Student constructions of formal theory: giving and extracting meaning

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*This paper describes an analysis of student constructions of formal theory in university mathematics. After a preliminary study to establish initial categories for consideration, a main study followed students through a twenty-week Real Analysis course, interviewing individuals at regular intervals to plot the growth of their knowledge construction. By focusing on the students constructions of definitions, arguments and images, two distinct modes of operation emerged—**giving meaning** to the definitions and resulting theory by building from earlier concept images, and **extracting meaning** from the formal definition through formal deduction. Both routes may be successful or unsuccessful in constructing the formal theory.*

Advanced mathematical thinking is so vast an enterprise that different individuals focus on different kinds of activities. One mathematician might focus on "thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs." Another may extend formal theory already developed by "getting and understanding the needed definitions, working with them to see what could be calculated and what might be true to finally come up with new 'structure theorems'," (MacLane, 1994, p. 190–191). The division of labour between those "guided by intuition" and those "preoccupied with logic" was noted by Poincaré (1913), citing Riemann as an intuitive thinker who "calls geometry to his aid" and Hermite as a logical thinker who "never evoked a sensuous image" in mathematical conversation (p, 212).

So how can we expect students to fully understand all the processes of advanced mathematical thinking when mathematicians themselves must specialise in only part of the total enterprise? This research project began with a preliminary study analysing written work and interviews with students to establish basic categories for analysis. It was founded on theory in the literature of advanced mathematical thinking (e.g. Tall, 1991, and subsequent developments). Few of the students concerned proved to have a grasp of the formal theory, exhibiting imagery already studied in the literature. The main study was designed to cover a wider spectrum of students, including highly successful ones. Students were interviewed at intervals on seven occasions through a twenty week first year course on Real Analysis. The methodology uses a form of theory construction following the style of Strauss (1987), Strauss & Corbin (1990). It begins by reviewing data and attempting to categorise it, re-evaluating the categorisations to fit the data collected until it falls into a natural structure that is grounded in the available data.

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Preliminaries

A preliminary categorisation was considered in which students:

1. become acquainted with the definition,
2. use the definition to deduce results,
3. use the results in further theorems to build up systematic theories.

This may be summarised under the successive headings:

1. DEFINITIONS,
2. DEDUCTIONS,
3. SYSTEMATIC THEORY.

However, the cognitive processes proved to be more intimately interconnected. To truly understand the nature of a definition requires the use of deductions to construct its implications. There is therefore an important interplay of the form

DEFINITIONS <—> DEDUCTIONS.

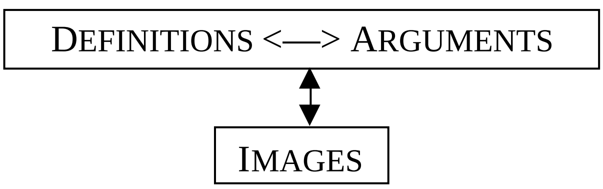
To take account of this observation, Bills & Tall (1998) defined:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.

In a preliminary study, Pinto (1998) analysed the final assessments of twenty student trainee teachers. Only *three* based any arguments on definitions and only *one* used a formal definition in an operable manner. The remainder gave informal justifications often based on a particular case. To take account of this spectrum, the heading DEDUCTIONS was modified to ARGUMENTS to include all types of justification, and the main study focused on DEFINITIONS, DEDUCTIONS and underlying MISCONCEPTIONS. The negative tone of the third category was later modified to focus on:

- DEFINITIONS,
- ARGUMENTS,
- IMAGES.

The first two headings were analysed in turn with each being related to underlying concept images as follows:



The students chosen for the main study were selected using a test designed to provide a full spectrum of students following a first year pure mathematics course including potential high and low achievers. The students were interviewed on seven occasions throughout a twenty-week course. All interviews were transcribed from the tapes and significant episodes selected to be coded and organised into a classification system. The initial coding system followed the plan of themes highlighted by the exploratory study:

- DEFINITIONS given by each student were classified as descriptive, correct formal or distorted formal,
- ARGUMENTS were categorised as being based on concept images or based on the formal theory presented,
- IMAGERY, as evoked by the students, was classified as to whether it was apparently constructed from the formal theory or not.

Given the differences between the informal approaches of the students in the preliminary study and the desired formal theory, the responses initially were classified as follows:

Approach:	DEFINITIONS	ARGUMENTS	IMAGERY
informal	descriptive	based on concept image	not constructed from definition
formal	formal (correct or distorted)	based on formal theory	constructed from definition

Table 1

This analysis, however, was revised when two distinct approaches were found to occur:

- ***giving meaning*** *to* the concept definition from concept imagery,
- ***extracting meaning*** *from* the concept definition by making formal deductions.

Although reminiscent of the earlier-mentioned approaches of research mathematicians, they differ because students are given the definitions as starting points. However, there are certain parallels. *Giving* meaning involves using various personal clues to enrich the definition with examples often using visual images. *Extracting* meaning involves routinising the definition, perhaps by repetition, before using it as a basis for formal deduction. This led to a new categorisation (table 2) where giving meaning could lead to formal theory or fail by remaining image-based, while extracting meaning could be done either reflectively or mechanically, leading again to a spectrum of success or failure.

Approaches	Concept Construction			
Strategies	Characteristics	DEFINITIONS	ARGUMENTS	IMAGERY
Giving meaning (building from *informal* ideas)	1. Reconstructing old knowledge to give new 2. Interpreting new knowledge in terms of old	1. Formal: • correct • distorted 2. Descriptive: • general • prototype • specific	1. Based on thought experiments: • formally presented • image-based 2. Rote-learned	1. Reconstructed with the formal theory 2. Old images retained 3. New ideas added as extra information 4. Conflict between old and new
Extracting meaning (building from *formal* theory)	Routinising: • reflective • mechanical where either may remain compartmentalised or later be linked to old knowledge	Formal: • correct • distorted	Based on formal theory: • meaningful • rote-learned	Based on formal theory: • compartmentalised • linked

Table 2

Students building operable definitionsby giving and extracting meaning

In the main study some individuals used both approaches at different times, but many showed a distinct preference for one approach. For instance of two highly successful students, Ross was categorised as an *extractor* of meaning and Chris, a *giver* of meaning.

In his first interview, Ross wrote down the definition as follows (Pinto, 1998):

A sequence (a_n) tends to limit L if, $\forall\ \varepsilon>0,\ \exists\ N(\varepsilon)\in\mathbb{N}$ s.t. $\forall\ n\geq N;\ |a_n - L| < \varepsilon$. (Ross, first Interview)

He explained that he coped by:

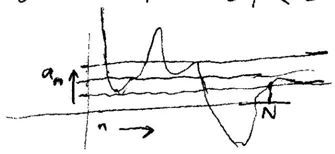
> "Just memorising it, well it's mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it get imprinted and then I remember it." (Ross, first interview)

Throughout the course he constantly attempted to prove results from formal definitions, seeing links with earlier established results until towards the end when he began to slip behind the pace of the lectures. At such times he might consider imagistic ideas but then always attempted to base his ideas on *extracting meaning* from the definition.

Chris, on the other hand, used imagery to support his thinking, drawing pictures to represent his main ideas. He wrote down the limit definition as he drew a picture, saying:

> "I don't memorise that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down." (Chris, first interview)

~~$\forall\ a_n \to L$ then there~~ exists
For all $\varepsilon>0$, there exists $N\in\mathbb{N}$
such that $|a_n - L| < \varepsilon$ for all $n\geq N$



> "I think of it graphically ... you got a graph there and the function there, and I think that it's got the limit there ... and then _ once like that, and you can draw along and then all the ... points after N are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that's the *n* going across there and that's a_n. ... Err this shouldn't really be a graph, it should be points." (Chris, first interview)

The slip in drawing a curve revealed him concentrating on more important ideas and (temporarily) neglecting others. He always seemed to be negotiating with his ideas. For instance, he considered an alternative definition in which increasing N caused ε to become small before rejecting it and settling on the standard form. He seemed to enjoy the tension of challenge and was constantly *giving* meaning from his concept images whilst reconstructing them to take account of the formal theory.

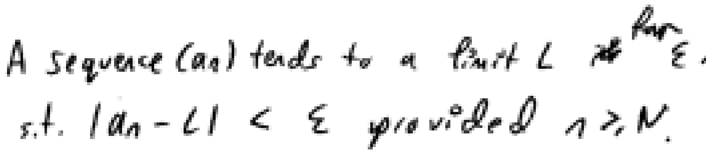
Both students could use the definition of limit in an operable manner in different ways. For instance, when asked about "non-convergence", Ross wrote down the limit definition and negated the quantifiers, while Chris wrote down the definition immediately as if thinking the ideas through in a mental experiment. Ross practised and thought through his proofs formally, Chris wrote formal proofs linked to thought experiments.

Less successful students

Many students on the course had difficulty with definitions. Robin tried rote learning:

> "It's just *memorising* the exact form of it, making the actual idea sort of *understandable* ..."

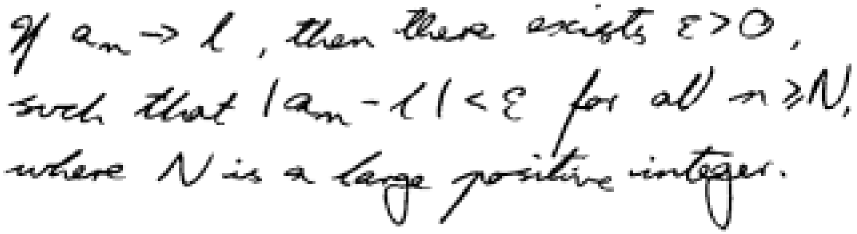
Despite this, he could not write the definition of convergence accurately:



(Ross, first interview)

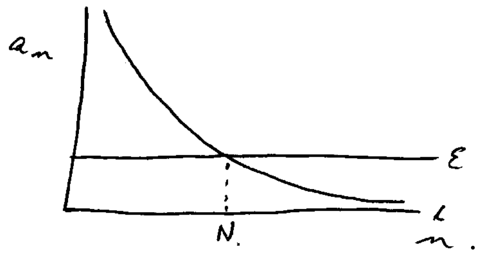
He attempted to extract meaning from the definition, but was unable to remember it accurately, let alone make it operable. For instance, in one problem he set N=5 without mentioning any relationship between ε and N.

Colin was also unsuccessful with the definition, writing:



(Colin, first interview)

He often attempted to support his ideas using a diagram:

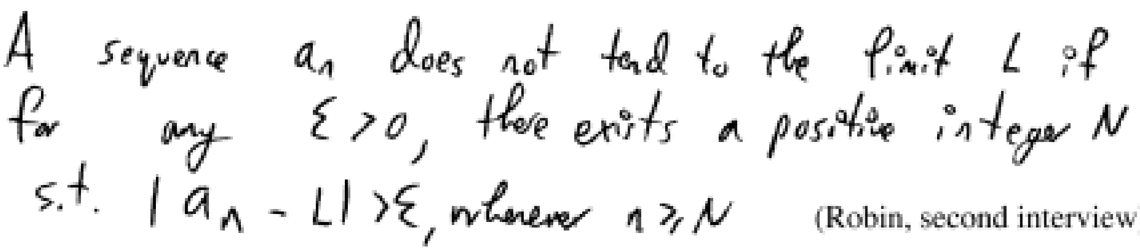


(Colin, first interview)

However, his pictures were highly specific and seemed to imprison him in their implied detail rather than provide the flexibility of thought available earlier to Chris. For instance, he denoted the limit by l yet wrote ε instead of l+ε, and considered the limit as a *lower bound* (a common concept image noted by Cornu, 1981, 1991). He explained:

> ".. umm, I sort of imagine the curve just coming down like this and dipping below a point which is ε ... and this would be N. So as soon as they dip below this point then ... the terms bigger than this [pointing from N to the right] tend to a certain limit, if you make this small enough [pointing to the value of ε]." (Robin, first interview)

Neither student could cope with non-convergence. Robin wrote:



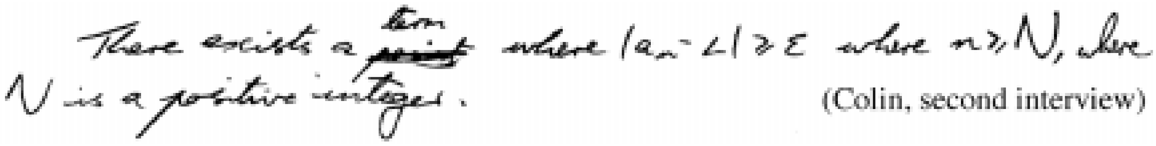
(Robin, second interview)

He leaves the original quantifiers unchanged, and only modifies the inner inequality $|a_n-L| < \varepsilon$ incorrectly to give $|a_n-L| > \varepsilon$. He is unable to treat the whole definition as a meaningful cognitive unit (Barnard & Tall, 1998), focusing instead only on the inner statement as something which he can attempt to handle.

Meanwhile Colin, said:

> "Umm ... I would just say there doesn't exist a positive integer because we can't work it out ... no ... you cannot find an integer N ...".

and wrote:


There exists a ~~point~~ term where $|a_n - L| \geq \varepsilon$ where $n \geq N$, where N is a positive integer.

(Colin, second interview)

Both students used pictures and symbols in their work, giving meaning on some occasions and attempting to extract it from the definition on others. However, Robin's main preference was to extract meaning from definitions which were regrettably often erroneous, whilst Colin preferred to attempt to give meaning using concept images which were too limited to build the general concept.

Other unsuccessful approaches

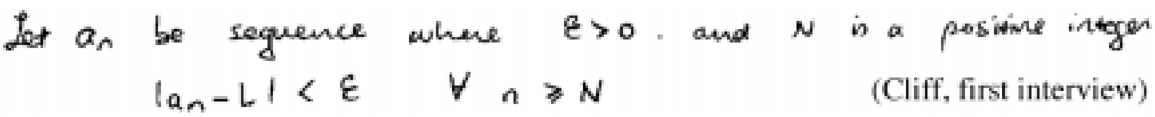
Two other classes of students were considered—*users of mathematics*, including those studying physics, statistics, economics, etc and *future teachers of mathematics*. Some of the users of mathematics were successful, others were more interested in mathematics only for its use, having little interest in formal proof which appeared too complex, even alien. Rolf (an applied mathematics student) wrote the definition as:


$(a_n) \to L$ if

$|a_n - L| < \varepsilon$ $\varepsilon \in \mathbb{Z}$, $\varepsilon > 0$ for $n \geq N$

(Rolf, first interview)

Cliff (a statistics student) wrote:


Let a_n be sequence where $\varepsilon > 0$. and N is a positive integer

$|a_n - L| < \varepsilon$ $\forall$ $n \geq N$

(Cliff, first interview)

Both definitions are distorted and restricted to the inner statement, with total absence of the two external quantifiers and the functional relation between ε and N. Rolf saw the definition as a process, which he attempted to memorise, and use as a criterion to check if a sequence is convergent or not. He tended to try to *extract* meaning from it, but failed. Cliff seemed to think of it as a dynamic *description* of convergence which he imagined occurring in time as N increases and ε decreases. He attempted to *give* meaning but is unable to do it successfully. Given their inadequate definitions, neither student could define non-convergence. Both subsequently resorted to rehearsing routine computations requested in previous examinations and tried to rote-learn them to pass the course.

The student teachers in the main study all replicate the imagistic meanings of those in the preliminary study. They have dynamic images of convergence with terms getting “arbitrarily close” which have often been reported in the literature (Cornu, 1991). For instance, Laura evoked many personal images for the idea of a limit with built-in conflict:

> “The number where the sequence gets to, but never quite reaches.”
>
> Let a_n be the sequence and L is the limit which it tends to. Then when some initial values are placed into the formula of the sequence the answers will never reach the value of L (negative or positive).
>
> “... oh, yes, I put ‘never reach’, and it *can* reach, and that will be the limit of it. ...”
>
> “... But it won’t never get bigger than the limit. The limit is like the top number it can possibly reach. And I put never reach.” (Laura, various sayings, first interview)

She was unable to write down the definition in any formal sense, although she had mental pictures that gave her imagistic meaning for some of the theorems. Any justifications she made involved attempting to *give* meaning using images. She was unsuccessful with the formal aspects of the theory as were all the other teacher-training students. Essentially, the idea of formal proof in analysis was alien to their day-by-day routine in teaching practice. As Laura explains:

> “I’m on another planet when it comes to Analysis. It seems just completely surreal to me. … it sparks in a lot of people in the group ... a lot of people. I don’t think there is anybody who understands it. And a lot of people are getting very frustrated, with it. I just want to throw books around the room and ... get up and leave.” (Laura, first interview)

Summary

In this paper we began by noting that mathematicians use different cognitive techniques to generate new theorems. Some work with formal definitions, carefully extracting meaning from them by deducing from them and gaining a symbolic intuition for theorems that may be true and can then be proved. Some have a wider problem-solving approach, developing new concepts that may be useful before making appropriate definitions to form a basis for a formal theory.

Students learning mathematics have a different problem. They come from elementary mathematics, deeply ingrained in the computation of arithmetic and the symbol manipulation of algebra using standard algorithms to solve certain types of problems. The forms of proof at this level (often called “demonstration” or “justification”) usually either involve algebra to give a symbolic description for a general arithmetic statement, or some kind of thought experiment focussing on a “typical” or “generic” case.

The transition from elementary mathematics to formal proof is a huge chasm for many students whose underlying concept image is unable to sustain the formalism. Many (including the majority of those in our sample preparing to teach mathematics) have informal images which dominate their thinking. Some remain entrenched with their old images and those that attempt to use the definition may only be able to cope with part of the structure, giving a personal definition that is not formally operable.

Success comes to those who achieve it in (at least) two ways, either *giving* meaning by working from the concept image, or *extracting* meaning by working formally with the definition. These two techniques can each be successful or unsuccessful. For the successful student, *giving meaning* involves constantly working on various images, reconstructing ideas so that they support the formal theory. The successful student who extracts meaning from the definition has a different task of building up a formal image based mainly on the proof activities themselves.

Those who fail to cope with formal proof but try to give meaning from their concept imagery may be able to imagine thought experiments which give generic proofs and an intuitive insight into some of the ideas, others may fail completely. And become extremely frustrated. Those who fail to extract meaning are unable to cope with the complexity of the definitions and be totally confused. A fall-back strategy to attempt to pass exams is to learn proofs by rote.

Teaching and learning formal proof remains an important component of theory building in advanced mathematical thinking. For future mathematicians it is essential. However, in using different approaches through giving or extracting meaning involves quite different sequences of construction. *Giving meaning* from concept images requires ongoing reconstruction of personal ideas throughout the course to focus on essential properties of the definition and to construct an integrated formal theory. *Extracting meaning* builds up ideas mainly from formal deductions with fewer links to other concept images and so avoids some possible conflicts at the time. However, this formal approach has its own difficulties and may end up with a formal theory unconnected to informal imagery. These different developments suggest that it may not always be possible to deal with different student approaches within a single teaching method.

The most serious finding is the negative effect caused by teaching formal proof in analysis has on future teachers which may have an implicit effect on their teaching of mathematics to the next generation.

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