USING THE FUNCTION MACHINE AS A COGNITIVE ROOT

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We describe students' developing understanding of function as an input/output process and as an object by tracing the internalization of the function machine concept as it relates to representations of functions We examine whether function machines serve as a cognitive root for the function concept for undergraduates enrolled in a developmental algebra course, in particular by providing a rich, foundational understanding of function.

Introduction

Students' use of expressions, tables, and graphs in understanding functions has been studied extensively over the past several decades. Much of the literature on students' concepts of function examines what they do not understand and their misconceptions, offering explanations as to why this might be so (Goldenberg, 1988; Janvier, 1987; Kaput, 1989; Tall & Bakar, 1992; Thompson, 1994). A *cognitive root* (Tall, McGowen, and DeMarois, in press) is a concept met at the beginning of a curriculum sequence that:

(i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence,

(ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,

(iii) contains the possibility of long-term meaning in later theoretical development of the mathematical concept,

(iv) is robust enough to remain useful as more significant understanding develops.

The function box can operate as a cognitive root because it is a concept that is meaningful to a significant number of students, including the majority of those who experience difficulty in mathematics and enroll for remedial college mathematics courses. The function machine box was introduced as a visual representation for the concept of function seen as an input/output *proces*s, in which, when a specific element is input, there is a single output for that input.

The introduction of the function machine as an input/output box enables students to have a mental image of a box that can be used to describe and name various processes often without the necessity of having an explicit process defined. Other forms of representation may be seen as mechanisms which allow an assignment to be made (by a table, by reading a graph, by using a formula, or by some other assignment procedure). By tracing the internalization of the function machine concept we address the question of whether use of the function machine representation leads to a rich, foundational understanding of function.

Modes of Inquiry and Data Sources

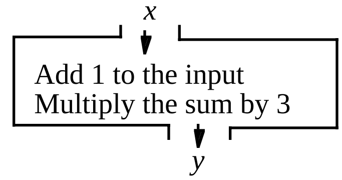
Data from two previous studies on the use of function machines (DeMarois, 1998, McGowen, 1998) are examined for evidence of the function box as a cognitive root. The subjects of these studies were undergraduate students enrolled in developmental courses that do not carry general education mathematics credit: either an Introductory or Intermediate Algebra course. Many students had encountered the content before, so these studies used a restructured curriculum centered on the concept of function using function machines. The two studies include: (a) quantitative methods of data collection used to indicate global patterns generalizable across populations to document changes in students' understanding and to measure improvements in their mathematical competencies; and (b) qualitative methods that add depth and detail to the quantitative studies which allowed the researchers to focus on individual students within the broad-based context of the quantitative studies.

All students were given pre- and post-course surveys to establish what they knew about functions initially and after sixteen weeks. Several students from each course participated in interviews subsequent to the course. At 5 week intervals in the Intermediate Algebra study, data routinely collected included student work, mid-term student interviews, and concept maps. Growth in students' understanding and improved flexibility of thought was documented in descriptions/explanations of their work throughout the semester in terms of an input/output process and their improved ability to (i) interpret and use ambiguous function notation, (ii) translate between and among various function representations, and (iii) view a function as an object in its own right. Various types of triangulation were used including data triangulation, method triangulation and theoretical triangulation (Bannister et al., 1996).

Examination of Data

A question asked on the pre- and post-course Introductory Algebra survey was:

Consider the diagram:



a. What are the output(s) if the input is 7?

b. What are the input(s) if the output is 18?

Table 1 indicates that two-thirds of Introductory Algebra students were able to interpret a function machine diagram flexibly at the beginning of the course. This suggests that the function machine representation is an accessible starting point for many students.

Table 1: Function machine input and output

Question	Pre-course number correct (% correct) n = 92	Post-course number correct (% correct) n = 92
a) Function machine: input given	62 (67%)	79 (86%)
b) Function machine: output given	44 (48%)	64 (70%)
Function machine: both parts correct	43 (47%)	61 (66%)

Students in both studies were asked on pre- and post-course surveys to find output given a graph and input. They were also asked to find input given a graph and output. The questions on the two surveys differ in some respects. The Introductory Algebra question displays a window indicating scale and the graph of a parabola. A correct response includes recognition that there are two answers to part (b). Given the form of the question, students are not required to interpret function notation in order to solve the problem. However, students were required to switch their train of thought to answer part (b). Their responses to both parts were considered a measure of their improved ability to think flexibly.

Consider the viewing window and graph copied from a TI–82 graphics calculator.

a. What are the output(s) if the input is 3?

Answer:________

b. What are the input(s) if the output is 0?

Answer:________

The Intermediate Algebra survey question asked students to determine output given the graph of a piece-wise function and an input and to determine input, given an output, using the same graph.

Given the graph

(8) Indicate what $y(8)$ = _________
What first comes to mind?

(9) If $y(x) = 2$, what is x? ________

What first comes to mind?

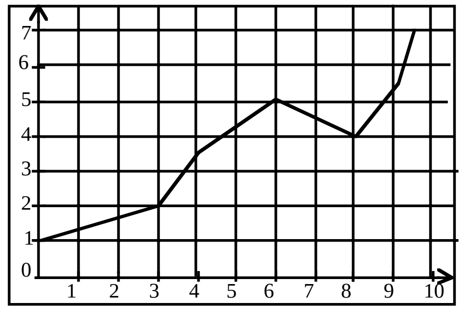


Table 2 displays the results of student responses to the survey questions:

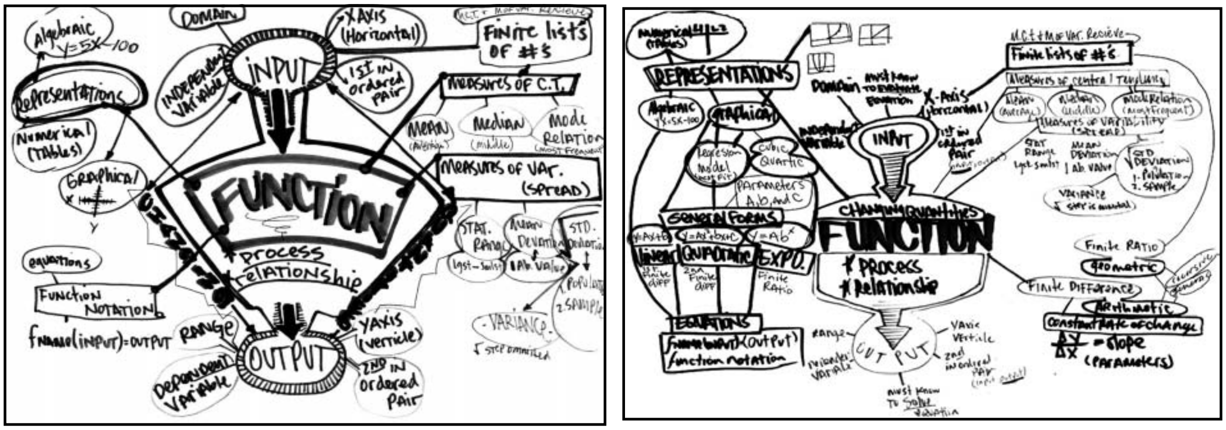
Table 2: Graph: input and output

Survey Question	Beginning Pre-course (% correct) n = 92	Beginning Post-course (% correct) n = 92	Intermediate Pre-course (% correct) n = 52	Intermediate Post-course (% correct) n = 52
Graph: input given	1% (1/92)	41% (38/92)	38% (20/52)	71% (37/52)
Graph: output given	0% (0/92)	22% (20/92)	17% (9/52)	46% (24/52)
Graph: pair correct	0% (0/92)	21% (19/92)	6% (3/52)	40% (21/52)

The results indicate that Introductory Algebra students demonstrated little connection between function machines and graphs, even when understanding of functions based on a function machine representation was demonstrated. Only 41% students were able to find output given input and only 22% were able to reverse the process at the end of the semester [DeMarois,1998]. Slightly more than 70% of Intermediate Algebra students were able to find output given input and 46% able to reverse the process by the end of the semester. Only 21% of Introductory Algebra students and less than half the Intermediate Algebra students (40%) were able to do both processes by the end of the semester. Yet, when one considers students' initial responses and their growth over the sixteen weeks, the results are encouraging. The average change in correct responses for the Intermediate Algebra students was statistically significant (two-tailed paired t-test, $p < 0.001$).

Concept maps done throughout the semester document how the function machine idea impacted students' developing concept image of function-as-process. Figure 1 illustrates how one student's concept image of function developed from the function machine as a cognitive root.

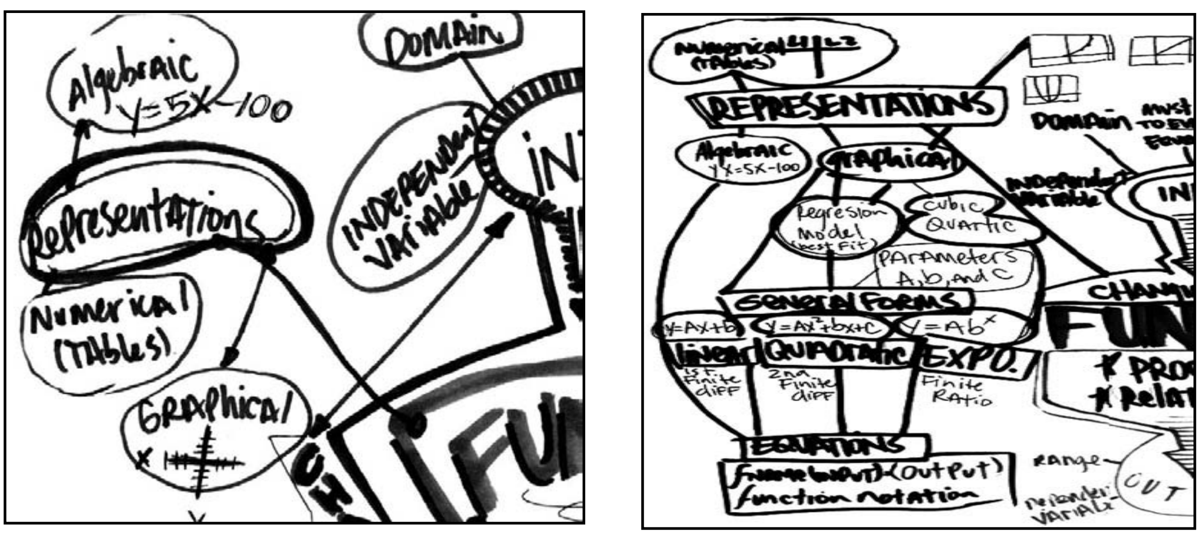
FIGURE 1: Concept Maps (Week 4 and Week 9): Cognitive Expansion



A closer examination of these maps illustrates the development of the notion of *representations* through cognitive expansion that has occurred over time. By Week 15 the student internalized the function-as-process concept. Evidence of the input/output cognitive root was still present in his final map, which was color-coded to indicate concepts connected with input or output.

FIGURE 2: Week 4, Week 9, & Week 15 Concept Map Close-ups: Representations

Week 4 Week 9



In an interview at mid-term, the student describes his growing ability to make sense of and interpret functional notation in terms of input and output:

> I'm learning how these algebraic models are set up and what the variables that they contain represent. I'm no longer just blindly solving for *x*, but rather understanding where *x* (input) came from and how it was found from the data given. Through this kind of learning I have developed an understanding for the use of function notation [$f(x)$ = output] and how it replaces the dependent variable, *y*.

On a journal (Week 9) he writes:

> I feel that I have really made sense of input and output when dealing with function notation. Problems such as #3 on the Unit II individual test used to look so unfamiliar to me, but now make perfectly good sense.

By the end of the semester, the student was able to translate flexibly and consistently among various representational forms (tables, graphs, traditional symbolic forms and functional forms) as well as express confidence in the correctness of his answers. In his final interview of the semester, the student spoke of his understanding of function notation:

> I think the most memorable information from this class would be the use and understanding of function notation. A lot of emphasis was put on input and output which really helped me comprehend some algebraic processes such as solving for *x*.

Conclusion

The evidence presented suggests that the function machine is a cognitive root for the function concept for the subject population and that function machines provide a foundation on which to further develop the function concept. Function machines impacted students' thinking and learning as evidenced in their work and by the vocabulary they used. They were able to interpret the instructions in a function machine diagram flexibly at the beginning of the courses. This suggests that the function machine representation is an accessible starting point for many students—a cognitive root at the beginning of a learning sequence that made sense as representative of the function input/output process.

For the successful students and for many of the somewhat successful students, references to input and output occur in their work and interviews throughout the semester—an indication that they use the function machine notion to organize their thinking as they work problems and interpret notation. Axes on graphs were labeled in terms of input and output as were questions using symbolic notation. The function machine provided students with access to the function concept and became a meaningful unit of core knowledge upon which to build subsequent understanding about functions. The concept maps document the cognitive expansion that occurred over time and provide evidence that the function machine as cognitive root is robust enough to remain useful as more significant understanding develops.

Further analysis of the data documents the profound divergence that occurred over time between the most successful and least successful students. Strikingly, the least successful students generally did not make use of the function machine notion except in limited instances. In contrast to the more successful students, the least successful students made very few references to function machines in their work or the vocabulary they used. The least successful students demonstrated little or no improvement in their ability to thinking flexibly. Such rigidity of thought extended to arithmetic computational processes. Their ability to reverse a train of thought appeared frozen, regardless of which representation was used. On the other hand, the most successful students demonstrated flexibility of thinking in their ability to use various representations. They were able to translate among representations, intelligently choosing among alternative procedures.

We continue to examine the usefulness of function machine as a cognitive root as students attempt to deal with the function concept. Work is on-going with students at the College Algebra level to determine how their development of the function concept compares with that of developmental algebra students, as is the search for possible cognitive roots for other mathematical concepts.

References

Banister, P., Burman, E., Parker, I., Taylor, M., & Tindall, C. (1996). *Qualitative Methods in Psychology: A Research Guide*. Buckingham: Open University Press.

DeMarois, P. (1998). *Facets and Layers of Function for College Students in Beginning Algebra*. Ph.D. Thesis, University of Warwick, Coventry, UK.

Goldenberg, P. (1988). Mathematics, Metaphors, and Human Factors: Mathematical, Technical, and Pedagogical Challenges in the Educational Use of Graphical Representations of Functions. *Journal of Mathematical Behavior*, 7(2).135–173.

Kaput, J. J. (1989). Linking representations in the symbol systems of algebra. In C. Kieran, and S. Wagner (Eds.). *Research Agenda for Mathematics Education: Research Issues in the Learning and Teaching of Algebra*. Hillsdale, NJ: Lawrence Erlbaum Publishers. 167-194.

McGowen, M. A., (1998). *Cognitive Units, Concept Images, and Cognitive Collages: An Examination of the Process of Knowledge Construction*. Ph.D. Thesis, University of Warwick, U. K.

Tall, D. O. & Bakar, M. (1992). Students' Mental Prototypes for Functions and Graphs. International Journal of Math, Education, Science, and Technology 23 (1). 39–50.

Tall, D. O., McGowen, M., & DeMarois, P. (2000) The Function Machine as a Cognitive Root for the Function Concept. *Proceedings of PME-NA*. Arizona (in press).

Thompson, P. (1994). Students, Functions, and the Undergraduate Curriculum. In Dubinsky, Schoenfeld and Kaput, (Eds.). *Research in Collegiate Mathematics Education. I. CBMS Issues in Mathematics Education. Vol. 4*. 21–44.