Making, Having and Compressing Formal Mathematical Concepts

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This paper studies the mathematical concept development of novice university students introduced to formal definitions and formal proof, with empirical data collected on "equivalence relation" and "partition". Before meeting the formal definitions of these concepts, the student will already have informal knowledge that allows some intuition as to their meaning. Our focus is on how the concepts are given a formal basis to become part of a formal theory. At first the definitions themselves will be used to establish that certain concepts satisfy the definitions and that other properties can be deduced from them. Later the statements of the theorems may be used to relate concepts without necessarily unpacking them. In this paper we consider responses to a questionnaire and categorise them as various kinds of informal and formal argument.

The introduction of formal proof in mathematics involves a significant shift from the computation and symbol manipulation of elementary arithmetic and algebra to the use of formal definitions and deduction. This shift alters the way in which language is used from an everyday informal register to a formal mathematical register (in the sense of Halliday, 1975), recently described by Alcock and Simpson (1999) as the "rigour prefix". This changes register from informal "loosely" speaking to formal "strictly speaking" in mathematics. The shift from informal to formal thinking is by no means easy. Intuitive deduction occurs using "embodied arguments" that build on bodily sensations, such as "inside inside" is "inside" (Lakoff & Johnson, 1999, p.32). Thus a formal statement such as "$A \subset B$, $B \subset C$ implies $A \subset C$" is simply "obvious". Everyday arguments often proceed by using prototypical examples referring to categories that have fuzzy boundaries (eg "short people") in the sense of Rosch (1973) and Lakoff (1987). Lakoff and Johnson argue that all thought, including formal deduction, is based on embodied conceptions, so that formal thought also includes informal elements.

The shift from informal to formal thought in mathematics does not occur in a single step. At first the mathematics is "definition-based" where proofs relate back to the fundamental definitions. It then becomes increasingly "theorem-based" as new proofs are based on previously established theorems that are no longer unpacked to the original definitions. Although the mathematician may see this as a simple programme of building up a formal theory, the student is faced with considerable cognitive reconstruction. First there is the shift from informal concepts—that already "exist" in the mind of the student and may be described verbally—to formal concepts in which the definition is given and the concepts must be constructed by deduction. Cognitively the concepts need to grow in interiority (Skemp, 1979), and become compressed (Thurston, 1990) so that they may be used imaginatively and efficiently. Even at this level, informal mental images of concepts such as "partition" may be used side-by-side with formal concepts.

In this paper we consider the cognitive growth of "equivalence relation" and "partition" at a time when students have been given the definitions and have been expected to operate in an increasingly "theorem-based" manner. We analyse whether they still operate informally, whether they are using definitions and theorems formally, or whether they have compressed the ideas of equivalence relation and partition as a single flexible cognitive unit (in the sense of Barnard & Tall, 1997).

Making and Having (Informal or Formal) Concepts

The shift to the formal mathematical register is a continual development in which the status of concepts becomes increasingly rich and formal. A formal course presents a sequence of theorems:

Theorem 1, Theorem 2, ..., Theorem N, ...,

interspersed with new definitions introduced as the theory becomes more extended. The purpose of Theorem N is to deduce the properties to be proved in its statement using the axioms, definitions and previous theorems 1, 2, ... , N–1. For the learner, a concept that needs to be proved one day becomes a concept that can be used without proof the next. Pirie and Kieren (1994) formulated the distinction between *making* images and *having* images. The formal course essentially expects the students to be *making concepts* during the proof activity, and then *having* these concepts for future development. In practice, students rarely "have" the concepts of definitions or axioms in a form that can be utilised formally as they prepare to "make" the next idea formal. Bills & Tall (1998) found that in developing the notion of "least upper bound" most of the students interviewed did not have an operable grasp of the formal definition during the ensuing parts of the course that implicitly required it. There were, however, instances of a student able to make use of an informal understanding as part of what appeared to be a formal proof.

Definition-based Formal Mathematics

Initially formal mathematical concepts are given in terms of definitions. For example, an equivalence relation may be defined as follows:

An *equivalence relation* on a <u>set</u> S is a <u>binary relation</u> on S that is
reflexive: $a \sim a$ for all $a \in S$
symmetric: if $a \sim b$ then $b \sim a$ for all $a, b \in S$
and *transitive*: if $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in S$. (Stewart & Tall, 1977)

This illustrates several difficulties faced by students being presented with formal definitions. To understand it, the student must already "have" the notion of "set" S and of "binary relation" on S. However, the first of these is not (and cannot be) given a formal definition at this stage. The second can be given informally using the informal notion of relation between two things which either holds or does not. It can also be introduced as a function from $S \times S$ to S, which now requires the notions of *cartesian product* $S \times S$ and *function*, each of which (especially the latter) has subtle cognitive difficulties (Sierpinska, 1992). The steady accumulation of concepts based on both informal and formal ideas can lead to a feeling of uneasiness in attempting to deal with them.

Theorem-based Formal Mathematics

Under this heading we consider those deductions that use the *results* of theorems without necessarily going right back to the definitions themselves. This occurs increasingly as more and more formal concepts are introduced. For example, a bijection is defined as follows:

> A function $f: A \to B$ is a *bijection* (or is a *one-to-one correspondence*) if it is both an injection and a surjection (to B). (Stewart & Tall, 1977)

Notice again that this definition requires the student to "have" the concept of injection and surjection, which in turn depend on the concept of function. However, soon after the definition is made, the following theorem is proved:

> The *identity* map is a bijection.
> The *composition* of bjections is a bijection.
> The *inverse* of a bijection is a bijection.

This theorem may be used in solving the following:

> A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection f: $X \to Y$. Is this relation an equivalence relation?

A definition-based deduction uses the original definitions of concepts. A theorem based deduction refers to theorems (in this case usually the three separate components, each matching one of the three parts of the definition of equivalence relation).

Compressed Concept-based Mathematics

Some students use their knowledge of concepts in a much more flexible and imaginative way, for instance by identifying the notion of equivalence class directly its corresponding partition within a single cognitive unit, enabling a given problem to be approached by linking directly to whichever properties are required at a given time.

Empirical Study

Thirty six students taking mathematics at one of the top five universities in the UK responded to questionnaires, 18 from a course for mathematics majors and 18 others taking mathematics in a course such as statistics or economics. Both courses covered the same material over a ten-week term with three lectures per week. The questionnaire was given out six weeks after the definitions of equivalence relation and partition had been formulated, with the subsequent time used to develop the formal theory. Two questions invited the students to say what they understood by given concepts, two more investigated the use of a definition, one in an informal context, the other in a formal context that could also involve definitions, theorems, or an alternative insightful view:

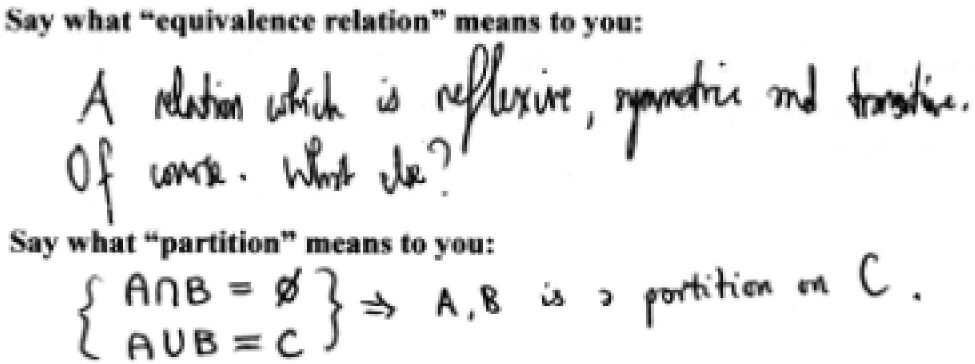
1. Say what "equivalence relation' means to you.
2. Say what "partition" means to you.
3. If M is the set of all mathematics students at Warwick, is the relation "has the same surname as" an equivalence relation?
4. A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection f: $X \to Y$. Is this relation an equivalence relation?

Student Responses

The student responses to the first two questions were analysed to see if the students gave some kind of operable definition or not. Definitions were classified as:

> *Formal/detailed*: giving an "essentially correct" formal definition in full detail,
> *Informal/outline*: either an informal verbal description, or "reflexive, symmetric, transitive",
> *Example*: giving a single specific or general example,
> *Picture*: using visual imagery in a drawing.

For instance, the following student gave an outline response to 1 and an example for 2:


Say what "equivalence relation" means to you:

A relation which is reflexive, symmetric and transitive.
Of course. What else?

Say what "partition" means to you:

$\left\{ \begin{array}{l} A \cap B = \emptyset \\ A \cup B = C \end{array} \right\} \Rightarrow$ A, B is a partition on C.

In question 1 the majority of students were able to give definitions, although many were informal, or simply specified "reflexive, symmetric, transitive", (table 1).

	Mathematics Majors (N=18)	Others (N=18)	Total (N=36)
Formal/detailed	5	6	11
Informal/outline	8	11	19
Total definitions	***13***	***17***	***30***
Example	1	0	1
Picture	0	0	0
Others	4	1	5
No response	0	0	0

Table 1: Responses to "equivalence relation"

Fewer students offered a definition of a partition, with less than half in the non-mathematics majors, (table 2).

	Mathematics Majors (N=18)	Others (N=18)	Total (N=36)
Formal/detailed	6	5	11
Informal/outline	8	2	10
Total definitions	***14***	***7***	***21***
Example	1	1	2
Picture	0	2	2
Others	2	6	8
No response	1	2	3

Table 2: Responses to "partition"

Question 3 was formulated in an informal context to see how the students would respond using the formal notion of "equivalence class", (table 3).

		Mathematics Majors (N=18)	Others (N=18)	Total (N=36)
Informal	Informal Definition	5	11	16
	Other	1	1	2
	No response	0	1	1
Formal *perhaps with some informal language*	Definition	9	4	13
	Theorem	0	0	0
	Partition	3	1	4

Table 3: Responses to the informal "surnames" question

Sixteen out of thirty six were classified as operating informally; they either reproduced the definition with no reference to the problem (eg $a{\sim}a$; $a{\sim}b \Rightarrow b{\sim}a$; $a{\sim}b,\ b{\sim}c \Rightarrow a{\sim}c$) or they responded in an informal prototypical manner:

refl : Smith = Smith ✓
trans: Smith (A's) = Smith (B's) = Smith (C's) ✓
symm: Smith (A's) = Smith (B's) ⇔ Smith (B's) = Smith (A's) ✓

Thirteen responded in a more formal manner, using set theoretic formalism, eg:

✓① $\forall m \in M$ $(m,m) \in \sim$ ⇒ You have the same surname as you. TRUE
✓② If $(m,n) \in \sim$ then m and n have the same surname
If $(n,p) \in \sim$ then n and p have the same surname
So m and p must have the same surname ⇒ $(m,p) \in \sim$
✓③ If $(m,n) \in \sim$ then m has the ... as n ⇔ n has the ... as m ⇒ $(n,m) \in \sim$

No responses used theorems (because the problem focused on the use of the definition), however, four responded in terms of partitions:

This partitions the set of maths students into sets containing students with the same surname. As there is a bijection between partitions and equivalence relations, this too is an equivalence relation.

Question 4 revealed a wider spectrum of responses, (table 4).

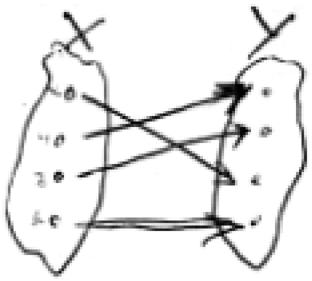
Twenty one gave some kind of informal response, including those who simply wrote down an outline definition:

refl symm trans

		Mathematics Majors (N=18)	Others (N=18)	Total (N=36)
Informal	Informal Definition	7	8	15
	Other	2	3	5
	No response	0	1	1
Formal *perhaps with some informal language*	Definition	2	0	2
	Theorem	5	6	11
	Partition	2	0	2

Table 4: Responses to the formal “bijection” question

Others were unable to make sense of the question, eg:



No, not necessarily as $x \sim x$ $\forall x$ doesn't always happen.

Only fifteen gave a formal response (often using at least some informal language). Of these, eleven were *theorem-based*, using the parts of the theorem concerning bijections:

Identity bijection $X \to X$ so $X \sim X$
$f\, X \to Y$, $g\, Y \to Z$ bijections $\Rightarrow \exists$ bijection $h: X \to Z$ $h = g \circ f$
So $X \sim Y$, $Y \sim Z \Rightarrow X \sim Z$.
$X \sim Y \Rightarrow \exists f : X \to Y$ bijection so $f^{-1}: Y \to X$ is also bijective
So $X \sim Y \Rightarrow Y \sim X$.

Of these eleven, only three explicitly mentioned the identity to establish the “reflexive” property, the others only asserting, at most, that there *is* a bijection. This is a most interesting phenomenon worthy of further study. The students seem to be more comfortable giving a *general* argument than using a specific case.

Just two students were *definition-based*, referring back to the definition of bijection:

There is a bijection mapping each element of X to itself Therefore reflexive.
If every element of X maps to a unique element of Y, then each element of Y maps to a unique element of X Therefore symmetric.
If every element of X maps to a unique element of Y, and every element of Y maps to a unique element of Z, then every element of X maps to a unique element of Z, therefore transitive.

Neither of these referred explicitly to the identity function to establish reflexivity.

Two students were classified as compressed concept-based, using the notion of partition as being equivalent to the notion of equivalence relation. One explicitly referred to the theory of cardinal numbers, although his response was not fully formal:

> This partitions the set of sets into sets which contain sets with the same cardinality (ie have a bijection between them). Thus, as this is a partition it is also an equivalence relation on cardinal numbers.

This particular student shows great flexibility in response, treating the concepts of equivalence relation and partition as a single cognitive entity, using whichever properties were appropriate at the time (eg in another question, not given here, he responded in terms of properties of equivalence relations). His definitions of equivalence class and partition were a model of structured precision. He defined equivalence class first at an outline level, then gave the full detail; he described a partition first in a precise verbal way and then repeated it in strict symbolism. He seemed able to operate at a level of his choice, preferring to work at a high conceptual level with the detail implicitly subsumed in his arguments. This is an example of someone working with the concepts concerned compressed as a single highly-connected cognitive unit.

Summary

The responses to the questionnaire reveal that after the students have been working formally with the notions of equivalence relation and partition for six weeks, more than half of them offered only *informal* responses. Less than half gave *formal* responses in terms of *definitions* or *theorems*. Four others gave broader conceptual responses to question 3, falling to 2 students in question 4 (the other two giving formal theorem responses). This confirms a picture in which the majority of students following a formal course at a highly rated university responded at an informal level after several weeks' experience of formalism. At the same time, two able students worked in a different way using the *compressed concept* that encompassed both equivalence relation and partition.

These findings relate closely to other theories. The definition/theorem approaches use *analytic* powers to *deduce* properties, the formal global category uses *synthetic* powers to *relate ideas together*. This is consonant with the theory of *extracting meaning* (from the definitions) and *giving meaning* (to the concepts) as formulated by Pinto & Tall (1999). It also relates to the process/object phenomena identified by Dubinsky and his colleagues in his APOS theory approach (eg. Asiala, Dubinsky *et al.*, 1997).

The research of Moore (1994) formulates a framework based on "definition - image - usage" and gives many fascinating insights into the usage of images and definitions in formal mathematics. In a sense his distinction between the use of images and the use of definitions has similarities with our focus on informal and formal thinking. However, his paper uses an interpretation of "concept image" which contrasts definition and image as distinct entities. For us the concept image *includes* the definition and its resulting related imagery. This allows us to formulate an ongoing change of the total concept image that steadily builds up the formal register.

Our research instrument—a single questionnaire applied at one point in a development—is too restricted a tool to give answers to other questions, in particular, whether there is a hierarchy running through the given categories. The evidence of Pinto & Tall (1999)—taken from an analysis course studied in the same institution—suggests that there is a spectrum of approaches. Some students do not go beyond the informal stage, some go through the stages in the given sequence as a hierarchy by *extracting* meaning from the definition. Others perform thought experiments from the very beginning, building up their theoretical perspective by modifying their images and *giving* meaning to the definition and its subsequent deductions.

This research, together with other sources mentioned in this paper, shows the difficulty of building the formal register in a first university course in mathematics. Perhaps matters could be improved by explicitly encouraging students to gain an overall view of the strategies involved in the transition to the formal register. However, using the memorable phrase of Sfard, (1991) this may involve a "vicious circle" where the strategy to understand formal proof is difficult to comprehend until the student has experienced formal proof itself. This learning strategy remains an investigation for another time.

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