possibility of selecting the most efficient solution in a given context. (We shall consider the enigmatic case $n = 0$ later.)
- A *procept* requires symbols to be conceived flexibly both as processes to do and concepts to think about. This flexibility allows more powerful mental manipulation and reflection to build new theories.

Development consists of increasingly sophisticated usage of symbols with differing qualities of flexibility and ability to think mathematically (Figure 2).

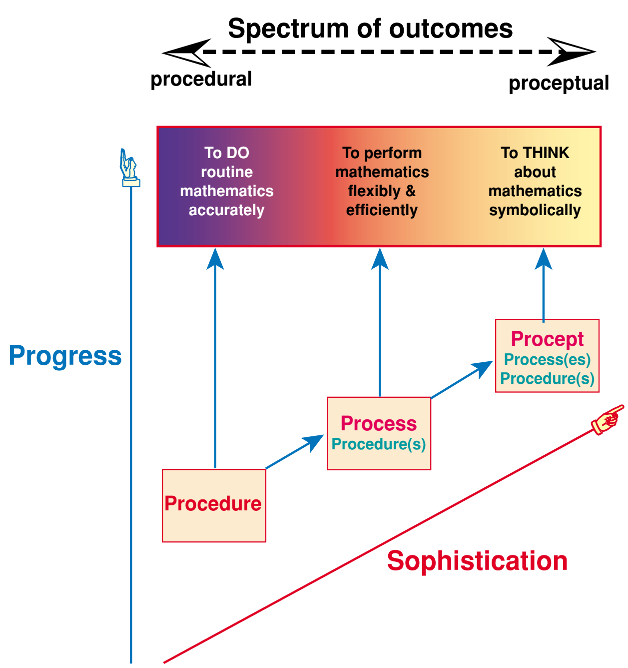

Figure 2. The development of proceptual thinking in mathematics (Tall, 1998).

Procedural thinking certainly has its value—indeed, much of the power of mathematics lies in its algorithmic procedures. However, a focus on procedures alone leads to increasing cognitive stress as the individual attempts to cope with more and more specific rules for specific contexts. Gray, Pitta, Pinto, and Tall (1999) give evidence to show that there is a continuing bifurcation in mathematical behaviour, with the less successful focusing more on procedures and the more successful students developing a hierarchy of sophistication through encapsulation of process into procepts.

The Power of Procepts

Human thinking capacity is enhanced by a range of activities unique to the human brain. One is the use of language, which allows the mind to conceive of hierarchies of concepts and focus at an appropriate level of generality to solve a given problem. Language also allows communication that shares the corporate knowledge of the species. Another major facility is the use of symbols in mathematics. The invention of the modern decimal number system and the development of algebra and calculus all build on a use of symbolism that specially fits the configuration of the human brain. The brain is a multi-processing system, with an enormous quantity of activity going on at the same time. The only way that the human brain makes sense of such pandemonium is to focus attention on things that matter and suppress other detail from conscious thought (Crick, 1994, pp. 63-64). To reduce the strain of cognitive processing, some activities become routinised so that they can be performed largely without conscious attention, although they may be brought to the fore when necessary.

To link the comparatively small, but dynamic, focus of attention to other brain activity requires two essential conditions. One is that there is a part of the activity sufficiently compressed to be held, perhaps as a token, in the focus of attention; the other is that this part has rich and immediate links with important related ideas. The notion of procept has precisely these properties. The symbol, as a token for the procept, can be used to focus either on the compressed concepts and their relationships, or attention can easily switch to the process aspects of the activity—preferably using routinised procedures operating in efficient ways that place minimal cognitive strain on overall brain function.

The phenomenal explosion of science and technology in recent centuries has owed much of its power to the construction of mathematical models followed by the use of mathematical techniques to compute and predict what is going to happen. I believe that the formulation of the notion of procept allows us to describe how this happens and that this notion is as fundamental to the cognitive psychology of symbol manipulation as the notions of set or function are foundational to mathematics itself. Each of these constructs—set, function, procept—has certain properties in common. Firstly, they are all three essentially simple concepts, underlying the theory in such a profound way that, paradoxically, although they are eventually seen as foundational to the theory, they have not been formulated explicitly until long into the practical development of the theory. For instance, although the concept of set may now be seen as being more fundamental than that of a function, it was not formally defined until long after. It is the classic *tool-object* dialectic (Douady, 1986): a concept is used as a tool for a considerable time until it takes on sufficient importance to be studied as an object in its own right.

When Eddie Gray and I first formulated the notion of procept, it occurred in a context where we were talking together about children doing arithmetic. We knew the theory of process-object encapsulation (or reification) that had arisen in the theories of Piaget , Dienes, Davis, Dubinsky, Sfard, and others (Tall, D. O., Thomas, M. O. J., Davis, G., Gray, E. M., & Simpson, A. (2000).). But we found that an addition such as 3 + 2 could equally well be conceived by children as the sum 3 + 2 and we had no word to describe the duality of symbol use. The word "procept" was suggested by one of us and the other quickly saw its power in terms of duality (as process and object), ambiguity (as process or object) and flexibility (to move easily from one to the other).

Fundamental Differences in Proceptual Structure

Subsequent developments have enabled us to formulate not just the way procepts enable the human brain to formulate and solve problems, but also the ways in which proceptual thinking works in one context (say whole number arithmetic) and is cognitively different in other contexts (say fractions, powers, algebra, limits). As a result, we can now formulate a broad theoretical overview of student learning difficulties that were previously seen as being specific to certain areas. To illustrate, let us look at the different ways procepts operate in arithmetic, algebra, and limits:

- Arithmetic procepts, such as 5 + 4, 3 ×4, $\frac{1}{2}+\frac{3}{5}$, and 1·54 ÷ 2·3, have computational processes and manipulable concepts.
- Algebraic procepts, such as 2 + 3x and $ax^2 + bx + c$, have potential processes (evaluation) and manipulable concepts. For instance, $a(b + 3)$ can be expanded to $ab + 3b$ and the latter factored back to $a(b + 3)$.
- Limit procepts, such as $\lim_{x \to a} \frac{x^3 - a^3}{x - a}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, have potentially infinite processes (of evaluation) and have some manipulable concepts (using rules or theorems on limits). For instance, $\sum a_n + \sum b_n = \sum (a_n + b_n)$. Except in a few special cases (for instance, the geometric series), limit procepts usually have no finite procedures of computation, although they may have algorithms to give approximations to the limit. A limit is therefore an example of a process which requires n algorithms to compute the concept, where $n = 0$.

Different kinds of procept bring different cognitive challenges. Experiences with arithmetic symbols, with built-in processes of computation, can lead students to find algebraic expressions difficult to contemplate; they find themselves being asked to manipulate symbols that represent processes which cannot themselves be evaluated (unless the numerical values of the variables are known). Limit procepts have a potentially infinite process of evaluation and carry a sense of "getting close", "getting large", or "getting small"; this can lead either to a sense of a limit never being reached or to the conception of a number line with infinitesimal and infinite quantities. We briefly consider the cognitive challenges presented by each of the three types of procepts.

Proceptual Problems in Arithmetic

Arithmetic has a fundamental proceptual structure in which the symbol for a whole number has a dual role as number concept and counting process (Gray & Tall, 1994). Addition is successively compressed from the triple counting processes of count-all (count one number, count the other, count the two together) through the transition of the double counting process (count one number then count on by the second) to the single count-on process (starting from the first number and count on by the second). Some facts are remembered (so-called "known-facts") and may be used flexibly to derive new ideas from those that are known (for instance, deriving 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14 or 23 + 5 = 20 + 3 + 5 = 28). Gray, Tall, Pitta, and Pinto (1999) show how there is a bifurcation in strategy between those who cling to the counting operations on objects and those who build (to a greater or lesser extent) a conceptual hierarchy of manipulable relationships between process and concept—a technique which Gray and Tall (1994) called *proceptual thinking*.

Proceptual Problems in Algebra

Long before the notion of procept was formulated, students were noted to have difficulty conceiving an expression such as 7+x as the solution to a problem. This to a problem was described as *lack of closure* (Collis, 1972). Davis, Jockusch, and McKnight (1978) remarked similarly that "this is one of the hardest things for some seventh-graders to cope with; they commonly say, 'But how can I add 7 to x, when I don"t know what x is?'" Matz (1980) commented that, in order to work with algebraic expressions, children must "relax arithmetic expectations about well-formed answers, namely that an answer is a number". Kieran (1981) similarly commented on some children's inability to "hold unevaluated operations in suspension". All of these can now be described as the problem of manipulating symbols that—for the students—represent potential processes (or specific procedures) that they cannot carry out, yet are expected to treat as manipulable entities. Essentially they see expressions as unencapsulated processes rather than manipulable procepts.

To address this problem, Tall and Thomas (1991) designed a course on programming expressions in BASIC and evaluating them numerically, showing, for example, that an expression such as A + 3 means "whatever A is, plus 3". Thus, if A = 2, the BASIC expression PRINT A + 3 yields the response 5. In this way, they were able to give a suspended meaning to an expression such as A + 3. They were also able to show the equivalence of expressions in the sense that, whatever the value of A is, the expressions 2*(A + 1) and 2*A + 2 always give the same answer.

This idea that different expressions could essentially be the same is, of course, a step in the compression from procedure to process and perhaps on to procept. This compression process has been beautifully demonstrated by DeMarois (1998), who worked with some students who were using graphing calculators in a remedial college pre-algebra course. The students were asked to consider the two functions given in Figure 3, to write the outputs of these two function boxes, and to say if they are the same function.

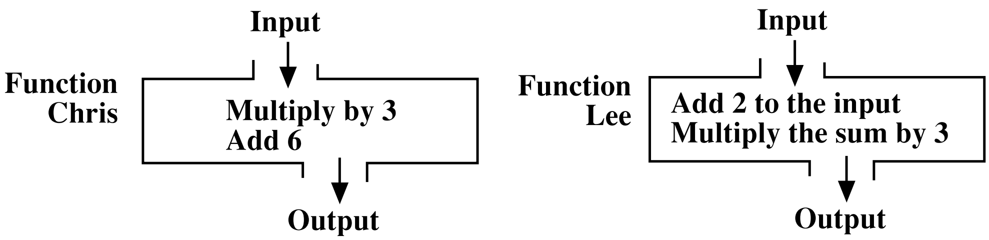

Figure 3. Two functions having different internal procedures but the same input-output relationship (DeMarois, 1998).

He selected three students as random representatives of those attaining the three grades, A, B, C. Their responses were as follows:

Grade A Student: $3x + 6$, $3(x + 2)$. Yes, if I distribute the 3 in Lee, I get the same function as Chris.

Grade B Student: $x3 + 6$, $(x + 2)3$. Yes, but different procedures.

Grade C Student: $3x + 6$, $x + 2$ (3×). No, you come up with the same answer, but they are different processes. (DeMarois, 1998).

The Grade A student responded in a manner which shows that she could manipulate the expressions and see that one is the same as the other—a procept interpretation. Students B and C wrote the algebra in idiosyncratic ways; B followed the precise verbal order, but C used the algebraic order for $3x + 6$ yet wrote the second expression in a way which can be read as "$x + 2$ (three times)". Although B and C used the terms procedure and process, respectively, B's response suggests a process interpretation and C's suggests a procedural interpretation.

More recently, Crowley (2000) has analysed the way in which remedial students study straight-line equations in a college course where they can use graphic calculators. Of those who succeeded, some continued to be successful in the following pre-calculus course. Crowley writes about these students:

> [They] had readily accessible links to alternative procedures and checking mechanisms [and] had tight links between graphic and symbolic representations. [They succeeded even though they] made a few execution errors. (p. 208)

Students who succeeded in the course, but had serious difficulties with the next, already showed different ways of working in the first course:

> They had links to procedures, but did not have access to alternate procedures when those broke down. They did not have routine, automatic links to checking mechanisms. They did not link graphical and symbolic representations unless instructed to do so. … They showed no evidence that they had compressed mathematical ideas into procepts. (p. 209)

Proceptual Problems in the Concept of Limit

It has long been known that students have difficulty coming to terms with the limit concept typically presented at the beginning of a calculus course. The early research on this topic is summarised in Cornu (1991). The conceptual difficulties can be clearly formulated using the notion of procept. At first a limit (say of a sequence of numbers) is seen as a process of obtaining a better and better approximation to the limit value. This process of getting close to but never reaching the limit gives rise to the mental image of a variable quantity that is arbitrarily small or arbitrarily close to a fixed quantity. This may then lead to the construction of a mental object that is infinitely small—a cognitive infinitesimal. Monaghan (1986) called such a conception of limit a *generic limit*: The limit object has the same properties as the objects in the sequence which is converging to it. For example, in the sequence $(1/n)$, all the terms are positive, so the generic limit is positive; it is also arbitrarily small. This conceptualisation leads to a concept image of the number line that contains infinitesimal quantities and is therefore at variance with the formal definition of the real numbers.

Symbol manipulators use representations of various kinds of numbers which are familiar to students, including integers, rationals, finite decimals, radicals such as $\sqrt{2}$ and $\sqrt[10]{7}$, and special mathematical numbers such as π, e. Students may have mental images of both repeating and non-repeating infinite decimals, but these can often represent "improper numbers which go on forever" (Monaghan, 1986). Such improper numbers are again viewed as potentially infinite processes rather than as number concepts. Students at this stage do not necessarily have a coherent view of the number line, which (to them) is populated by a range of different kind of creatures, some familiar, some less familiar, and some downright peculiar.

Failure of Some Computer Approaches to Conceptualize the Limit Procept

The number system used on computers is at variance both with the formal real number system and also with the cognitive number system of students who conceptualise infinitesimal elements. This fact, and various idiosyncrasies of computer languages, can have unforeseen conceptual implications that need to be considered in using computers in learning environments.

For over fifteen years, my department taught a course in BASIC programming designed to give students intuitions to form a basis for the later formalities of mathematical analysis, including the limit concept. Li & Tall (1993) investigated this assumption and found it flawed. A significant problem was that the computers (Acorn RISC machines), although relatively fast, took a considerable time to compute some of the exotic functions the students programmed. For instance, the students were asked to program the sum of N terms of a series where the nth term was defined to be $1/n^2$ if n is prime, $1/n^3$ if n is not prime and even, and $1/n!$ if n is not prime and odd. The procedure to check whether a number is prime must be performed for every term and takes longer as the number grows larger. It is almost instantaneous for $N = 10$ or $N = 100$, but it takes far longer for $N = 1\ 000$ or $N = 10\ 000$, the latter taking almost an hour on the Acorn. A side-effect of this phenomenon is that it reinforces the students' sense that the process goes on and on without end. The majority willingly accepted that the sequence was bounded above, but, when a vote was taken, most of the class believed that the series would not tend to a limit, because, no matter what value was considered below the upper bound, the sequence might eventually creep over it. These students refused to accept the completeness axiom for the real numbers. Their argument was: "How can we accept a fundamental axiom that we know not to be true?"

The reader is almost certainly willing to accept the completeness axiom. However, this does not mean that the students were wrong. Although there is only one complete ordered field—the field of real numbers—there are many, many other ordered fields. In particular, there are many ordered fields which include the real numbers and have infinitesimal and infinite elements. Even though students' imagery for the number line need not be expressible coherently as an ordered field, when their number systems include the possibility of infinitesimals, these systems cannot be complete. Essentially the students were being asked to accept the axioms for a system (complete ordered field) for which they had no mental model.

The course was designed to give practical experience of summing series, in order to give intuitions for the formal ε-N definition of convergence. The method was to set the numbers to be given to a certain number of decimal places (say four) and then find for what value of N successive terms became constant to this accuracy. Clearly, if more decimal places were required (say six), then more terms would need to be computed to obtain this greater accuracy. This experience of seeking an N to give a desired accuracy was then discussed in class in relation to the formal ε-N definition.

There were some growth in the students' conceptions. For instance, the students were asked to complete the following sentences:

- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ tends to___________
- The limit of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is ___________

They responded in the manner described in Table 1.

Table 1
Responses to "Tends to" and "Limit" Items on the Convergence of a Sequence

Test	Responses to "tends to" / "limit" items						
	0 / 0	$0 / \frac{1}{\infty}$	$\frac{1}{\infty} / \frac{1}{\infty}$	0 / ?	2 / 2	0 / 2	0 / 1
Pre-test (N = 25)	0	11	1	5	0	2	2
Post-test (N = 23)	8	3	3	0	4	0	2

Interviews revealed that all of these responses have natural explanations: 2 is the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\ldots$, 1 is the largest term (just as a speed limit is the maximum allowable speed), and $\frac{1}{\infty}$ is the generic limit ($\frac{1}{n}$ with n is infinite). The most common response on the pre-test suggests a conceptualisation in which the sequence tends to 0 (which also happens to be a limiting value in the sense of being a lower bound) but the limit is the arbitrarily small value $\frac{1}{\infty}$. There was some positive change from pre-test to post-test in that the most common response was now the standard response. However, the teacher involved (myself) could hardly call this a real success.

A full session was dedicated to discussing how an infinite decimal can be seen as the limit of the sequence of finite decimal approximations. Specific attention was given to explaining why the infinite decimal $0\cdot\dot{9}$ (nought point nine repeating) is the limit of the sequence ($1-1/10^n$), and is therefore 1. Despite this discussion, the meaning of $0\cdot\dot{9}$ remained essentially unchanged two weeks later (see Table 2).

Table 2
Distribution of Responses to the Question "Is $0\cdot\dot{9}$ *Equal to 1?"*

Test	Yes	No	?	No response
Pre-test (N = 25)	2	21	1	1
Post-test (N = 23)	2	21	0	0

Interviews revealed that students continued to conceive of $0\cdot\dot{9}$ as a sequence of numbers getting closer and closer to 1 and not a fixed value, because "you haven't specified how many places there are" or "it is the nearest possible decimal below 1". The programming experiences did not change this view; the limit object could not be constructed *exactly* in this environment, so old process ideas remained without the limit becoming a mental object.

The responses to two other questions gave further insight:

- Can you add 0.1 + 0.01 + 0.001 + ... and go on forever to get an exact answer?
- 1/9 = 0.1 repeating. Is 1/9 equal to 0.1 + 0.01 + 0.001 + ...?

The favoured response on both pre-test and post-test was "No" to the first question and "Yes" to the second (see Table 3).

Table 3
Distribution of Responses to Two Items on the Series 0.1 + 0.01 + 0.001 + ...

Test (N)	Response combination						
	Y / Y	Y / N	N / N	N / Y	N / ?	? / N	- / Y
Pre-test (25)	4	0	1	18	0	1	1
Post-test (23)	2	2	2	14	1	0	0

These data indicate that the majority regarded 0.1 + 0.01 + 0.001 + ... = 1/9 to be false but 1/9 = 0.1 + 0.01 + 0.001 + ... to be true. Reading from left to right, the first statement seems to represent a potentially infinite process which can never be completed, whereas the second shows how 1/9 can be divided out to get as many terms as are required. Interviews suggested shades of meaning often consistent with this view, with students again seeing the expression 0.1 + 0.01 + 0.001 + ... as a process, not a value.

We thus see that the underlying beliefs of students reveal a widespread variety of concept images of the nature of number and limit that are only partially solved by attempts at teaching. Other research confirms this viewpoint (e.g., Williams, 1992).

One of the major underlying reasons for students' unwillingness to accept a definition in axiomatic mathematics is that, up to this point, the student was used to having definitions that describe already existing concepts that are familiar to them. They are not, at this stage, willing to accept an axiom that does not correspond to something that they already know. The problem with using the completeness axiom as a basis for the theory of convergence is that it does not agree with students' previous experience. How can such an axiom be the basis of a theory, if it is clearly untrue?

The Principle of Selective Construction

To reduce the cognitive strain on the learner in a mathematics curriculum using computers, I formulated the *Principle of Selective Construction* (Tall, 1993a). This principle leads to the design of software that allows the learner to focus on part of the theory whilst the computer invisibly carries out the underlying processes. This aim can be accomplished in a range of ways. For instance, the approach to differential equations advocated earlier (Figure 1) focuses on providing a manipulable visual representation in which the computer carries out the algorithms and draws pictures while the student focuses on getting a global sense of the theory. At another time, the student might implement the algorithms using a simple computer program in order to focus on the details of a numeric approach, or on a study of available symbolic methods for solution.

The principle is implicitly used in Gray and Pitta's (1997) work with Emily. The software performs the calculation and Emily can concentrate on the relationships without having to carry out the intervening counting processes which were an essential part of her personal thinking processes. This imaginative use of the calculator shows the poverty of the UK numeracy strategy to discourage the use of calculators for young children. It is not the technology that is at fault, but the use to which it is often put.

Another use of the principle of selective construction arises in Michael Thomas's approach to the introduction of algebra (Thomas, 1988; Tall & Thomas, 1991). In one activity, the pupils enact a physical game in which variables are represented by squares drawn on a sheet of card which can be labelled with a letter and a number stored inside. A typical instruction is to put a = 2 (place a number 2 in the box marked "a") and then PRINT a + 3 (find the number in the box marked "a" and print out the sum of it and 3). This approach concentrates on the process aspect of evaluation of expressions. On the other hand, writing simple programs, such as "INPUT a: PRINT 2*(a+1), 2*a+2", gets the computer to carry out the process of calculation so that the student can concentrate on the equivalence of the same outputs from different procedures of calculation. Focussing separately on symbol as process and on equivalent concepts provided a long-term improvement in conceptualisation of expressions as manipulable concepts.

It is my contention, however, that, once the student has made sense of one particular aspect in a stable way, it should be refined and related to other aspects to give a more stable global schema. In the School Mathematics 16–19 project (1991), my method of seeing the gradient of a graph in terms of local straightness (Tall, 1985) was implemented to allow students to look at graphs such as $y=x^2$ and $y=\sin x$ and to see the changing gradient built up on a computer screen. However, because the students were considered weak at symbol manipulation, $y=x^3$ and more general polynomials were done just by looking at the graphs using the software, failing to drive home the relationship between visualisation and symbolism at an early stage.

Technical and Formal Aspects of Mathematics Using Computers

So far we have considered numeric, symbolic, and visual aspects of mathematics. As the experience of the student grows, the time may come to move on to the more formal aspects of mathematical proof. The observation above, that some students are unwilling to accept definitions as axiomatic unless they agree with their current experience, shows that there is a distinct bridge to make between observed properties that are described and assumed properties that are defined is part of the problem.

There is also a fundamental difference between the view of proof, as seen from an expert viewpoint, and the developing cognitive awareness of proof and its ramifications as seen by the growing student. A preliminary model of cognitive growth (see Figure 4) includes the usual three major representations afforded by the computer—numeric, symbolic, and graphic—which build on the individual's cognitive tools of sensori-motor activity, visual imagery, and verbal expression and communication. Also included in the diagram is a severe cognitive gap between an appreciation of the technical aspects of mathematics which build on visual and proceptual abilities and the formal aspects based on axiomatic definition and proof.

Skemp (1971) underlined the difference between what he termed the *expansion* of cognitive structure by taking in new ideas which require little modification and the *reconstruction* of old ideas which becomes necessary to fit with new experiences that clash with previous experience. The latter includes the difficult transition to axiomatic theories where concept definitions are used as axiomatic starting points to construct mathematical theories. As students grow from what I term the technical aspects of mathematics to the formal aspects, there is a huge cognitive reconstruction required which can cause such difficulties that few students make the transition (Figure 5). Technical mathematics, on the other hand, in the main extends the use of symbols and visual elements found in more elementary mathematics. There is some cognitive reconstruction involved, to be sure, as we have already seen occurs between arithmetic, algebra and limits. However, there is ample evidence that students who are unable to cope with the lesser reconstructions necessary for technical aspects often find that the transition to formal proof presents insurmountable difficulties.

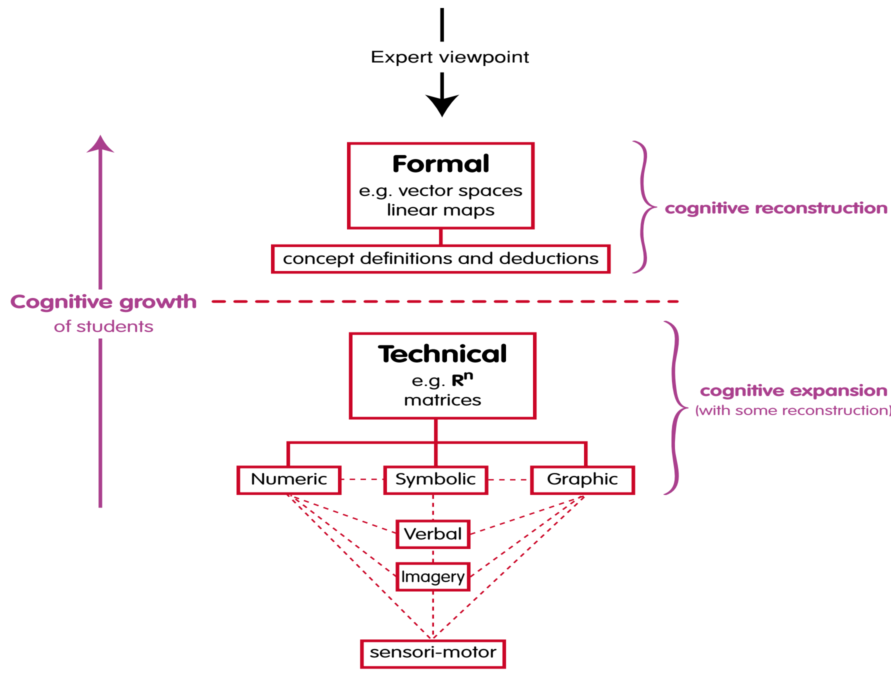

Figure 4. Cognitive growth in mathematics: A preliminary model.

Figure 5 shows a revision of the preliminary model to allow for the fact that not all advanced users of computers need aim for the formal aspects of mathematics. For many users of mathematics, the technical aspects are enough to support their needs. Their concept imagery—built up from their mental structure and experiences, from the sensori-motor through imagery and language on to the proceptual and visual aspects—provide a great deal of insight which enables them to use the computer imaginatively. There may very well be subtle misconceptions in their concept imagery, but this may not affect the technical usage for which the computer is being used. Few (if any) individuals have a range of concept imagery about numbers that is totally coherent. Think, for example, of the standard notion that the real line consists of a set of points. If the points have zero size, how can it be possible that even an uncountable number of them fill up the line? Mathematicians solve this problem by not thinking about it. They focus on the axioms of a complete ordered field and make deductions from them without concerning themselves with the question of visual representation. After all, as Bourbaki asserted in the quote on the first page of this paper, "a mathematician can always be fooled by his visual apparatus".

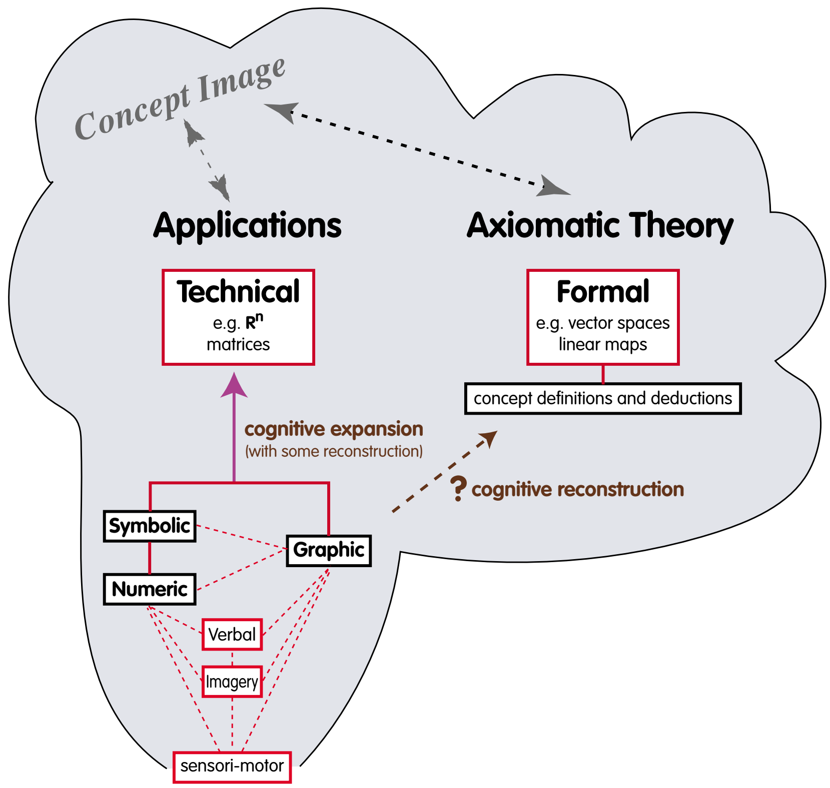

Figure 5. Cognitive expansion and reconstruction in applications and axiomatics.

However, as Pinto (1998) has found, some students build formal theories from the axioms by formal deduction and others are successful by reconstructing their concept imagery to give images that support the concept definitions and their deductions. Both routes can be successful, although many students remain so firmly fixed in their perception and experience that they never achieve the transition to formal mathematical thinking. It may therefore be appropriate for some students to be supported by visual images and others to concentrate more on formal deduction. It would also be wise to realise that that there are many who benefit from technical aspects of computer usage but do not gain a great deal from studying formal proof.

Computer Experimentation and Proof

Computer environments are particularly valuable in encouraging experimentation which helps, before any formal theory is developed, to give a sense of a given phenomenon and suggest what kind of properties are involved. A good example is the experimentation of Feigenbaum in iterating the logistic equation $f(x) = \lambda x(1-x)$ for various values of λ (Gleick, 1987). Starting at an initial population x_1 and calculating $x_{n+1} = \lambda x_n(1 - x_n)$ for $n = 1, 2, \ldots$ gives successive population values $x_1, x_2, \ldots$. Starting with $0 < x_1 < 1$, the sequence of successive populations settles down into a pattern of convergence that does not depend on the starting point x_1, but produces radically different behaviours as λ increases. Feigenbaum used an HP-65 calculator to iterate the equation and track the behaviour of the iterated population.

For $0 < \lambda < 1$, the population diminishes to zero. For $1 < \lambda < 3$, the population stabilizes on a value depending on λ. For λ just above 3, the odd terms tend to a limit k_2 and the even terms to k_2 with the population settling down close to a limit cycle $\{k_1, k_2\}$ where $f(k_1) = k_2$ and $f(k_2) = k_1$. As λ increases, there occur a succession of values $\lambda_1 = 3$, $\lambda_2 \approx 3.4495, \ldots , \lambda_n, \ldots$ at which the limit cycle doubles in length. Between λ_1 and λ_2 the population oscillates between 2 values, between $\lambda_2$andλ_3 the population cycles between 4 values, and so on, the number of values doubling each time.

Feigenbaum noticed that the differences $d_n = \lambda_n - \lambda_{n-1}$ between successive terms were such that d_n / d_{n+1} tended to a limit approximately equal to 4.669. Thus the sequence $\lambda_1, \lambda_2, \ldots , \lambda_n, \ldots$ converged in a manner comparable to the sum of a geometric series and was therefore convergent.

When Feigenbaum later did the same kind of calculations iterating the function $f(x) = \sin x$, he found that the same constant appeared again. Other examples suggested an underlying universality that was common for all such iterations. At first he could not explain what was happening, but then:

> When inspiration came, it was in the form of a picture, a mental image of two small wavy forms and one big one. That was all—a bright, sharp image etched in his mind, no more, perhaps, than the visible top of a vast iceberg of mental processing that had taken place below the waterline of consciousness. It had to do with scaling, and it gave Feigenbaum the path that he needed. (Gleick, 1987, p. 175)

Feigenbaum used his own visual imagination to give meaning to the underlying numerical behaviour he had discovered. But he was unable to provide a formal mathematical proof. He remained on the *technical* side of advanced mathematical thinking, supported by visual imagery. His articles were rejected by several journals. He failed to satisfy the Bourbaki demands of rigour. Yet he had amazing insight which led to a whole new branch of mathematics. Only later was a mathematical proof formulated (Lanford, 1982) and even this depended on numerical calculations that could not be performed unaided without a computer.

Intuitive Bases for Formal Ideas

Feigenbaum's example shows that new ideas may require several different types of mind to see the subtle inner truths—empirical exploration supported by computer carried out by an individual with rich intuitions, then quite separate mathematical verification and proof by a pure mathematician. It underlines the fact that computer exploration has a role that is important in its own right and that some individuals may benefit from it to gain insight into mathematical ideas.

Chae & Tall (in press) observed a course in which able students began with an initial exposure to theoretical aspects of period doubling and then moved on to computer experimentation to gain further insight. For instance, they calculated the fixed point of the logistic function $f(x) = \lambda x(1-x)$ for $0 < \lambda < 3$ theoretically and studied the behaviour of cycles of length 2 that occurred for values of λ just larger than 3. In particular, they focused on the idea that if $\{k_1, k_2\}$ is a cycle of length 2 for $f(x)$, then k_1 and k_2 are fixed points for $f^2(x) = f(f(x))$ (because $f(k_1) = k_2$ and $f(k_2) = k_1$, so $f^2(k_1) = f(f(k_1)) = k_1$ and $f^2(k_2) = k_2$). This relates the bifurcation of f from two to four to a corresponding bifurcation of f^2 from one to two (see Figure 6).

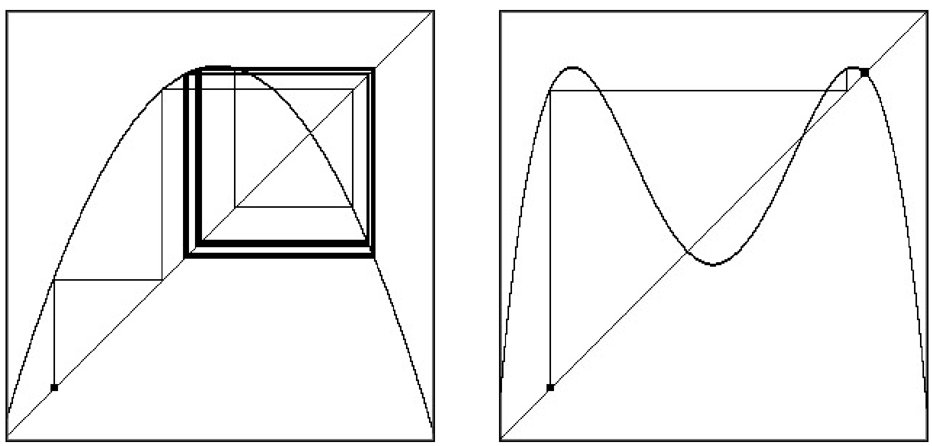

Figure 6. The bifurcation from period 2 to period 4 for f (left picture) corresponds to a bifurcation from period 1 (fixed point) to period 2 for f^2 (right picture).

In this way, students were given new insight into successive period doublings. They were then invited to experiment and attempt to observe as accurately as possible where the period doublings occur. They were then asked to calculate successive ratios to see if the values were consistent with geometric convergence. Eight out of the 12 students involved were able to relate numeric, algebraic, and visual aspects of the theory in a flexible manner. The other 4 had various difficulties, including the problem that the visual display did not give a very accurate picture of what was going on.

This example exposes a significant problem with the use of the computer (with pictures made of discrete pixels and numeric calculations having an inherent inaccuracy) in setting up intuitions for formal mathematics. The demands of Bourbaki are great and many individuals (including Feigenbaum) may gain informal insight into problems without ever being able to formulate the theory in formal terms.

Informal Insight into Formal Theories

Given that a full formal theory is vital for some, but less feasible for others, I designed an experimental analysis course using software to explore more subtle theory in an intuitive way. This included using intuitive approaches to formal definitions, such as the idea that a real function is differentiable if it is “locally straight” and it is continuous if, when pulled horizontally, a small portion “pulls out flat”. This approach is fundamentally more embodied and less technical than the approaches taken in reform college calculus texts (Tall, 1993b) where, for instance, local linearity is not used at all. The notion of local linearity introduces the symbolic form of the best linear approximation to the graph. It deals first with the gradient at a point and then generalises to the gradient function at every point. My approach simply zooms in on a curve, maintaining the same scales for the x and y axes, to see how steep the graph is. The global gradient function is introduced by encouraging the student to look at the changing gradient of the curve in a global sense, long before the local notion of tangent is considered.

Most college calculus courses soon turn to procedures for symbol manipulation and quickly focus on graphs of functions given by formulae. My approach looks more carefully at the underlying theory in an intuitive sense before proceeding. This employs a graph-plotter with a number of special facilities (Tall, Blokland, & Kok, 1990, Blokland, Giessen, & Tall, 2000). One is the inclusion of functions such as the “blancmange function” $bl(x)$ (Tall, 1982) which is everywhere continuous but is so wrinkled that it is nowhere differentiable. We then define the “nasty function” $n(x)=bl(1000x)/1000$; it is a tiny function (less than 1/1000 high) which is hardly visible at normal magnifications yet is nowhere differentiable. If $f(x)$ is a differentiable function, the graph of $f(x) + n(x)$ looks virtually the same as $f(x)$ at normal magnifications yet is nowhere differentiable. Just by looking at a graph, you cannot see if it is differentiable or not! You need either to be able to zoom in at higher and higher magnifications to see the wrinkles, or, better still, you need an explicit way of calculating the function to any desired accuracy.

The software also has functions to calculate numerical gradients and numerical areas. For instance, the symbol *area*(*expression*, a, b, h) calculates the area under the graph of the function given by *expression* from a to b with step size h (using the mid-ordinate rule, adjusting the width of the last strip to be less than or equal to h). As an example, *area*(sin$x$, 0, x, 0.1) calculates the area under sin$x$ from 0 to x with step 0.1. A function notation allows us to define new functions; for example, $A(x)$ = *area*($bl(x)$, 0, x, 0.01) defines the (mid-ordinate) area under the blancmange function from 0 to x with step 0.01.

In my course, a week after the students were shown an outline proof that the area function $A(x)$ can always be calculated for a continuous function $f(x)$ and satisfies $A'(x)=f(x)$, they were introduced to the graph of the area under the continuous blancmange function. One of the students spontaneously exclaimed, “Oh, so the area function is differentiable once but not twice!” There was a class discussion between the students at this point and the other students expressed themselves happy with this idea. At the time, I was extremely amazed by the fluent discussion of a concept that most students following a traditional course would never be able to envisage. Which traditional student would have an experience that allowed them spontaneously to imagine and talk about a function that is differentiable once but not twice?

The students were then introduced to the *pièce de resistance* of the software: a routine to draw a model of a function that takes one value on the rationals and another on the irrationals. (The technicalities are given in Tall, 1993b). The approximate area was then calculated under the graph of the following function:

$y = 0$ if x is irrational and $y = 1$ if x is rational

This time, given the step h, the rule used takes a random step smaller than h, then a random point in each interval to calculate the Riemann sum. When the area function $A(x)$ was drawn for this function, it was a good approximation to $A(x) = x$. Clearly, as the algorithm chose a random value of the original function, it was more likely to choose a value for which y is 1 than a value for which y is 0. A student suggested this was because "most random numbers must be irrational". This led to a discussion of the cardinality of the rationals and irrationals, in terms of talking informally about how likely it would be that a randomly generated decimal would repeat. Thus intuitive images were motivated which intimated subtle properties of more sophisticated theories (in this case Lebesgue integration), all done using language related to a carefully prepared technological environment.

None of these discussions led to formal proofs, but at the end of the course the students astonished the external examiner with their ability to visualise and discuss concepts of analysis that he found difficult to imagine. Regrettably, the course was new and experimental and it was not possible for me to repeat it under more controlled and properly observed conditions. However, the way in which the students responded to a new way of looking at mathematical analysis remains impressed in my memory.

The moral of the story is that it is possible, with a well-designed microworld, to build an environment for exploring highly subtle theories in an informal way. However, I do not see the computer microworld as the sole agent in facilitating student exploration and peer discussion. The role of the teacher as mentor is vital—to draw out ideas from students and to encourage them to express verbally what they see occurring visually. I take this approach not to control the students' learning but to enhance it, by focusing on big ideas that are consonant with the formal theory of mathematicians. I do not produce Bourbakis, but I do encourage students to think for themselves in a subtle and insightful way.

Concluding Reflections

In this paper, we have seen how, in mathematical thinking, sensori-motor and visual insight can grow to complement numeric and algebraic symbolism. We have seen the power of symbols as procepts but also the subtleties within their usage that can cause various levels of mathematical misunderstandings. Cognitive development, through the operational procepts of arithmetic via the potential procepts of algebra and the potentially infinite limit procepts in the calculus, leads inevitably to epistemological obstacles, which some may cope with by mental reflection but others may find impossibly difficult to grasp.

We have seen that much of the current use of computer software offers an environment where the computer provides internal calculation and manipulation, presenting the results of these algorithms in various numeric, symbolic, and graphic representations. This has a technical focus that can provide a powerful environment for doing mathematics and, with suitable guidance, to gain conceptual insight into mathematical ideas.

We can use new technologies in imaginative ways that were previously impossible to contemplate. However, if we are to use technology in teaching mathematical concepts, we need to observe carefully what it is that students actually learn during the process. The evidence is that they learn by building up mental images in ways that are consistent with what they do and what they observe whilst using the technology. The experience can have insightful aspects that support the theory, but it can also lead to a variety of other mental imagery that may differ from the mathematical ideals currently held by experts.

We are already seeing that the availability of technology is changing the nature of advanced mathematical thinking. The experimental explorations of a Feigenbaum that were rejected a generation ago are now seen as a positive way of advancing mathematical knowledge. Using computer technology and the principle of selective construction, the sense of mathematics can be explored before the formalism is developed. It is a way of thinking which students—supported by a guiding mentor—can share.

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