THE FUNCTION MACHINE AS A COGNITIVE ROOT FOR THE FUNCTION CONCEPT

David Tall	Mercedes McGowen	Phil DeMarois
University of Warwick Coventry CV4 7AL, UK	William Rainey Harper College Palatine, Illinois	Mt. Hood Community College Portland, Oregon
david.tall@warwick.ac.uk	mmcgowen@harper.cc.il.us	demaroip@mhcc.cc.or.us

The concept of function is considered as foundational in mathematics. Yet it proves to be elusive and subtle for students. In this paper we suggest that a generic image that can act as a cognitive root for the concept is the *function box*. We see this not as a simple pattern-spotting device, but as a concept that embodies the salient features of the idea of function, including process (input-output) and object, with the various representations seen as methods of controlling input-output.

The notion of “function” has often been used as an organizing principle in the teaching of mathematics (Yerushalmy and Schwarz, 1993). However, the subtlety of the function concept with its various representations and process-object duality proves to be highly complex, leading not only to a concept with wide ranging powers, but also with widespread misunderstandings (see for example, Dubinsky and Harel, 1992; Sfard, 1992; Cuoco, 1994; Thompson, 1994). We consider how the function concept may be introduced in a manner which is potentially more meaningful across a wide spectrum of students with differing abilities and needs. In doing so, we develop general principles that relate to other theories of cognitive development in mathematics education. This will therefore have wider implications at a theoretical level, particularly at this point in time as technology affords us entirely new ways of interacting with and constructing conceptual ideas.

The APOS theory of Dubinsky and his colleagues, for example, sees cognitive development in the light of Piaget’s theory of reflective abstraction. APOS theory suggests that the individual first performs *actions* (on already existent objects) that are then interiorized into *processes*, later to be encapsulated into *objects* to be built into a wider cognitive *schema*. The embodied theory of Lakoff and Johnson, (1999) on the other hand suggests that *all* thought is built upon embodied perceptions and actions. A vast proportion of the brain is dedicated to vision, for the perception and analysis of objects. It is therefore natural for the brain to construct cognitive concepts not only through encapsulation of processes, but also by focusing on objects and their properties. We contend further that even the encapsulation of a process to a mental object does not occur *only* by a shift in which a process becomes conceived as an object. We suggest that other embodied mental connections are involved. For instance, a *symbol* may act as a pivot between process and concept (Gray and Tall, 1994). More generally, encapsulation will involve a much wider range of mental structure, including visual images, properties, relationships, perceptions, actions, emotions, and so on, which are already present in the mind. These will be modified and integrated as part of a conceptual object-schema that links and retains both process-driven and object-focused aspects.

Thompson (1994) suggests that an appropriate initial focus builds not from the various representations, but from a meaningful context that embodies the function concept:

> I agree with Kaput that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function. We should instead focus on them as representations of *something* that, from the students' perspective, is representable, such as aspects of a specific situation.
>
> (Thompson, (1994), p.39, (our italics))

We suggest that the "something", rather than being a variety of different contexts from which the student is expected to abstract the function aspect, could usefully be a generic embodied image that exhibits as many of the important aspects of the function concept as possible. We also intend that such an initial image should be appropriate for a wide spectrum of students.

Tall (1992, p.497) defined a *cognitive root* to be "an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built." An example of a cognitive root, is the notion of "local straightness" in calculus.

To help us formulate our theory, another relevant construct is the notion of *cognitive unit* —"a piece of cognitive structure that can be held in the focus of attention all at one time" together with its immediately available cognitive connections (Barnard and Tall, 1997, p. 41). Its power "lies in it being a whole which is both smaller and greater than the sum of its parts — smaller in the sense of being able to fit into the short term focus of attention, and greater in the sense of having holistic characteristics which are able to guide its manipulation." (Barnard, 1999, p. 4).

This allows us to propose a refined definition of the notion of cognitive root:

Definition: A *cognitive root* is a concept that:

(i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence,

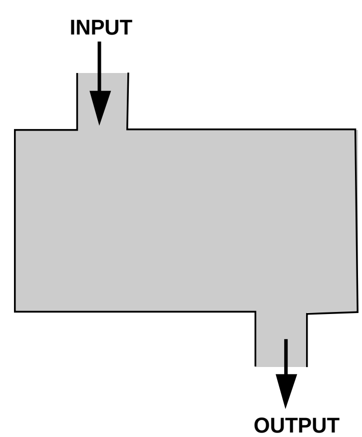
(ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,

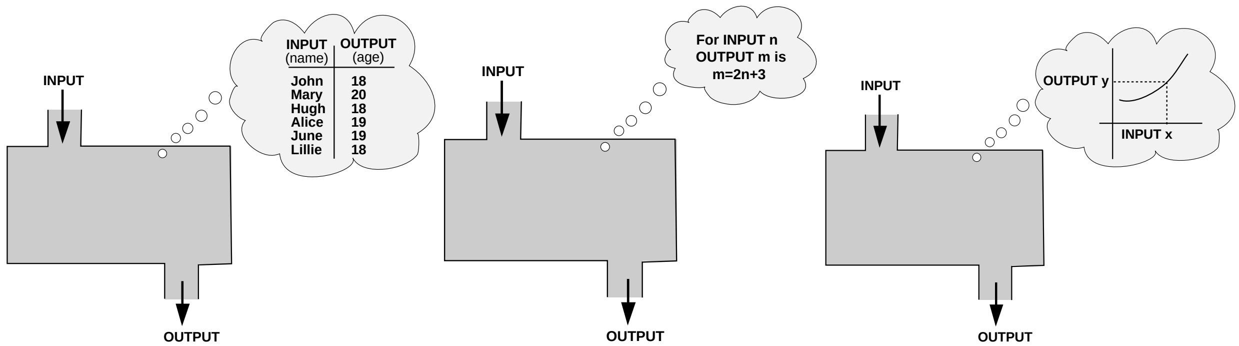
(iii) contains the possibility of long-term meaning in later developments,

(iv) is robust enough to remain useful as more sophisticated understanding develops.

A cognitive root certainly does not guarantee that every student will progress to later theoretical developments, but it has the advantage of embodying ideas which are potentially meaningful at the time (in the sense of Ausubel, Novak and Hanesian, 1968) and lay groundwork for possible later theories. As the theory develops, the cognitive root will become more sophisticated with a richer interior structure and more appropriate links to other related concepts. Some reconstruction will undoubtedly be necessary as old ideas are seen in a new light. At such times these changes may be threatening to some learners. What is important is that the curriculum designer is aware of reconstructions and their related difficulties and takes account of them in the learning sequence. It is hoped that a firmly based cognitive root will allow the learning sequence to build from meaningful foundations that may be enriched and adjusted whilst maintaining the strength of the entire structure.

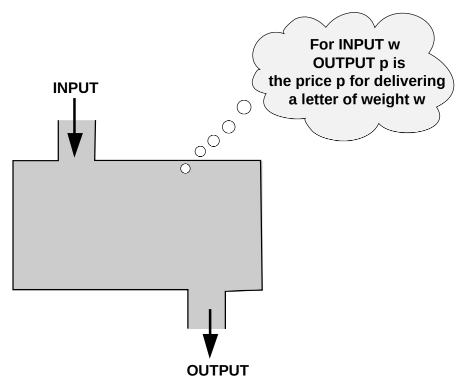
Given the complexity of the function concept, we seek a cognitive root that embodies both its process-object duality and also its multiple representations. A highly likely candidate is the *function machine* as an *input-output* box. This already has iconic, visual aspects, embodying both an *object*-like status and also the *process* aspect from input to output. The usual *representations* of function (table, graph, formula, procedure, verbal formulation, etc) may also be seen as ways of representing or calculating the inner input-output relationship:





The function box as a table, a formula and a graph

We note that the function box is often used in the early stages of the curriculum. However, this is usually as a "guess my rule" problem, to guess the internal formula expressing the rule. This activity gives rise to the epistemological obstacle that *all functions are given by a formula*. The function box may be used in a different way to retain greater generality through everyday examples with functions given by a procedure rather than a simple formula, for instance, the cost of delivering a letter of given weight.



We hypothesise that the function box in this wider context is an *embodied* version of the more general function concept. It can be imagined and represented in various ways that link directly to human perception and sensation. It allows simple interpretations of profound ideas, for instance that two function boxes are "the same" if they have the same output for each input in the domain, regardless of the particular inner workings of the box. We interpret this perception of two function boxes being the same as occurring at the *process* level in the sense of Dubinsky.

Empirical data to test the use of the function box is given in detail in a parallel presentation in these proceedings (McGowen, DeMarois and Tall, 2000). The data arose, not from a curriculum with a specific focus on the function box as a cognitive root, but from the performance of students on a college course based on various representations of linear relations using function boxes, linear equations, graphs and tables. This showed that 49% of the students began the course operating at a process level for the function box, on a par with their use of a numerical table, but considerably higher than their process use of algebra (20%) or of a graph (1%). (DeMarois, 1998, p. 147). An in-depth study of several individuals revealed successful students expanding their concept maps of function building on the function-box, whilst less successful individuals worked superficially on the current topic of study, making few long-term links (McGowen, 1998, p. 174–179). A parallel study of the growing sophistication of the meaning of a function box revealed a similar spectrum of student performance. A successful student operated at an object-level, a mid-range student acted at a process level, and an unsuccessful student was only able to use the function box in a step-by-step manner, without attending to the possibility of different procedures giving the same input-output process (DeMarois, 1998, p. 173–5).

Cognitive obstacles with the function box

The function box, as with any other initial starting point, gives rise to a range of cognitive obstacles requiring cognitive reconstruction in later developments. A major weakness is that it does not have an explicit range or domain. The domain can be introduced in a "natural" way as "the set of possible inputs", and in contexts such as real functions, there is a "natural" range, namely the real numbers. This may later embody a belief that a function will always have a "natural" domain and range, rather than the domain and range being *specifiable* in the definition. It would therefore be an advantage at an early stage to embody the function box as an input-output arrow taking the elements from a specific domain *A* into a range *B* to attempt to move closer to the formal definition.

In developing our theory, we note that the function concept itself is rarely a concept of study. Instead, the term "function" usually applies to a special kind of function—linear, quadratic, trigonometric, given by a formula, differentiable etc. We refer to such concepts as "function plus", where the "plus" refers to the relevant additional properties which significantly change the nature of a function. (For instance, a linear function only requires two pairs of input-output values to determine it uniquely). Sometimes the "plus" is extremely subtle; the graph of a real function incorporates the *order* of the real numbers on the two axes. Attempting to represent it only as sets without order would be foolish indeed! For such reasons, we see an important role for the function box as a cognitive root *before* considering specific types of function. In this new age of technology, we also consider the importance of the study of a wider range of functions that is now available in spreadsheets, symbolic manipulators and graphic calculators.

Comparison with other theories

The approach advocated has much in common with other theories, however, it reveals a significant underlying difference: the cognitive foundation of mathematical concepts is here based on meaningful scaffolding involving thought experiments with generic objects—in this case a "function box". The difference with the theories of Thompson and Kaput is a matter of emphasis. Our starting point builds out from the function-box metaphor, while their viewpoints focus either on a specific problem or on the links between several related representations.

We take a different position from the development sequence suggested by an (over-simplistic) interpretation of APOS theory. The first (Action) stage, is described as "… a reaction to stimuli which the subject perceives as external," (Czarnocha, Dubinsky, Prabhu, and Vidakovic 1999). The theory seems to intimate an initial stage in which the student does not, and cannot, have a view of the broad future development of the theory. The full schematic (S) part of the theory is, essentially, impossible to envisage until the student has reached the later stage (O) of encapsulation of objects. More recent interpretations of APOS (e.g. Czarnocha *et al*., 1999) suggest a broader dialectic in which "the development of each level influences both developments at higher and lower levels," but even this manifestly ignores the richer embodied activity of the brain (Lakoff & Johnson, 1999). Gray, Pitta, Pinto and Tall, (1999) show that a focus on objects in arithmetic leads to the less successful remaining with images and procedures, whilst the more successful develop a reflective hierarchy from primitive imagery to the powerful use of more refined mathematical ideas. At a later stage successful individuals often focus far more on the powerful higher levels with little emphasis on more primitive detail, however, this does not mean that such a level does not require a more primitive scaffolding at an early stage. It is our belief that the use of an embodied image can provide a foundation for the widest range of students, giving a good insight for some and laying a firm foundation for more subtle, highly compressed modes of thought that form the basis for more sophisticated mathematical thinking.

References

Ausubel D.P., Novak J.D. and Hanesian H. (1968). Educational Psychology: A Cognitive View (2nd Edition 1978), Holt, Rinehart and Winston.

Barnard, A. D. and Tall, D. O. (1997). Cognitive Units, Connections, and Mathematical Proof. In E. Pehkonen, (Ed.), Proceedings of the 21st Annual Conference for the Psychology of Mathematics Education, Vol. 2 (pp. 41–48). Lahti, Finland.

Barnard, A. D. (Tony). (1999), Compressed units of mathematical thought, Journal of Mathematical Behavior, 17 (4), 1–4.

Cuoco, A. A. (1994). Multiple Representations of Functions. In Kaput, J. J. and Dubinsky, E. (Eds.), Research Issues in Undergraduate Mathematics Learning (pp. 121–141). Washington, D.C.: MAA.

Czarnocha, B., Dubinsky, E., Prabhu, V., Vidakovic, D., (1999). One theoretical perspective in undergraduate mathematics education research. In O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of PME, Vol. 1, (pp. 95–110), Haifa, Israel,.

DeMarois, P. (1998). Facets and Layers of Function for College Students in Beginning Algebra, Ph.D. Dissertation, University of Warwick, Coventry, UK.

Dubinsky, E. and Harel, G. (1992). The Nature of the Process Conception of Function. In Harel, G. and Dubinsky, E., The Concept of Function: Aspects of Epistemology and Pedagogy (pp. 85–106). Washington, D.C.: Mathematical Association of America.

Gray, E. M. and Tall, D. O. (1994). Duality, ambiguity and flexibility: a “proceptual” view of simple arithmetic. Journal for Research in Mathematics Education, 25 (2), 116–140.

Gray, E.M., Pitta, D., Pinto, M, and Tall, D. O. (1999). Knowledge Construction and diverging thinking in elementary and advanced mathematics, Educational Studies in Mathematics, 38 (1–3), 111–133.

Lakoff, G. and Johnson, M. (1999). Philosophy in the Flesh. New York: Basic Books.

McGowen, M. A., (1998). Cognitive Units, Concept Images, and Cognitive Collages: An Examination of the Process of Knowledge Construction, PhD Thesis, University of Warwick, UK.

McGowen, M., DeMarois, P. and Tall, D. O. (2000). Using the Function Machines as a Cognitive Root. Proceedings of PME-NA, Arizona.

Sfard, A. (1992). Operational Origins of Mathematical Objects and the Quandary of Reification-The Case of Function. In Harel, G. and Dubinsky, E., The Concept of Function Aspects of Epistemology and Pedagogy (pp. 59–84). Washington, D.C.: MAA.

Tall, D. O. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof. In Grouws, D. A. (Ed.), Handbook of Research on Mathematics Teaching and Learning (pp. 495–511). New York: Macmillan Publishing Company.

Thompson, P. (1994). Students, Functions and Curriculum. In Dubinsky, E., Schoenfeld, A., and Kaput, J. J. (Eds.) Research in Collegiate Mathematics Education, I. (pp. 21–44). AMS.

Yerushalmy, M. & Schwartz, J. L. (1993). Seizing the Opportunity to Make Algebra Mathematically and Pedagogically Interesting. In Romberg, T. A., Fennema, E., and Carpenter, T. P. (Eds.), Integrating Research on the Graphical Representation of Functions (pp. 41–68). Hillsdale, NJ: Lawrence Erlbaum Associates.

The authors wish to thank the Reverend Gary Davis for his insightful comments in the preparation of this article.