# Technology and Cognitive Growth in Mathematics

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## Introduction

At this conference we are considering the use of new technologies in mathematics. Many presenters will show the use of a wide variety of technological environments — symbol manipulators, geometry environments, graphical facilities, spreadsheets, statistical packages, interactive productivity tools, electronic books, multi-media, and so on. These enable us to solve old problems in new ways and to confront entirely new scenarios. My own part in this discussion is to help focus on what is going on in the individual's mind when *using* computer software. A vital part of such a study includes the thinking of *professional mathematicians*; this paper will focus on how *students* think and learn with technology.

## Mathematical Thinking

Thinking processes amongst professional mathematicians vary enormously, from those who are logically based—building from axioms in a coherent and steady way—to those who use a wide range of intuitions to suggest entirely new mathematical theories, with others who combine the two in various ways. There are those who are theoretically based, developing mathematics for its intrinsic value and those who see it as a tool for use in a wide range of applications.

Recent work of Dehaene (1997) and subsequent announcements in the press reveal his evidence from brain scans that individuals doing arithmetic cover a spectrum from those who use only the language centre (including mathematical symbols) in the left brain whilst others combine this with the visual cortex to support their thinking. This counsels us that there are genuine differences between individuals, which need to be considered in using technology for providing an environment for learning.

Some activities in mathematical thinking are explicitly seen as part of mathematics, such as using numeric, symbolic and graphical methods to carry out computations or represent mathematical ideas, and axiomatic definitions and deductions to build up formal theories. There are other, deeper, human activities that act as a basis for all thought. These include the use of verbal language,

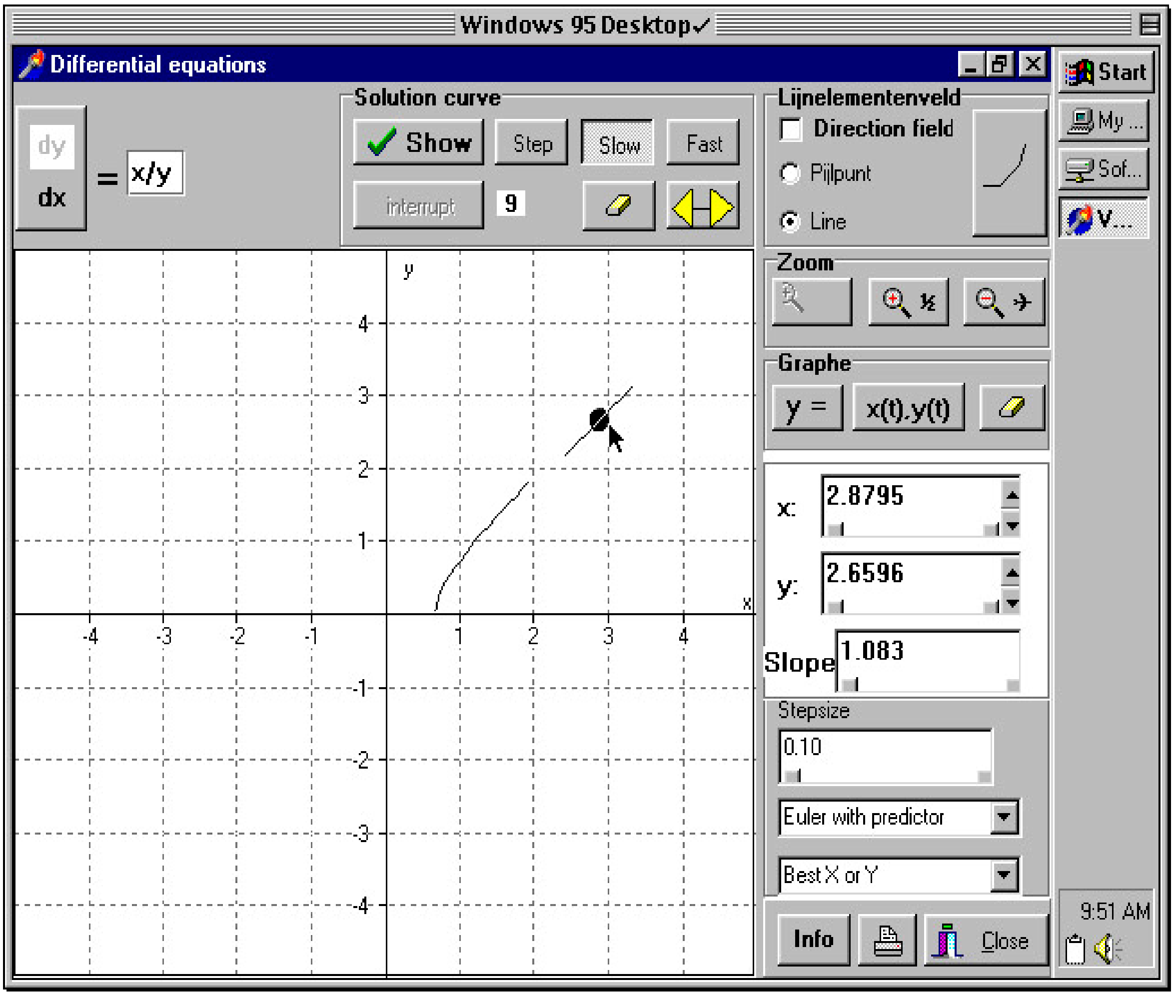


Figure 1 : Enactively building the solution of a differential equation

visual imagery, and sensori-motor activity (physical sensations and bodily movements).

All of these play an important part in the computer interface. For example, the sensori-motor system involves stored actions that allow decisions to be made intuitively using mouse and keyboard. These low-level cognitive actions also provide support for high-level theoretical concepts. Consider, for example, figure 1 which shows software to build graphical solutions to (first order) differential equations by using the mouse to move a small line segment whose slope is determined by the differential equation. A click of the mouse deposits the segment and the user may fit line segments together to give an approximate solution.

Such an activity can be performed intuitively with little knowledge of the theory of differential equations. *Yet it already carries in it the seeds of powerful existence* theorems—that if the slope is defined at a point, then there is a unique solution in that direction, and that the likely places where things might go wrong is at (or near) places where the gradient is not defined. Thus the intuitive interface can give advance organisers for formal theory, especially to those individuals who naturally build on visual imagery.

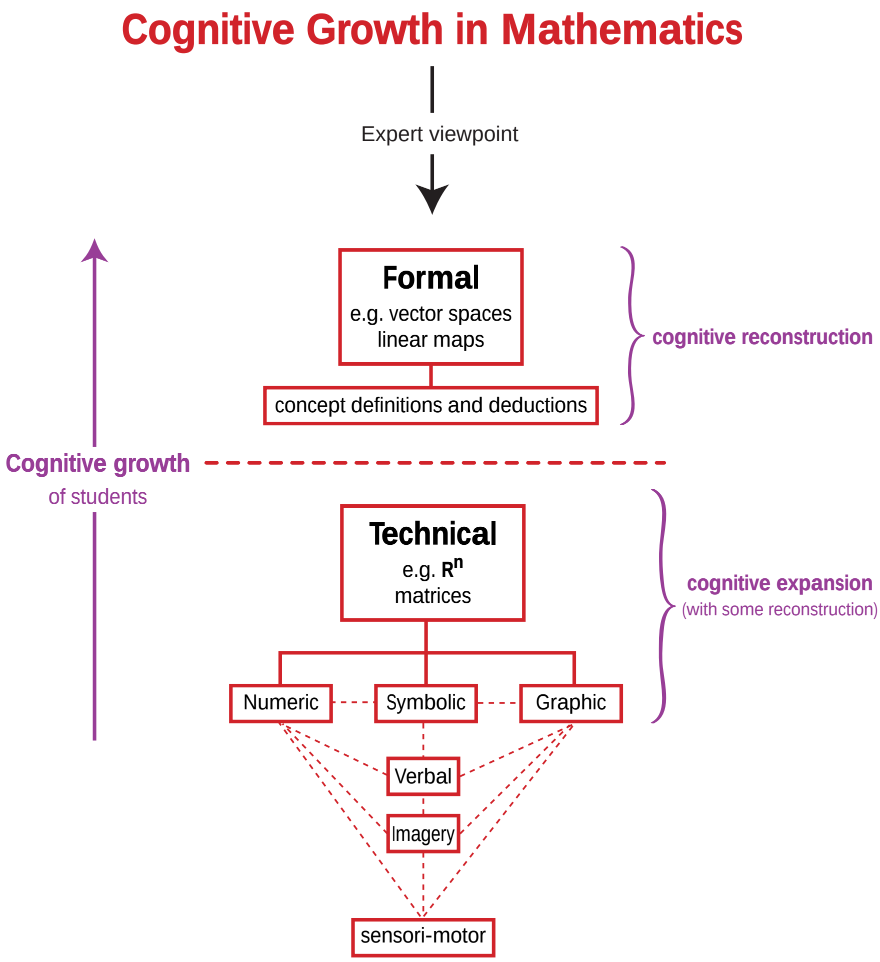


Figure 2: Cognitive development in advanced mathematics

In the same way, those interested in the applications of mathematics—using statistical packages, spreadsheets, or symbol manipulators to solve problems involving real data sets—may benefit from intuitive insight gained by exploring the possibilities inherent in the software.

The cognitive growth of advanced mathematical thinking builds not only on the underlying human cognitive abilities, but also on the interface and facilities provided by the software (figure 2). These include the usual three major representations afforded by the computer—numeric, symbolic and graphic—which can be used to extend the individual's perceptions of mathematical ideas. It is important to underline that coping with new ideas can occur in two radically different ways which Skemp (1971) calls the *expansion* of current ideas with little modification and the more difficult *reconstruction* of old ideas to fit with new experiences that clash with previous experience. The latter includes the difficult transition to axiomatic theories where concept definitions are used as axiomatic starting points to construct mathematical theories. As students grow through what I term the *technical* aspects of mathematics to the *formal* aspects, there is a huge cognitive reconstruction required which can cause such difficulties that few students make the transition (figure 3).

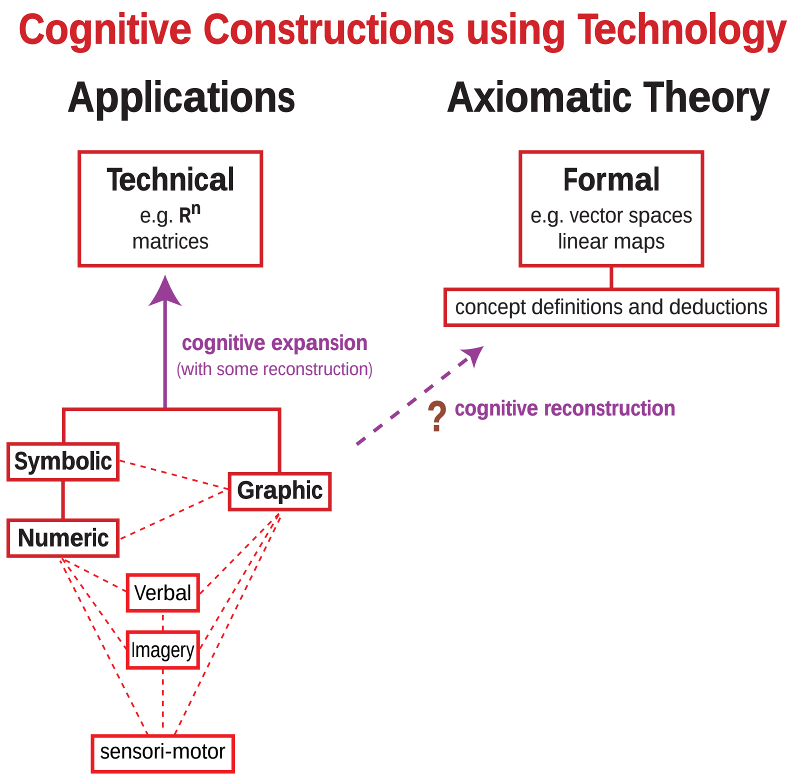


Figure 3: Is the use of technology going to emphasise the *technical* rather than the *formal*?

In using technology, we must ask precisely what imagery does it give to the user? Does it, for example, emphasise the *technical* at the expense of the *formal*? We may also ask if this matters. It may not matter at all to users of mathematics but it may have serious implications for those who wish to generate the pure mathematicians of the future.

In the remainder of this presentation I will report empirical research into the way in which students build concepts when using technology. First I shall consider the role of numeric and symbolic representations which are profoundly affected by the use of technology. Before the advent of the computer the individual had to perform all the computations of arithmetic and manipulations in algebra by hand (or, more appropriately, by brain). Now calculations and manipulations can be carried out effortlessly by the computer. This is wonderful for the *expert*, but what does it do for the mathematical thinking of the novice?

I will show a bifurcation between those who use symbolism in a procedural way, who may not build up a manageable cognitive structure to think about novel mathematical problems, and those who see symbolism as a flexible pivot between mathematical concept and mathematical process.

Second, I will consider links between visual and symbolic representations. Visual representations (especially those under control of the individual) may give helpful conceptual insight of value in its own right, or sometimes (but not always!) offer useful links to symbolic and formal theories.

## Symbols as process and concept

I have spoken and written often in the last decade on the vital role of symbols in arithmetic, algebra and calculus. This does not just involve precise definitions, but also has a measure of duality between concept and process, even an *ambiguity* of meaning which gives great power to switch between different conceptions (Gray & Tall, 1994, Gray *et al*, in press). This uses the notion of *procept* as a combination of symbol, process and concept in which the symbol can represent both and be used as a mental pivot between them.

There is a spectrum of usage of symbolism that can be described as follows:

(a) a ***procedure*** consists of a finite succession of actions and decisions built into a coherent sequence. It is seen essentially as a step-by-step activity with each step triggering the next.

(b) the term ***process*** is used when the procedure is conceived as a whole and the focus is on input and output rather than the particular procedure used to carry out the process. It may be achieved by $n$ procedures ($n \geq 0$) and affords the possibility of selecting the most efficient solution in a given context.

(c) a ***procept*** requires the symbols to be conceived flexibly as **pro**cesses to *do* and con**cepts** to *think about*. This allows for more powerful mental manipulation and reflection to build new theories.

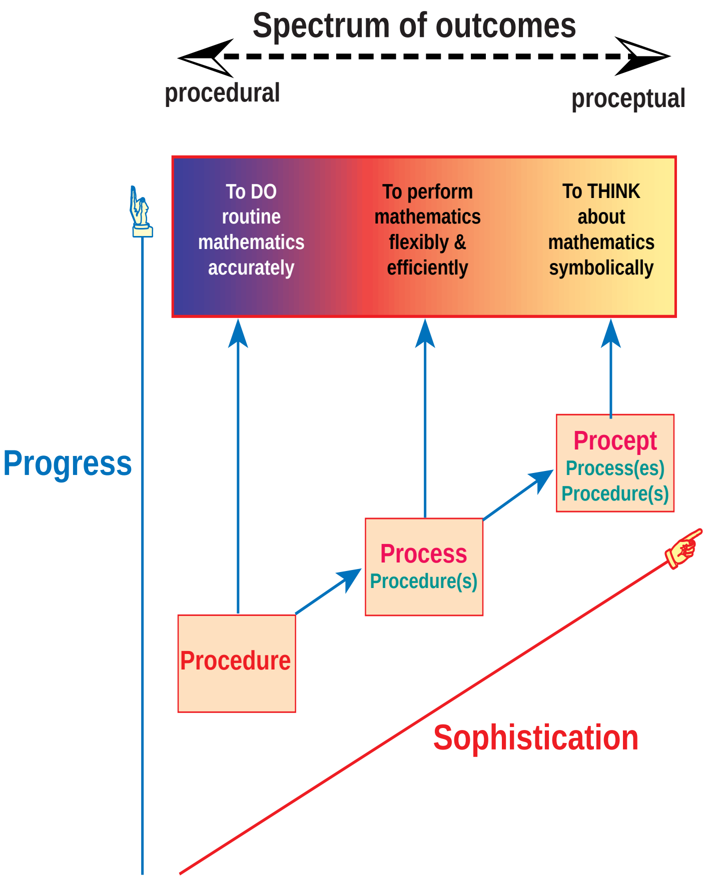


Figure 4: Differing levels of sophistication in using symbols

Different uses of symbolism give rise to differing levels of flexibility and ability to think mathematically (figure 4). This is *not* to say that procedural thinking does not have its value. Indeed, much of the power of mathematics lies in its algorithmic procedures. However, a focus on procedures *alone*, without conceptual linkages between them leads to increasing cognitive stress as the individual learns more and more disconnected pieces.

The inability to cope conceptually seems to increase throughout the curriculum. My own perception of these difficulties is that the underlying procepts act in very different ways, so that the learner, who has internal methods of processing the ideas, finds new ideas strangely conflicting with inner beliefs. This, I believe, leads to a lack of connections and the desire to learn procedures solely to pass examinations. A brief outline of the differences between arithmetic, algebraic and limit concepts is shown in figure 5.

***Arithmetic Procepts***

$5+4$, $3\times 4$, $\frac{1}{2}+\frac{2}{3}$, $1\cdot 54 \div 2\cdot 3$,

have **computational processes** and **manipulable concepts**.

***Algebraic Procepts***

$2+3x$, $ax^2+bx+c$

have **potential processes** (evaluation), and **manipulable concepts**.

***Limit Procepts***

$$\lim_{x\to a}\frac{x^3-a^3}{x-a},\ \sum_{n=1}^{\infty}\frac{1}{n^2}$$

have **potentially infinite processes** (of evaluation). These may have *no* finite procedures of computation (which is equivalent to *n* algorithms where *n* may be 0). The **concepts** are **(sometimes) manipulable**, (using rules or theorems of limits).

Figure 5: different kinds of symbol use

This reveals the discontinuities that the growing individual may sense in passing from arithmetic to algebra to analysis. Unlike the symbols in arithmetic which have built-in processes of computation, algebraic symbols (expressions) have only a *potential* process of evaluation (when the numerical values of variables are known) and so may "seem strange" to the developing student. Yet they are expected to manipulate these “strange” things as meaningful objects. Limit procepts have a *potentially infinite* process of evaluation and carry a sense of “getting close” or “getting large” or “getting small”; this often leads to the conception of *a number line with infinitesimal and infinite quantities.*

## Empirical evidence

### 1. Example from USA College Algebra

The students involved in this experiment from the doctoral thesis of DeMarois (1998) followed a reform curriculum in (pre-)college algebra based on the function concept using graphing calculators. A question that revealed the spectrum of responses indicated in figure 5 is as follows:

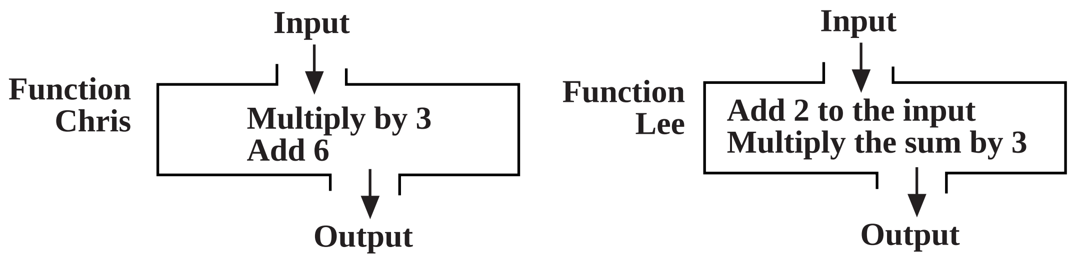


Figure 6: Write the outputs of these two function boxes and say if they are the same function.

The responses given by students achieving grades A, B, C were as follows:

- ***Grade A Student***: **3x+6, 3(x+2)**

  Yes, if I distribute the 3 in Lee, I get the same function as Chris. **(procept)**

- ***Grade B Student***: **x3+6, (x+2)3**

  Yes, but different processes [procedures]. **(process)** in my terminology

- ***Grade C Student*: 3x+6, x+2 (3x)**

  No, you come up with the same answer, but they are different processes [procedures]. **(procedure)**

(Phil DeMarois, PhD Thesis, University of Warwick, 1998)

Although these responses show the A student being proceptual, the B student processual and the C student procedural, the situation is actually much more diverse than this (see next example). In practice the most successful students reveal a greater tendency to use proceptual methods, but may also use processes or procedures, whereas the least successful often use limited procedures.

**2. Example from Malaysia (standard syllabus)**

The students in this study were taking a traditional symbolic calculus course without computers. These students were taken from the $50^{th}$ to the $90^{th}$ percentiles in the national ability range. Some of them used mainly the rules of calculus in a direct, procedural, manner, others were more flexible. This is illustrated using the following:

> Determine the derivative of $\dfrac{1+x^2}{x^2}$,

The direct application of the quotient rule gives:

$$y = \frac{1+x^2}{x^2},$$

$$\frac{dy}{dx} = \frac{(2x)(x^2)-(2x)(1+x^2)}{(x^2)^2} = \frac{2x^3-2x-2x^3}{x^4} = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

A little *conceptual preparation* to develop an alternative, more efficient approach involves simplifying the quotient to $x^{-2}+1$ and obtaining the derivative as $-2x^{-3}$.

| Student's grade | Procedural rule **[PROCEDURE]** | Conceptual preparation **[PROCESS]** |
|---|---|---|
| A | 2 | 10 |
| B | 6 | 6 |
| C | 8 | 4 |
| Total | 16 | 20 |

The difference between A and C grade students is significant at the 5% level (using a $\chi^2$ test).

(Maselan Bin Ali, PhD Thesis, University of Warwick 1997)

## 7. Example from British School Students using *Derive*

In this small study (Sun, 1993), a group of nine 16-year olds were taught calculus with Derive and nineteen others were taught separately using a standard course (with 9 of them selected to match the Derive group).

There was a clear difference between the ways in which the two groups responded to problems. In general the Derive-based students often used Derive to perform algorithms, but they were also likely to explain their ideas in terms of the sequence of buttons to press:

> Please find the following limits if they exist. If there is no limit, write 'no'.
>
> Please explain your results.
>
> (a) $\lim\limits_{x\to\infty} \dfrac{2x+3}{x+2}$
>
> (b) $\lim\limits_{x\to 1} f(x)$ where $f(x) = \begin{cases} 3 & \text{if } x = 1 \\ \dfrac{x^2-1}{x-1} & \text{if } x \neq 1 \end{cases}$

In question (a), 8 out of 9 Derive students used the software correctly to compute the result, the other "did not like computer systems". Of the non-derive students, only 12 out of 19 obtained the limit by dividing numerator and denominator by $x$ and letting let $1/x \to 0$. Three others substituted numbers, four did not respond.

Question (b) was novel to all students. Of the Derive students, 6 out of 9 focused on the second formula and computed the limit without making any theoretical comment. Of the non-Derive students, eleven attempted to address the discontinuity, two gave a solution focusing only on the rational expression and six left it blank.

The following question revealed interesting differences in responses:

> Please explain the meaning of $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

*None* of the Derive students gave any theoretical explanation; four related the sequence of keystrokes for a specific formula of their choice. *All* of the non-Derive students (who had discussed this notation in their course) gave a satisfactory explanation, at a general level, without using any specific examples.

This little experiment does not *prove* anything. However, it does intimate that the use of the software in the experiment focused on the specific procedure of calculating solutions (which is sometimes advantageous) but is, in this experiment, less successful with the conceptual ideas.

## 4. Example of British Student Teachers programming in structured BASIC

These students programmed sequences and series as numerical functions (which could be given recursively, or in terms of a for:next loop) and investigated the convergence of various sequences to build up an experiential basis for the theory. Initially they had a variety of images of the process of "tending to" a limit and the limit concept itself. The course seemed to improve their understanding of the limit as a number in one example, but failed to change their notion of the infinite decimal "point nine recurring".

> Complete the following sentences:
>
> 1, 1/2, 1/4, 1/8, … tends to ____________
>
> The limit of 1, 1/2, 1/4, 1/8, … is ____________

The responses are as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Tends to … | 0 | 0 | 1/∞ | 0 | 2 | 0 | 0 |
| Limit is … | 0 | 1/∞ | 1/∞ | ? | 2 | 2 | 1 |
| Pre-test (*N*=25) | 0 | **11** | 1 | 5 | 0 | 2 | 2 |
| Post-test (*N*=23) | **8** | 3 | 3 | 0 | 4 | 0 | 2 |

(The response "2" almost certainly indicates the sum of the *series* $1+\frac{1}{2}+\frac{1}{4}+\ldots$. The response "1" for the limit may be the "largest term" which "limits" the upper value attained.)

The most commonly occurring response changed from

"tends to 0, limit 1/∞" to "tends to 0, limit 0",

suggesting 1/∞ as an arbitrarily small quantity, is being replaced by the numeric limit 0. However, "0·9 repeating" did not change its image:

| ***Is*** $0\cdot\dot{9}=1$ ***?*** | Y | **N** | ? | no response |
|---|---|---|---|---|
| pre-test (*N*=25) | 2 | **21** | 1 | 1 |
| post-test (*N*=23) | 2 | **21** | 0 | 0 |

(Lan Li, MSc Thesis, University of Warwick, 1992)

### What are the links between symbolic and visual?

There is considerable evidence showing the value of visual representations helping conceptualisation (eg the work of Heid (1988), Palmiter (1991), Tall (1990, 1992, 1997), Uhl (1999)). There is also considerable evidence in the literature that students do not necessarily link symbolic and visual representations in the way an expert might expect (Tall, 1997). For instance, in asking the question:

| | |
|---|---|
| Find roots and asymptotes of: $f(x) = \frac{x(x-4)}{(x+2)(x-2)}$ | Caldwell (1995) |

produced many responses such as 0.01 and 3.98 using a graphing calculator. The students drew the graph, looked at the picture and saw the numerical values displayed without relating back to the precision of the algebra.

Another study revealed students unable to reconcile the picture given on a graphing calculator with images related to the algebra:

| | |
|---|---|
| Draw a graph of $f(x) = \frac{x^2+2x-3}{2x^2+3x-5}$ | Boers and Jones (1993) |

This has a removable discontinuity at $x$=1. More than 80% of the gave an inappropriate response, for instance drawing an asymptote suggested by the zero in the denominator, despite the graphic evidence of the calculator:

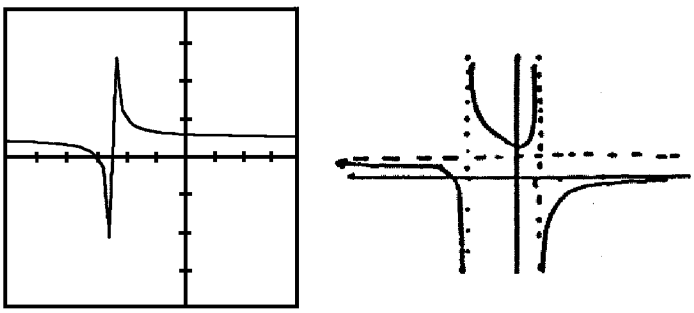

Figure 7: Graphic calculator display and student graph

### Summary

Although we can use new technologies in imaginative ways to carry out processes that were previously impossible, if we are to use it in *teaching* mathematical concepts, we need to observe what it is that students actually *learn*. The evidence is that they learn by building up mental imagery that operates in ways that are somewhat different from the mathematical ideals held by experts. In particular there is a spectrum of different ways in which the students conceptualise mathematics that must be taken into account.

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