TECHNOLOGY AND VERSATILE THINKING IN MATHEMATICS

David Tall
University of Warwick
<david.tall@warwick.ac.uk>

Today's technology gives us a great opportunity to complement the subtlety of human thought with the power and accuracy of modern computers. In this presentation I consider fundamental modes of human thinking to see how enactive, visual and symbolic methods can be used in a versatile way with the support of well-designed software. My analysis focuses on the use of symbols to think about mathematics and to do mathematical procedures and how visual enactive software can be used to enhance our conceptual thinking processes. In particular, I consider a theoretical cognitive development in arithmetic, algebra and the calculus and reflect on empirical research to show how the computer can be used both well and badly in supporting mathematical learning.

Introduction

The turn of a century is a time for looking forward and reflecting on the past. Given this is the turn of the millennium, perhaps we should begin by looking back long into our previous history. We stand now at a time of great innovation in technology, moving on at an exciting, even bewildering pace. But as we do so, we should reflect that the human brain which conceives these new ideas was not designed top-down to be able to process complex mathematical thoughts. It has evolved from the brain of an upright hunter-gatherer that lived in the grasslands of Africa over a hundred thousand years ago. Our sophisticated brain is therefore built on primitive perception and physical action. It may even be, as is contended by Lakoff and Johnson (1999) that *all* our thoughts are *embodied*, in that they build from our human interaction with the world.

The major part of our brain is taken up with vision, perception of space and causing actions on what we see and sense outside us. We have opposed thumbs that allow us to manipulate objects in subtle ways. By a happy accident, our upright stance moves our larynx into a position that enables us to complement our mental powers by making sophisticated sounds that give us language. This provides a formidable facility for constructing our thoughts and communicating them to others. In recent millennia, and in particular, in the period since the invention of calculus in the seventeenth century, modern *homo sapiens* has developed a method of writing and manipulating mathematical symbols. These symbols are peculiarly fitted to the working of the brain. They are compact tokens that allow us to think about them and the relationships between them, and they can be manipulated, using the brain's natural ability to practice a sequence of activities until they become automatic and need little conscious attention to carry them out. In this way, the symbols of arithmetic, algebra and symbolic calculus enable us both to *think about* mathematics and also to *do* mathematics by way of calculation and manipulation of symbols.

This sophisticated symbolic ability provides us with a sequential/verbal-symbolic mode of thinking complemented by our more primitive holistic visuo-spatial senses. The term *versatile thinking* was used by Tall and Thomas (1991), following Brumby (1982), to refer to the “complementary combination of both modes, in which the individual is able to move freely and easily between them, as and when the mathematical situation renders it appropriate.”

Technology offers new ways of operation to utilise our versatile thinking processes in more powerful ways. The computer can perform algorithms at enormous speed with great fluency and accuracy, but it lacks the brains multi-processing ability to put old ideas together in new ways. Computer software not only offers arithmetic computation and symbolic manipulation, the results of these calculations can be represented visually, allowing the versatile brain to use visually presented results to see conceptual linkages.

In this presentation, I therefore focus on various aspects of mathematics, involving symbolic concepts and processes on the one hand and global visual representations on the other. In particular I shall consider how arithmetic, algebra and the calculus benefit the individual thinker by being provided in a computer environment that enhances and encourages versatile thinking.

Symbols and technology

The algorithms of arithmetic proved very amenable to being programmed and we have had four-operation calculators with us for most of the last half of the last century. In the nineteen-eighties numeric calculation on computers became enhanced by symbolic manipulation. There was widespread belief that the computer could do away with all the unnecessary clutter of calculation and manipulation allowing the individual to concentrate more on the essential ideas. Computers and calculators in business remove much of the tedium of calculation. An individual with no arithmetic skills beyond typing in numbers can enter the cost of items in a shop and the machine will give the total and even issue the correct change. Furthermore entry of information can be simplified by scanning bar-codes and stock-control can be handled by referring the items sold to the stock database to allow replacements to be ordered automatically. Soon even more of the economy will be taken over by technological means.

However, simply *using* technology, does not necessarily mean we *understand* what is happening. We may even lose some of the facility we had before. Hunter, Monaghan and Roper (1993) found that a third of the students using a computer algebra system could answer the following question before the course, but not after:

> ‘What can you say about u if $u=v+3$, and $v=1$?’

As the students had no practice in substituting values into expressions during the course, the skill seems to have atrophied. It is a warning given by the old adage ‘if you don’t use it, you lose it,’ a saying supported by physical evidence in the brain that unused pathways will tend to decay.

Other evidence also suggests that the use of symbol manipulators to reduce the burden of manipulation may just replace one routine paper-and-pencil algorithm with another even more meaningless sequence of keystrokes. Sun (1993)—reported in Monaghan, Sun & Tall (1994)—describes an experiment in which nine highly able 16/17 year old students taking a ‘further mathematics’ course had unlimited access to the software *Derive*. When asked to find a limit such as

$$\lim_{x\to\infty}\frac{2x+3}{x+2},$$

eight out of nine Derive students used the software procedure to produce they answer ‘2’. They claimed they knew no other method, even though they had been shown the technique of dividing top and bottom by x to get

$$\lim_{x\to\infty}\frac{2x+3}{x+2}=\lim_{x\to\infty}\frac{2+(3/x)}{1+(2/x)}=\frac{2+0}{1+0}=2.$$

Meanwhile, in a comparable group of 19 students using only paper-and-pencil methods, twelve used the simplification method given above, three substituted various numbers and four left it blank. This slender evidence intimates two things. First, almost all the Derive students obtained the ‘correct result’, whereas a considerable minority of the others failed, showing the power of the software. Second, the Derive students appeared simply to be carrying out a sequence of button presses, showing the distinct possibility of using technology with a lack of conceptual insight.

This phenomenon was repeated when the students were asked to explain the meaning of

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Here both the Derive group and one of the paper-and-pencil groups had been part of a discussion of the meaning of the notation in their mathematics lessons. *All* of the non-Derive group gave a satisfactory theoretical explanation of the concept. *None* of the Derive group gave *any* theoretical explanations. Four of them gave examples by replacing $f(x)$ with a polynomial and going through the sequence of key-strokes to calculate the limit. Although the curriculum may include initial theoretical activities (and most introductions to calculus begin with some discussion of the limit concept), what the students learn is dependent on the way they sense and interpret those activities that they are actually involved with at the time. In the case of these students using symbol manipulators, they have routinized the button-pressing activities and carry them out as automatic routines.

From this data we can see that students learn what they do. If they press buttons, they learn about button-pressing sequences. What is therefore important is to build a sense of meaning through reflection on the underlying mathematics. It is here that a *versatile* approach may prove of real value.

Sensori-motor and Visual aspects

Underlying the more conscious aspects of doing mathematics, there are other, deeper, human activities that provide an essential basis for all thought. The most primitive of these involve sensori-motor activity (physical sensations and bodily movement) and visual imagery. They play an important part in the computer interface. For example, the sensori-motor system allows decisions to be implemented intuitively using the mouse and keyboard.

These low-level cognitive actions also provide support for high-level theoretical concepts. Figure 1 shows software to build graphical solutions to (first order) differential equations by using the mouse to move a small line segment whose slope is determined by the differential equation. A click of the mouse deposits the segment and the user may fit line segments together to give an approximate solution.

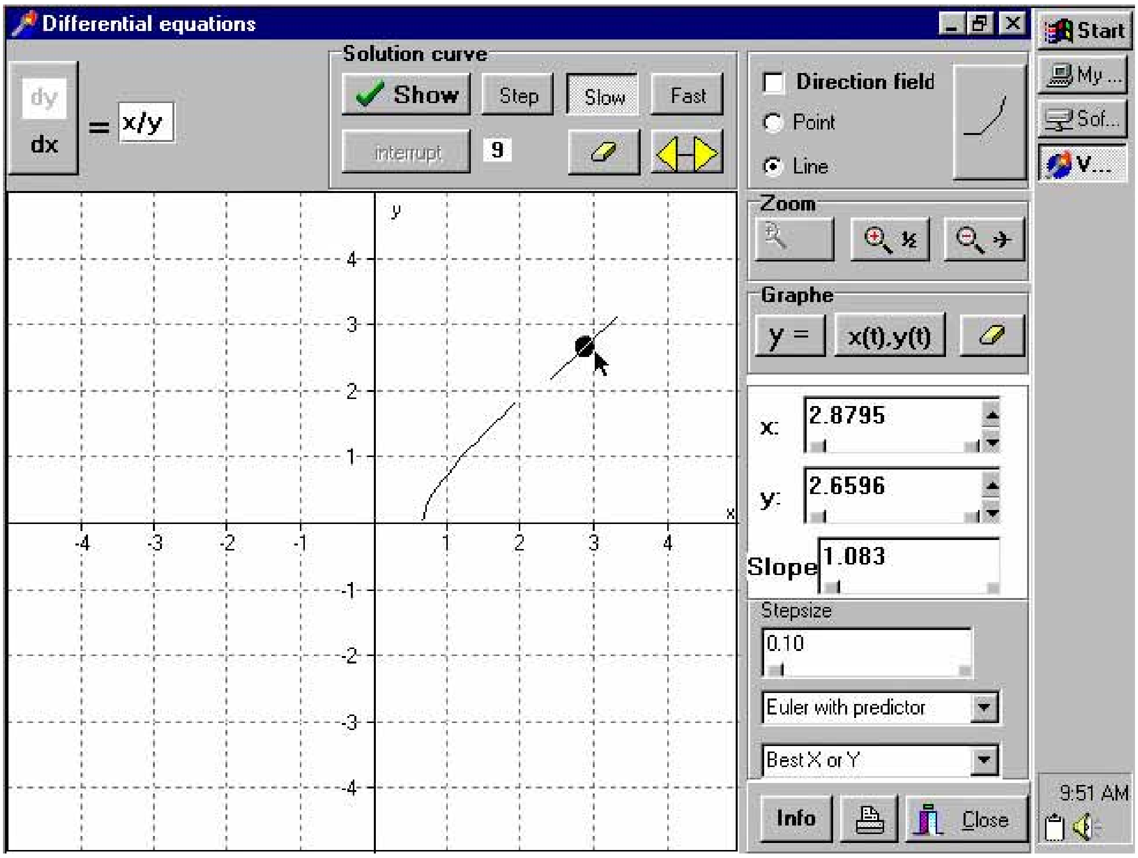

Figure 1: enactive software to explore the solutions of differential equations (Blokland, et al, 2000)

Such an activity can be performed intuitively with little knowledge of the theory of differential equations. Yet it already carries in it the seeds of powerful ideas about possible existence theorems—that a typical first order differential equation will have a unique solution through each point, and following the changing direction will build into a global solution curve. By considering selected examples it is be possible to look at the wider view of what happens to a whole range of solution curves and to see their behaviour. In this way an intuitive interface can give advance organisers for formal theory, especially to those individuals who naturally build on visual imagery.

Symbolism as a mental pivot between process and concept

Symbols used in a range of mathematical contexts give *Homo sapiens* an incredibly simple way of dealing with quantities for calculation, problem solving and prediction. Many symbols simply act as a *pivot* between the symbol conceived as a concept (such as number) and a process (such as counting). (figure 2).

symbol	***process***	***concept***
4	counting	number
3+2	addition	sum
–3	subtract 3 (3 steps left)	negative 3
3/4	sharing/division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\, dx$	integration	integral
$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$ $\sum_{n=1}^{\infty} 1/n^2$	tending to limit	value of limit
$(x_1, x_2, \ldots, x_n)$	vector shift	point in n-space
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n

Figure 2: Symbols as process and concept

Gray & Tall (1994) refer to the combination of symbol representing both a process and the output of that process as a *procept* (figure 3).

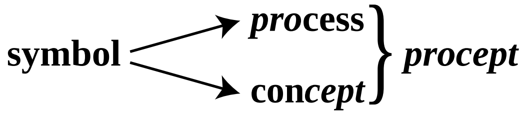

Figure 3: the symbol as pivot between process and concept forming a procept

The procept notion has been given increasingly subtle meaning since its first formulation (Gray & Tall, 1991). It is now seen mainly as a *cognitive* construct, in which the symbol can act as a *pivot*, switching from a focus on process to compute or manipulate, to a concept that may be *thought* about as a manipulable entity. We believe that procepts are at the root of human ability to use mathematical ideas in arithmetic, algebra and other theories involving manipulable symbols. They allow the biological brain to switch effortlessly from thinking about symbols as concepts to using them for doing mathematical processes in a minimal way.

There are several different ways in which the symbolism is used :

(a) a *procedure* consists of a finite succession of actions and decisions built into a coherent sequence. It is seen essentially as a step-by-step activity with each step triggering the next.

(b) the term *process* refers to when the procedure is conceived as a whole and the focus is on input and output rather than the particular procedure used to carry out the process. It may be achieved by *n* procedures ($n \geq 0$) and affords the possibility of selecting the most efficient solution in a given context.

(c) a *procept* requires the symbols to be conceived flexibly as processes to do and concepts to think about. This allows for more powerful mental manipulation and reflection to build new theories.

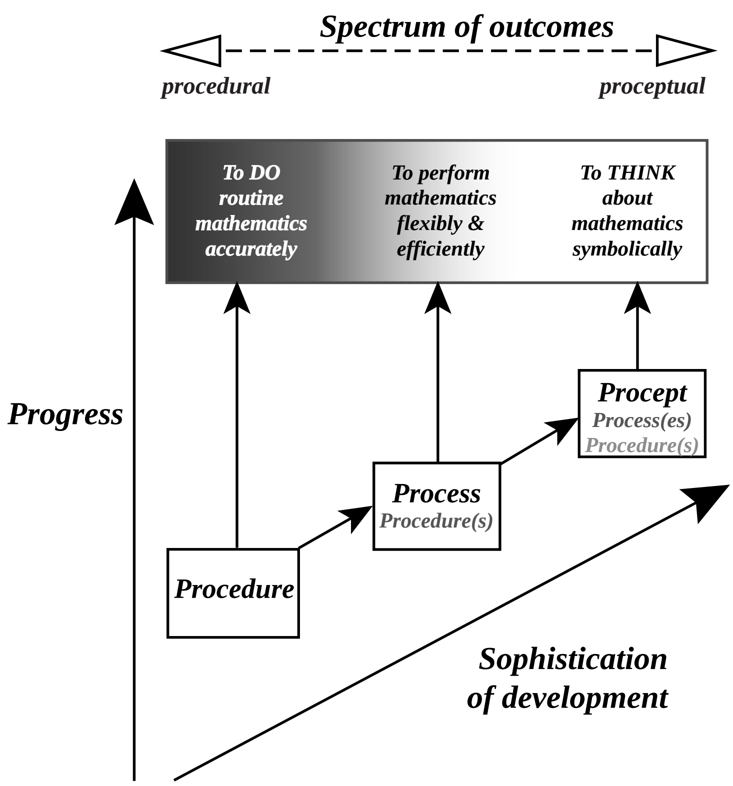

Figure 4: A spectrum of performance in the carrying out of mathematical processes

Different uses of symbolism give rise to differing levels of flexibility and ability to think mathematically (figure 4). This is not to say that procedural thinking does not have its value. Indeed, much of the power of mathematics lies in its algorithmic procedures. However, a focus on procedures alone, without conceptual linkages between them, leads to increasing cognitive stress as the individual learns more and more disconnected pieces.

The difficulty in thinking conceptually seems to increase throughout the curriculum. My own perception of these difficulties is that the underlying procepts act in very different ways, so that the learner, who has internal methods of processing the ideas, finds new ideas strangely conflicting with inner beliefs. This, I believe, leads to a lack of connections and the desire to learn procedures solely to pass examinations.

This spectrum has been beautifully demonstrated by DeMarois (1998) who worked with students using graphing calculators in a remedial college pre-algebra course. They were asked to consider the problem given in figure 5, relating to two functions having different internal procedures but the same input-output relationship.

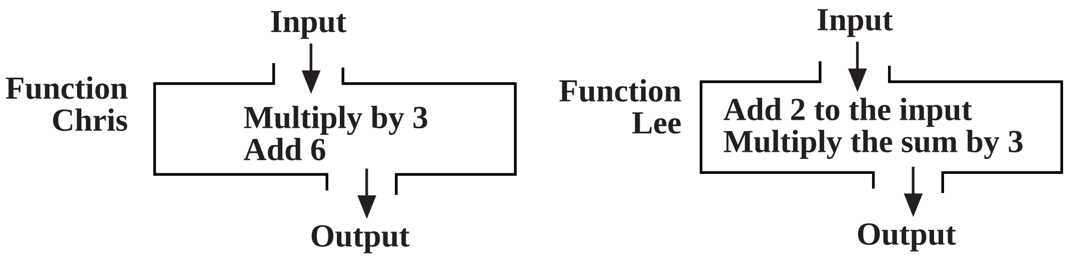

Figure 5: Write the outputs of these two function boxes and say if they are the same function.

The responses given by students achieving grades A, B, C showed characteristics of the three above distinct levels of sophistication. They were as follows:

> •*Grade A Student*: **3x+6**, **3(x+2)**
> Yes, if I distribute the 3 in Lee, I get the same function as Chris. (*procept*)
>
> •*Grade B Student*: **x3+6, (x+2)3**
> Yes, but different procedures (corresponding to our notion of *process*)
>
> •*Grade C Student*: **3x+6**, **x+2 (3x)**
> No, you come up with the same answer,
> but they are different processes (corresponding to our notion of *procedure*)
>
> (DeMarois, 1998, p.171, 174–5)

The grade A student is responding in a manner which shows that she can manipulate the expressions and see that one is the same as the other. Her fluency of manipulation of the expression here and in other contexts reveals her flexible ability operating with expressions as *procepts*. Students B and C write the algebra in idiosyncratic ways, with

B following the precise verbal order and C using the algebraic order for 3*x*+6 yet writing the second function expression in a way which can be read as ‘*x*+2 (three times).’ Although B and C use the terms procedure and process in their own way, B says ‘yes’ (they are the same function) suggesting a *process* interpretation and C says ‘no’ (they are not the same function) suggesting a *procedural* interpretation.

Fundamental differences in proceptual structure

Despite the power of procepts, students often have difficulty with them. This is in part often due to their working at a level which causes them greater cognitive stress, for example, working at a procedural level and being unable to put all the ideas together to be able to cope with more complex problems. However, there is also another highly significant factor. In various different parts of the curriculum, symbols as process and object operate in very different ways. Thus the student who may very well have a good grasp of the ideas in one context suddenly finds that things are confusing at the next stage. This may be illustrated by looking at some of the different types of procept in arithmetic, algebra and limits.

(1) ***arithmetic procepts***, 5+4, 3×4, $\frac{1}{2}+\frac{2}{3}$, 1·54÷2·3, all have built-in algorithms to obtain an answer. They are *computational*, both as processes and even as concepts. For instance in the sum 8+6, the concept 6 can be linked to the operation 2+4, which can be combined in the sum 8+2+4 to give 10+4 which is 14. A deeper analysis would reveal more differences between operations with whole numbers, negative numbers, fractions and decimals.

(2) ***algebraic procepts***, such as 2+3*x*, can only be evaluated if the value of *x* is known. Thus an algebraic procept has only a *potential process* (of numerical substitution) and yet the algebraic expressions themselves are expected to be treated as *manipulable concepts* using the usual algebraic rules.

Meaningful power operations such as

$$2^3 \times 2^2 = (2\times 2\times 2)\times(2\times 2) = 2^5$$

can act as a cognitive basis for the power law

$$x^m \times x^n = x^{m+n}$$

valid for all real x and for whole numbers m, n. A student can have a meaningful interpretation of this symbolism, yet be totally confused when the symbols are used in a different way as:

(3) ***implicit procepts***, such as the powers $x^{\frac{1}{2}}$, x^0 or x^{-1}, for which the original meaning of x^n no longer applies, but is implicit in the generalised power law.

Attempting to give the 'same meaning' to these ideas does not work, for we can hardly speak of 'half an x multiplied together', or 'no xs multiplied together' (surely no xs must give zero). Thinking of x^{-1} as 'minus one xs' seems as foolish as talking about 'minus one cows'. These powers can, however, be given an implicit meaning by assuming the power law as an axiomatic basis for deduction. Thus, for $m = n = \frac{1}{2}$, we get

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1 = x$$

from which we may deduce that $x^{\frac{1}{2}} = \sqrt{x}$. Of course, such a meaning is being deduced from a law that we do not *know* is true, but one that we *assume* as an axiom—a new way of doing mathematics that is powerful for those willing and able to follow it through, but puzzling for those who attempt to give the symbols the same meaning as before.

Later examples in the curriculum are:

(4) ***limit procepts***, such as $\lim_{x \to a} \frac{x^3 - a^3}{x - a}$ or $\sum_{n=1}^{\infty} \frac{1}{n^2}$. These have ***potentially infinite processes*** 'getting close' to a limit value, but this may not be computable in a finite number of computations. Limits are often conceived as 'variable quantities' which get arbitrarily close to a limiting value, rather than the limit value itself.

(5) ***calculus procepts***, such as $\frac{d(x^2 e^x)}{dx}$ or $\int_0^{\pi} \sin mx \cos nx \, dx$. These are more familiar in the sense that they (may) have finite operational algorithms of computation (using various rules for differentiation and integration).

Although limit procepts are often introduced as the first idea in calculus, we shall see that this use does not fit with the embodied brain that invariably interprets the notion of limit as a potentially infinity process rather than a manipulable concept. Instead the students are much happier with calculus procepts that involve a process of computation and an *answer*, albeit in the form of an expression rather than a number.

The Principle of Selective Construction

To reduce the cognitive strain on the learner in the mathematics curriculum using computers, I formulated the *Principle of Selective Construction* (Tall, 1993). This proposes the design of software that embodies selected aspects of the theory for the learner to explore whilst the computer carries out other essential processes internally. It can be accomplished in a range of ways. For instance, the approach to differential equations advocated earlier (figure 1) provides a manipulable visual representation where the computer carries out the algorithms and draws the pictures leaving the student to concentrate on building a solution. The interface enables the user to gain an embodied visuospatial sense of the theory in which hand and eye coordinate to build the direction of the solution curve. Having gained this insight, the student could then profit by focusing on other aspects, such as the internal numerical procedures to produce approximate solutions, or the study of available symbolic methods to give solutions in particular cases.

Another use of the principle of selective construction arises in Michael Thomas's approach to the introduction of algebra (Thomas, 1988; Tall and Thomas, 1991). In one activity the pupils enact a physical game, away from the computer, using two large sheets of cardboard. One represents 'the screen' on which instructions are given. The other is 'the store' on which squares are drawn which can be labelled with a letter for the variable and the numerical value stored inside. Typical instructions might be to put A=2 (place a number 2 in a box marked 'A') then PRINT A+3 (find the number in the box marked 'A' and print out the sum of it and 3, to output 5). This concentrates on the *process* aspect of evaluation of expressions and uses an embodied approach that builds on human activity. On the other hand, writing simple programs on the actual computer, such as INPUT A: PRINT 2*(A+1), 2*A+2, gets the computer to carry out the process of calculation so that the student can concentrate on the *equivalence* of the same outputs from different procedures of calculation. This combination of separate focus on symbol as process and as (equivalent) concepts proved to give a long-term improvement in conceptualization of expressions as manipulable concepts (Tall & Thomas, 1991).

In the next few sections we look at the developing curriculum in arithmetic, algebra and the calculus. In each case we begin with the cognitive processes involved and then consider how the computer can be used to promote versatile thinking using the principle of selective construction.

Long-term considerations in arithmetic, algebra and calculus

Manipulating symbols in arithmetic

Whole number arithmetic has a fundamental proceptual structure in which the number symbol has a dual role as number concept and counting process (Gray & Tall, 1994). Addition is successively compressed from the triple 'count-all' process (count one collection, count the other, count the two together) through the transition of the double counting process of 'count-both' (count-one number then count-on the second) to the single 'count-on' process (count-on the second number starting from the first). Some facts are remembered (as 'known-facts') and may, or may not, be used flexibly to 'derive facts' from those already known (for instance, deriving 8+6 from 8+2+4=10+4=14 or 23+5=28 as 20+3+5, and so on). Gray Pitta, Pinto & Tall (1999) show how there is a bifurcation in strategy between those who cling to the counting operations on objects and those who build (to a greater or lesser extent) a conceptual hierarchy of manipulable relationships between process and concept. The former become imprisoned in thinking about manipulating objects in a way which does not generalise to larger numbers whilst the latter have an internal engine to derive new facts from old that forms a foundation for mental arithmetic.

The successive compression from triple-counting to the single count-on, to the flexible use of remembered known facts is a long journey taking several years to become efficient in arithmetic. The use of calculators and computers in early mathematics has been perceived by the British government and their experts as being less successful than it might be. So much so that, in the English National Curriculum, the use of calculators with young children has been discouraged in the hope that their absence will enable children to build mental arithmetic relationships.

Perhaps this is more to do with the misuse of the calculator (for performing calculations without having to think) than it is to any inherent defect in the apparatus itself. Used well, to *reflect* on mathematical ideas, the calculator can be very beneficial as Gray and Pitta (1997) showed in their work with a slow-learner having difficulties with arithmetic.

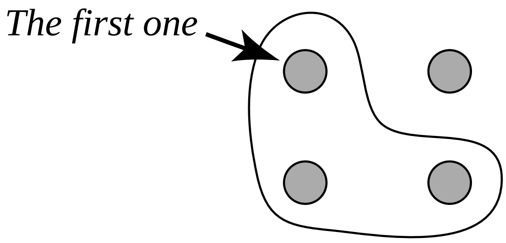

figure 6: Emily's image of 4–3.

Emily, aged 8, was identified as one of the weakest four children in arithmetic in a year group of 104. She 'seemed to associate counting with fingers with the use of a particular sequence of fingers' and she found this difficult. She explained, 'sometimes I get into a big muddle with them … I am not concentrating on the sum. I am concentrating on getting my fingers right which takes a while.' When she did simple arithmetic mentally, she imagined manipulating arrays of counters, for instance, she explained the sum 4–3, saying '… there's two dots above each other and then there's … the first one, the one below and the next to it are being taken away and there is only the first one left.' (Figure 6.) With these direct manipulations of fingers or mental objects, she was under considerable cognitive stress handling small numbers and had great difficulty with numbers larger than ten. At this stage she clung to counting and was a candidate for long-term failure. However, Gray and Pitta planned a series of activities using a graphic calculator which displayed both the arithmetic operations to be carried out and the numerical results. (Figure 7).

4 + 5	
	9
4 + 4 + 1	
	9
3 + 2 + 4	
	9

figure 7: ways of making 9 on a graphic calculator.

Emily was given a personalised workbook asking her to find ways to make 9, or to make 9 starting with 4, or with 10. Her combinations included standard ones such as 4 + 5, 6+9, but also more compliex ones such as 4+4+1, 5+6–2, 5+1+1+2. In her interviews she was encouraged to talk about her discoveries without using the computer. Only once did she ever refer to seeing ‘dots’ and nine months later she was operating mentally, seeing numbers ‘flash’ into her mind. This insight shows her making strides towards compression that are enhanced by her calculator work. She was seeing relationships between *numbers* in her mind’s eye.

The principle of selective construction is implicit in Gray and Pitta’s work with Emily. The software performs the calculation and Emily can concentrate on the relationships without having to carry out the intervening counting processes. Were she to continue to focus on counting, it would have become too onerous for her to develop much further.

This imaginative use of the calculator shows the poverty of the UK numeracy strategy to discourage the use of calculators for young children. It is not the *technology* that is at fault, but the *use* to which it is put.

Proceptual problems in algebra

Long before the notion of ‘procept’ was formulated, students were noted to have difficulty conceiving an expression such as ‘7+*x*’ as the solution to a problem—aphenomenon described by Collis (1972) as ‘lack of closure’. Davis, Jockusch and McKnight (1978) remarked similarly that ‘this is one of the hardest things for some seventh-graders to cope with; they commonly say “But how can I add 7 to *x*, when I don’t know what *x* is?”’ Matz (1980) commented that, in order to work with algebraic expressions, children must ‘relax arithmetic expectations about well-formed answers, namely that an answer is a number.’ Kieran (1981) similarly commented on some children’s inability to ‘hold unevaluated operations in suspension.’ All of these can now be described as the problem of manipulating symbols that—for the students—represent potential processes (or specific procedures) that they cannot ‘do’, yet are expected to treat as manipulable entities. Essentially they see expressions as unencapsulated processes rather than manipulable procepts.

As mentioned earlier, this was tackled by Thomas (1988) using a combination of physical activity to give an embodied sense of the process and the use of the computer to focus on the relationships between the concepts.

In the work of McGowen (1998), DeMarois (1998) and Crowley (2000), American college students using graphic calculators to support their development of algebra show the spectrum of performance from procedural competence to flexible use of mental linkages. By getting her students to draw concept maps of their knowledge at various points in the course, McGowen found that the less successful simply focussed on the current work, attempting to use procedures as they were given, but the more successful built concept maps which built incrementally on previous work to give a highly integrated conceptual system. One successful student explained his methodology:

> While creating my [final] concept map on function, I was making strong connections in the area of representations. Specifically between algebraic models and the graphs they produce. I noticed how both can be used to determine the parameters, such as slope and the *y*-intercept. I also found a clear connection between the points on a graph and how they can be substituted into a general form to find a specific equation. Using the calculator to find an equation which best fits the graph is helpful in visualizing the connection between the two representations. I think it's interesting how we learned to find finite differences and finite ratios early on and then expanded on that knowledge to understand how to find appropriate algebraic models. (McGowen and Tall, 1999, p. 284).

There is evidence that this concentration on making links in the concepts in a course on straight line graphs using graphic calculators can radically affect performance on later courses. Crowley (2000, pp. 209, 210) found that those who continued to be successful 'had readily accessible links to alternative procedures and checking mechanisms' and 'had tight links between graphic and symbolic representations'. They succeeded even though they 'made a few execution errors.' Others who succeeded in the earlier course but 'had serious difficulties with the next,' had passed the first course whilst showing underlying weaknesses in conception. 'They had links to procedures, but did not have access to alternate procedures when those broke down. They did not have routine, automatic links to checking mechanisms. They did not link graphical and symbolic representations unless instructed to do so. … They showed no evidence that they had compressed mathematical ideas into procepts.'

Proceptual problems in the concept of limit

It has long been known that students have difficulty coming to terms with the limit concept presented at the beginning of a calculus course. The early research on this topic is summarised in Cornu (1991). The conceptual difficulties can be clearly formulated using the notion of procept. At first a limit (say of a sequence of numbers) is seen as a *process* of giving a better and better approximation to the limit value. This process of 'getting close' but 'never reaching' the limit gives rise to the mental image of a variable quantity that is 'arbitrarily small' or 'arbitrarily close' to a fixed quantity. This may then lead to the construction of a mental *object* that is 'infinitely small'—a cognitive *infinitesimal.* Monaghan (1986) called this a 'generic limit'. It is a cognitive concept wherein the limit object has the same properties as the objects in the sequence which is converging to it. Thus in the sequence $(1/n)$, all the terms are positive, so the generic limit is positive. It is also arbitrarily small. This leads to a concept image of the number line that has infinitesimal quantities included and is therefore at variance with the formal definition of the real numbers.

Symbol manipulators use a variety of representations for numbers, including integers, rationals, finite decimals, radicals such as $\sqrt{2}$, $\sqrt[10]{7}$, special mathematical numbers such as π, e. Students conceive of different kinds of number in subtly different ways. For instance they may be feel secure in ‘proper numbers’ such as whole numbers and fractions. But they may regard infinite decimals, both repeating and non-repeating, as ‘improper numbers’ which ‘go on forever’ (Monaghan, 1986). Procept theory classifies these ‘improper numbers’ as ‘potentially infinite’ processes rather than as number concepts. We therefore see that many students do not have a coherent view of the number line. It is populated by a range of different kind of creatures, some familiar, some less familiar, and some downright peculiar.

Reconstructing these views of numbers is not something that has proved at all easy (Williams, 1991). My own consideration is that the potentially infinite process in the limit procept causes great conceptual difficulty and I have preferred to attack the design of the calculus curriculum by making the notion of limit *implicit* in the software, whilst encouraging the student to focus visually on what is happening.

The calculus

Technology offers new ways of approaching the calculus in a versatile way which are becoming widespread around the world. My own approach builds on using visual software to give an embodied sense of the underlying mathematical concepts. There are four procepts in the calculus: the notion of *limit* (in a variety of different forms), *change* (function), *rate of change* (derivative), *cumulative growth* (integral). However, given the evidence that the formal notion of limit is not a sensible place to start when students are learning calculus, I concentrate on the last three (figure 8).

Procept		
Change: FUNCTION	doing	calculating values
	undoing	solving equations
Rate of change: DERIVATIVE	doing	differentiation
	undoing	anti-differentiation, solving differential equations
Cumulative growth: INTEGRAL	doing	integration
	undoing	fundamental theorem of calculus

Figure 8: Three procepts in functions and calculus

If we analyse these in greater detail, we find that there is a spectrum of approaches, from a real-world meaning of the terms, to various aspects that can be represented on the computer in graphic, numeric and symbolic ways, and on to the formal theory of analysis. Figure 9 shows my own analysis of the procepts and their representation in the calculus, taken from Tall (1997).

The first column (real-world calculus) has been exemplified in computer software by Kaput (see the *SimCalc* website http://tango.mth.umassd.edu). (The approach via SimCalc allows the movement of a child to be tracked by a sensor and then displayed as a graph to give a powerful embodied link between child and computer.) The final column does not concern us here as it relates to the formal theory discussed in an extension of this presentation (Tall, to appear) which carries the ideas of embodied thinking and the use of technology through to formal mathematical thinking. I will therefore concentrate on the three representations: graphic, numeric, symbolic.

Graphic representations on the computer already embody the principle of selective construction. The numerical calculations that produce the pictures are performed internally while the graphic software provides an environment for visuospatial

		Representations				
		Visuo-spatial	**Graphic**	**Numeric**	**Symbolic**	**Formal**
Procepts		*Enactive* observing experiencing	*Qualitative* visualizing conceptualizing	*Quantitative* estimating approximating	*Manipulative* manipulating limiting	*Deductive* defining deducing
Change: FUNCTION	doing	distance, velocity etc. changing with time	**graphs**	numerical values	algebraic symbols	set-theoretic definition
	undoing	solving problems	visual solutions where graphs cross	numerical solutions of equations	solving equations symbolically	intermediate value & inverse function theorems
Rate of change: DERIVATIVE	doing	velocity from time-distance graph	**visual steepness**	numerical gradient	symbolic derivative	formal derivative
	undoing	solving problems e.g. finding distance from velocity	visualize graph of given gradient	numerical solutions of differential equations	antiderivative —symbolic solutions of differential equations	antiderivative —existence of solutions of differential equations
Cumulative growth: INTEGRAL	doing	distance from time-velocity graph	**area under graph**	numerical area	symbolic integral as limit of sum	formal Riemann integral
	undoing	computing velocity from distance	know area – find graph	know area —find numerical function	Symbolic Fundamental Theorem	Formal Fundamental Theorem
		REAL-WORLD CALCULUS	THEORETICAL CALCULUS			ANALYSIS

Figure 8: A spectrum of representations in functions and the calculus

exploration. In a logical development, the numeric and symbolic would almost certainly precede the graphic. In cognitive terms, a graphic approach to the calculus is part of a versatile learning sequence. It enables the learner to visualise and conceptualise the concepts in an embodied manner that can form a foundation for future development, be it a technical support in applications, formal epsilon-delta analysis, or the infinitesimal calculus of non-standard analysis.

In figure 9, I have highlighted three parts of the graphic column: ***graphs***, ***visual steepness*** and ***area under the graph***. It would be a nice ending to my presentation to be able to show how a versatile use of these three conceptions can make the calculus deeply meaningful. However, although there are great benefits from such an approach, research has continually shown difficulties with the conception of function.

Mathematically the formal notion of function could not be simpler: there are just two sets A and B and, for each element x in A, there is precisely one corresponding element y in B which is denoted by the symbol $f(x)$.

The inner workings of our embodied brain colour this simplicity with a wide range of implied meanings linked to our perceptions and actions. This includes the idea that functions always seeming to have a formula, and a graph with a familiar smooth shape. In addition, experience with linear, quadratic, trigonometric and exponential functions focus on a wide range of different properties for each family of functions. For instance a linear function has an intercept and a slope that completely determine it. Study of the quadratic function involves factorization and 'completing the square'. Trigonometric functions involve radians, the sign of trig functions in each quadrant, relationships between sine, cosine and tangent, together with formulae for sin(A+B), cos(A+B), tan(A+B), and so on. Exponential functions have the power law and the relationship with logarithms and their properties. The learner is therefore bombarded more with the *differences* between all these examples than the underlying function properties that they share. Essentially the function concept itself is hardly ever the focus of attention. Instead the human brain makes links involving the special calculations and manipulations that are actually the main focus of study.

The plight of the function concept in terms of graphs is sealed by the research of Cuoco (1994). The students he studied who had been taught using the traditional notion of function as formula and graph hardly *ever* saw the graph as a process of assignment (for given x, move up to the graph and across to the y-axis to read off $f(x)$.) On the other hand, he showed that programming functions in computer languages involves imagining inputs, carrying out an internal process and giving an output. The *name* of the function procedure allows it to be treated like a manipulable object.

The process usually associated with a graph is the process of drawing it, or tracing along the curve from left to right. The approach used in SimCalc, with *time* as the variable, therefore encourages a bodily sensation of seeing the covariation of, say, distance with time. This therefore gives an embodied approach to the graphs of functions, although there will need to be further development along the line to replace the variable time and its human sense of duration for other variables.

A *Graphic Approach to the Calculus* begins with the notion of a graph representing the relationship between an independent and dependent variable. The study of straight line functions is a necessary pre-requisite to be able to *see* the gradient of the function when the axes have equal scales and to be able to imagine that changing the scales on the axes might change the visual slope in the picture, but not the numerical value of the gradient as *y*-step over *x*-step.

Zooming in on a graph, retaining equal scales reveals that most of the graphs we know look less curved the more we zoom in, until they look like a straight line under high magnification. If a small square centred on a point on the curve reveals an approximate straight line when magnified, moving the square along the curve and noting the changing gradient of the curve appeals to the embodied idea of 'looking along the curve and *seeing* its changing gradient'. Thus, in one go, the student can look along the curve and see the gradient changing as a function of *x*. (In the embodied sense that, as I point at successive places on the graph, the gradient can be seen to change as *x* changes.)

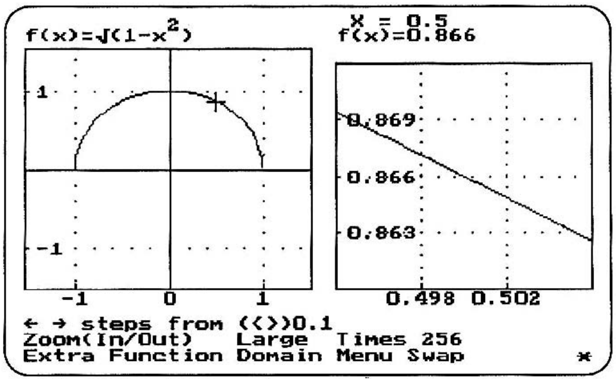

Figure 10: a circle is locally straight
(drawn with Tall et al, 1990)

Students are often quite surprised when they see that a circle–the archetypal 'curved' curve– is locally straight. (Figure 10), but then, as they realize that the curvature gets less as the circle gets bigger, this becomes less strange. Using software to magnify the curve can give graphic insight into the phenomenon.

Many mathematicians who hear me say things this way are puzzled and/or angry. For them the gradient must first be approached as a limit at a point, and then, when the limit is achieved, the point is varied and the limit value at the point for all points is the derivative function. For this reason, mathematicians often introduce the notion of 'local linearity' which means finding a linear function at a point on the graph which is the best linear approximation to the graph. Frankly, this is far more complex. Local linearity is defined first at a point in a functional symbolic manner, then the point is varied. Local straightness is a purely embodied visuo-spatial conception of the changing gradient of the graph itself.

Students who follow a locally straight approach are far better at drawing the gradient curve at a given point. They can just look along the curve and *see* the changing gradient and sketch the requisite curve. For instance, given the graph of cos$x$, as x increases from zero, the gradient of the graph starts at zero, moves increasingly negative until at $\pi/2$ the gradient is (about) –1, then it becomes less negative till the gradient is zero at π. Looking along the curve reveals the gradient function looking like the graph of sin$x$ upside down, suggesting the gradient is –sin$x$ (figure 11).

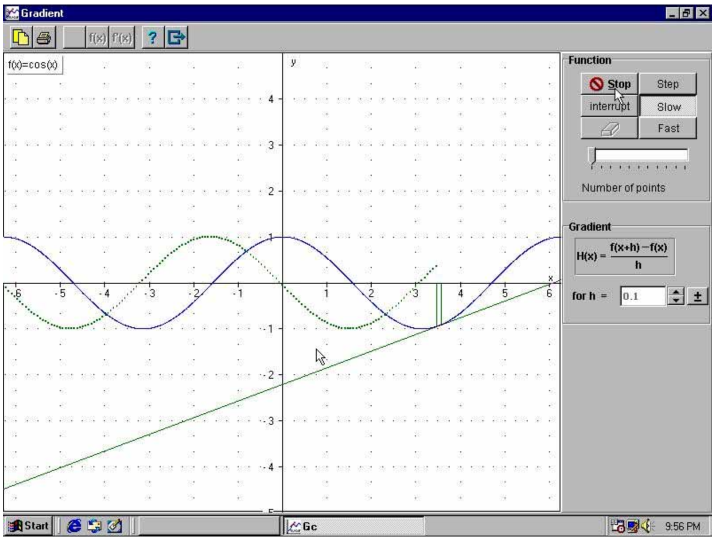

Figure 11: the gradient of cosx (drawn with Blokland et al (2000)

The graphic approach to the calculus, first by magnification, and then by moving along the curve to trace out the changing gradient gives an *embodied* view of the rate of change. The principle of selective construction is being used again because the software computes the gradient accurately and allows the user to interpret the graphs using knowledge of standard graph shapes.

It fits exactly with the opening idea of solving a first order differential equation. If the first derivative of a function is given, then the function has a locally straight graph, so the original graph can be (approximately) reconstituted to build up a solution curve that has the given gradient.

I also include the graph of the *blancmange* function *bl(x)* and other similar functions which are fractals which reveal the same detail at successive levels of magnification. They never look straight at any magnification, and so they are *nowhere* differentiable. (This can be proved by an embodied visuo-spatial method that itself can be turned into a formal proof (see Tall, 1982)). I can even take a small copy of *bl(x)*, say $n(x)=bl(1000x)/1000$. The functions sin$x$ and sin$x$ + $n(x)$ differ by less than 1/1000, so to a regular scale with the x-range, say from –5 to +5, will reveal no difference between the two graphs on the computer screen. And yet the first is differentiable *everywhere* and the second is differentiable *nowhere*. Thus a graphic approach to the calculus offers insight into far deeper ideas about differentiability (figure 12).

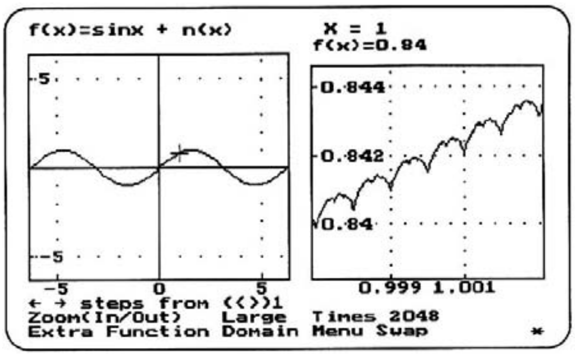

Figure 12:A ‘smooth-looking curve’ that magnifies ‘rough’.

I do not have the space here to give full details of the extent to which a graphic approach to calculus can lead on to the most subtle of formal ideas. Suffice it to say that, by thinking carefully, one can develop embodied visual intuitions that can contain the seeds of highly subtle formal mathematics. It is mainly a question of looking at the right pictures in the right way. As Benoit Mandelbrot said:

> When I came into this game, there was a total absence of intuition. One had to create an intuition from scratch. Intuition as it was trained by the usual tools—the hand, the pencil and the ruler—found these shapes quite monstrous and pathological. The old intuition was misleading. … I’ve trained my intuition to accept as obvious shapes which were initially rejected as absurd, and I find everyone else can do the same.
>
> (Mandelbrot, quoted in Gleich, 1987, p.102).

In a publication associated with this conference (Tall, to appear), I follow up this quotation to see how visual intuition can support formal ideas in advanced mathematical thinking. This, and other related papers can be found on my website:

http:warwick.ac.uk/staff/David.Tall

A selection of relevant papers includes:

Tall (1985, 1991, 1992, 1993, 1995, 1997) and Di Giacomo & Tall (2000).

Summary

In this presentation, I have presented a theory of mathematics built on the underlying vision and action of the human species, complementing theory with a range of empirical evidence. This shows that learning mathematics involves building up mental imagery in a sequence which is somewhat different from a formal logical development. In the last century, Kaput wrote:

> Anyone who presumes to describe the roles of technology in mathematics education faces challenges akin to decribing a newly active volcano – the mathematical mountain is changing before our eyes… (Kaput, 1992, p. 515.)

The volcano has been smouldering for a long time now and a shape of the future seems to be developing. In this future I am confident that what will make mathematics work to its best advantage are the qualities which make us human. But beyond that, I consider it foolish to attempt to say where the next millennium will take us, for even a decade is a long time at our present rate of development. Having taken part in previous crystal-ball gazing operations (such as the Mathematical Association Report on ‘Computers in the Mathematics Classroom’ (Mann & Tall, 1992), I now know that our vision of tomorrow soon becomes the history of yesterday. We then quoted the following as an example of forward thinking from a previous report:

> It is unlikely that the majority of pupils in this age range will find [a computer] so efficient, useful and convenient a calculating aid as a slide rule or book of tables.
>
> (Mathematical Association, *Mathematics 11 to 16*, 1974)

At this historical point I shall therefore refrain from suggesting what will happen even in the near future. But one thing seems sure. As we stand at the beginning of a new millenium (starting January 1st, 2001), being a mathematician and an educator has never been more exciting.

References

Blokland, P., Giessen, C, & Tall, D. O. (2000). Graphic Calculus for Windows. (see www.warwick.ac.uk/staff/David.Tall.)

Brumby M.N. (1982). Consistent differences in Cognitive Styles Shown for Qualitative Biological Problem Solving, British Journal of Educational Psychology, 52, 244-257.

Collis, K. F. (1972). *A study of the relationship between formal thinking and combinations of operations*. University of Newcastle, NSW, Australia.

Cornu, B. (1991). Limits'. In D. O. Tall (ed.), *Advanced Mathematical Thinking*, (pp.153-166), Kluwer: Crowley (2000).

Cuoco, A.A.: 1994, 'Multiple Representations for Functions'. In E. Dubinsky, A. H. Schoenfeld & J. Kaput (eds.), *Research Issues in Undergraduate Mathematics Learning*, MAA Notes 33, (pp.121-140). Washington DC: MAA.

Crowley, L. (2000). *Cognitive Structures in College Algebra*, PhD thesis, University of Warwick.

Davis, R. B., Jockusch, E., McKnight, C. (1978). Cognitive processes in learning algebra, *Journal of Children's Mathematical Behavior*, 2 (1).

DeMarois, P. (1998). *Aspects and Layers of the Function Concept*, PhD Thesis, University of Warwick.

Di Giacomo, S. & Tall, D. O. (2000). Cosa vediamo nei disegni geometrici? (il caso della funzione blancmange) *Progetto Alice* (to appear). [English version: What do we "see" in geometric pictures? (the case of the blancmange function) is available from www.warwick.ac.uk/staff/David.Tall.]

Gray, E.M. & Pitta, D. (1997). Changing Emily's Images, *Mathematics Teaching*, 161, 38–51

Gray, E. M. Pitta, D., Pinto, M., & Tall, D.O. (1999). Knowledge construction and diverging thinking in elementary & advanced mathematics. *Educational Studies in Mathematics*. 38 (1-3), 111–133

Gray, E. M. & Tall, D. O. (1991). Duality, ambiguity & flexibility in successful mathematical thinking, *Proceedings of PME XV*, Assisi, 2, 72–79.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic, *Journal for Research in Mathematics Education*, 26, 115–141.

Hunter, M., Monaghan, J. D. & Roper, T. (1993). The effect of computer algebra use on students' algebraic thinking. In R. Sutherland (Ed.), *Working Papers for ESRC Algebra Seminar*, Institute of Education, London.

Kaput, J. J. (1992). Technology and Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, New York:MacMillan.

Kieran C. (1981). Pre-algebraic notions among 12 and 13 year olds, *Proceedings of the Fifth Conference of the Psychology of Mathematics Education*, Grenoble, 158–164.

Lakoff, G. and Johnson, M. (1999). *Philosophy in the Flesh*. New York: Basic Books.

Mann, J. F. A., and Tall, D. O. (eds) (1992). *Computers in the Mathematics Curriculum, A Report of the Mathematical Association*. Leicester: Mathematical Association.

Mathematical Association, (1974). *Mathematics 11 to 1*6. Leicester: Mathematics Association.

Matz, M. (1980). Towards a computational theory of algebraic competence, *Journal of Mathematical Behavio*r, 3 (1), 93–66.

McGowen, M. & Tall, D. O. (1999). Concept Maps & Schematic Diagrams as Devices for Documenting the Growth of Mathematical Knowledge. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME*, Haifa, Israel, 3, 281–288.

McGowen, M. (1998). *Cognitive Units, Concept Images and Cognitive Collages*, PhD Thesis, University of Warwick.

Monaghan, J. D. (1986). *Adolescent's Understanding of Limits and Infinity*. PhD Thesis, University of Warwick.

Monaghan, J., Sun, S. & Tall, D. O. (1994), Construction of the Limit Concept with a Computer Algebra System, *Proceedings of PME 18*, Lisbon, III, 279-286.

Sun, S. (1993). *Students' Understanding of Limits and the Effect of Computer Algebra Systems.* M.Ed. thesis, Leeds University, U.K.

Tall, D. O., (1982). The blancmange function, continuous everywhere but differentiable nowhere, *Mathematical Gazette*, 66 11–22.

Tall, D. O. (1985). Understanding the calculus, *Mathematics Teaching* 110, 49-53.

Tall, D. O. (1991). Recent developments in the use of the computer to visualize and symbolize calculus concepts, *The Laboratory Approach to Teaching Calculus* , M.A.A. Notes Vol. 20, 15–25.

Tall, D. O. (1992). Visualizing differentials in two and three dimensions, *Teaching Mathematics and its Applications*, **11** 1, 1–7.

Tall, D. O. (1993). Interrelationships between mind and computer: processes, images, symbols In D. L. Ferguson (ed.), *Advanced Technologies in the Teaching of Mathematics and Science*, (pp. 385–413). New York: Springer-Verlag.

Tall, D. O. (1995). Visual Organizers for Formal Mathematics. In R. Sutherland & J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education*. Springer-Verlag: Berlin.52–70.

Tall, D. O. (1997) Functions and Calculus. In A. J. Bishop et al (Eds.), *International Handbook of Mathematics Education*, 289-325, Dordrecht: Kluwer.

Tall, D. O. & Thomas, M. O. J. (1991). Encouraging Versatile Thinking in Algebra using the Computer, *Educational Studies in Mathematics*, **22** 2, 125–147.

Tall, D. O. (to appear). Cognitive development in advanced mathematics using technology.*Mathematics Education Research Journal*.

Thomas M. O. J. 1988: *A Conceptual Approach to the Early Learning of Algebra Using a Computer*, Ph.D. thesis, University of Warwick

Williams, S. R. (1991). Models of Limit Held by College Calculus Students, *Journal for Research in Mathematics Education* **22**, 237–251.

Note: Information about Graphic Calculus, and all the papers involving David Tall as author or co-author may be obtained from the website: **www.warwick.ac.uk/staff/David.Tall.**