Reflections on APOS theory in Elementary and Advanced Mathematical Thinking

David Tall

Mathematics Education Research Centre
University of Warwick, UK
e-mail: David.Tall@warwick.ac.uk

*What are the processes by which we construct mathematical concepts? What is the nature of the cognitive entities constructed in this process? Based on the theories of cognitive construction developed by Piaget for younger children, Dubinsky proposed **APOS** theory to describe how **actions** become interiorized into **processes** and then encapsulated as mental **objects**, which take their place in more sophisticated cognitive **schemas**. He thus takes a method of construction hypothesised in (elementary) school mathematics and extends it to (advanced) college/university mathematics. In this paper I respond to Dubinsky's theory by noting the need for cognitive action to produce cognitive structure, yet questioning the primacy of action before object throughout the whole of mathematics. Biological underpinnings reveal cognitive structures for object recognition and analysis. I use this to suggest that APOS theory has already shown its strength in designing undergraduate mathematical curricula but question its universal applicability, in particular in geometry, and, more interestingly, in the formal construction of knowledge from definitions to deductions in advanced mathematical thinking.*

Introduction

The purpose of this paper is to respond to the research forum presentation of Ed Dubinsky (Czarnocha et al, 1999) on APOS theory as "one theoretical perspective in mathematics education research". It is clearly more than this, offering a major contribution to mathematics education at the undergraduate level. Indeed, in the Calculus Reform in the United States in the late 80s, it formed the basis of the *only* curriculum project that had a coherent cognitive perspective.

My response will analyse APOS theory within wider realms of mathematical learning and thinking, in particular a comparison of its roles in various contexts in elementary mathematical thinking (EMT) and its extension to advanced mathematical thinking (AMT). (These acronyms were introduced by Gontran Ervynck who was responsible for the formation of the AMT Working Group at the Psychology of Mathematics Education Conference in 1985.) AMT referred initially to "mathematics learning and teaching at 16+" including the activities of mathematicians in research. The work of Dubinsky and his colleagues has focused on undergraduate mathematics (RUME), in particularly in developing suitable working practices (eg co-operative learning) and learning sequences (genetic decompositions) in a wide range of specific mathematical areas, including discrete mathematics, logic, calculus, linear algebra, group theory. In this paper I shall concentrate on the role of APOS theory.

Presented at PME23 Haifa, Israel, July, 1999
Published in O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME, Haifa, Israel*, 1, 111–118.

The biological foundation of action-process-object-schema

I wish to begin by showing that the broad brush-strokes of APOS theory seem to have a deep underlying biological structure. Figure 1 shows a simplified model of three stages of brain development as a result of successive stimuli (which could be perceptual or reflective). Stage represents an external stimulus to neuronal group 1, which is sufficiently strong to fire neuronal group 2 but not group 3. The firing causes the link between 1 and 2 to become more sensitive for a period of hours or days (so that we are more likely to recall recent events). If the connection is reactivated, it becomes more easily fired until it reaches as stage where any excitation of 1 also fires 2. This *long-term potentiation* of the neuronal connections builds new structures. The combined strength of 1 and 2 may now cause group 3 to be excited, and so on. In this way an external stimulus can cause a firing between two states perceived initially as separate, then joined together, then part of more complex neuronal groupings that can fire in more complex situations. The broad action-process-object-schema therefore has a natural biological underpinning.

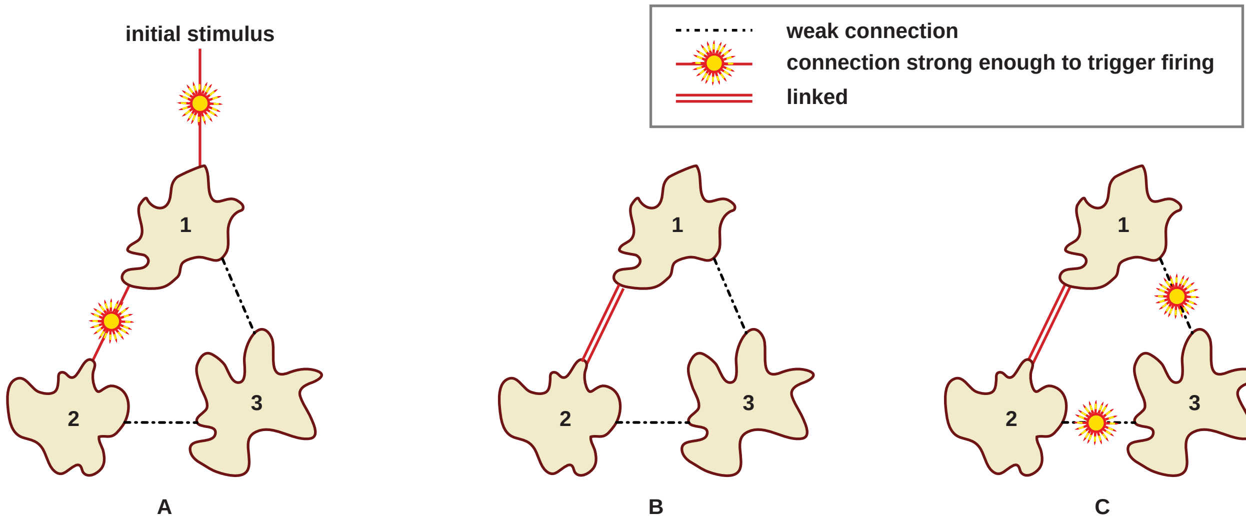

Figure 1: building memories in the brain by long-term potentiation (Carter, 1999, p. 160)

However, suppose that the first stimulus is a perception of an *object*. Then the same sequence of diagrams could represent a growing relationship with the properties of that object and the building of a more complex system of properties and connections. Both of these theories occur within the description of APOS in Czarnocha et al, (1999), There is a sub-sequence beginning with actions and moving to objects, and a further sub-sequence beginning with objects and moving to a broader schema.

The primacy of actions and objects

APOS theory begins with actions and moves through processes to encapsulated objects. These are then integrated into schemas—consisting of actions, processes and objects—which can themselves be encapsulated as objects. This suggests a primacy of action over object. At a fundamental level it absolutely clear that cognitive actions are required to construct cognitive objects. This I refer to as an application of “strong

APOS", in which the actions and processes are *any* cognitive actions or processes (conscious or unconscious), not just (conscious) mathematical ones.

However, even with this strong interpretation of the theory, the primacy of actions needs to be questioned. Dubinsky and his co-workers have made an impressive effort to formulate everything in action-process-object language. However, the urge to place this sequence to the fore leads to a description that, to me, soon becomes over-prescriptive.

The part of development that uses the triad theory of Piaget and Garcia (1983) moving from object to schema describes the initial object as "an encapsulated process" or a "thematized object" to maintain the primacy of the APOS sequence. The first stage of the triad, denoted *intra*, is simply described as "focus on a single object", followed by *inter* (study of transformations between objects) and *trans* (schema development connecting actions, processes and objects). Such a description (based on the language of action, process and object) seems to be at pains to avoid other more widely used terms such as "inter" including the study of *properties* of objects, or "intra" being concerned with *relationships* between them. The term "transformation" is one that I sometimes find impenetrable. Sometimes it has a mathematical meaning, but at others it seems obscure. In comparing the size of one object with another, is there a transformation of objects in some sense, or does the child just declare one is bigger because it pokes out beyond another?

APOS theory even formulates the notion of "permanent object" as arising through "encapsulating the process of performing transformations in space which do not destroy the physical object" (Dubinsky *et al*., 1988, p.45). Thus the permanent mental object in the mind is created by a physical or perceptual action on an external object, to maintain the primacy of process over object.

Looking closer at the structure of the brain suggests a distinctly different scenario. The research of Hubel & Wiesel (1959) revealed single neurons in a cat's brains that respond to orientation of an edge. Similar experiments with other animals revealed the same phenomenon. In the brain of Homo sapiens, in addition to a specific area of the cortex that builds a point by point copy of the visual field, other brain modules specialize in a variety of analytic activities, including the perception of edges, orientation, movement, colour, binocular vision, and so on. Homo sapiens and many other creatures therefore have complex systems for the visual perception and analysis of *objects*. One may attempt to state that such objects only arise as the result of cognitive actions, and this use of *strong* APOS cannot be denied at a theoretical level. However, the practicalities of consciousness tell a different story. When an individual is looking at an object, the *conscious* experience is that of the object being seen, not of the multi-faceted unconscious processes by which the internal brain processes information.

Furthermore the brain has a highly subtle collection of modules to detect object properties. Recent research has shown that animals and very small babies have primitive brain modules that distinguish between one object, two objects, and perhaps more. This distinction occurs in babies at a far earlier age than Piaget's theories predict. (See "Piaget's errors, p.44 *et seq*. in Dehaene, 1997).

The primitive brain also operates in other ways that impinge on high mathematical and philosophical constructions. Lakoff & Johnson (1999, p.16) hypothesize that the primitive "embodied mind" plays a role in *all* our thought, including what may be perceived as logical deduction when brain modules sense such things as "inside inside" is "inside" (ibid, p.32). This occurs not through logical deduction but because the brain models just observe that "it is". Furthermore, intuitive deduction occurs using "embodied arguments" such as the use of "prototypical" exemplars rather than strict quantification (ibid. chapter 7). The objects of the world and the embodied structures in the brain have a full role to play in learning and thought.

The previous discussion, whilst failing to deny the primacy of *cognitive* action to construct *cognitive* concept, indicates that the brain observes objects, and what seem to be primitive mathematical and logical concepts in ready-made brain modules. This seriously questions a rigid Action-Process-Object-Schema strategy in every curriculum. Even the APOS curriculum has sub-sequences building on objects.

APOS Theory in Elementary Mathematical Thinking

To see the relevance of APOS in mathematics, I begin with its source in EMT. Here Piaget spoke of three modes of abstraction: *empirical abstraction* from objects of the environment, *pseudo-empirical abstraction* from actions on objects in the environment and then *reflective abstraction* from mental objects.

Geometry includes many acts of empirical abstraction focusing on objects, beginning with an intra stage coming to terms with the nature of objects themselves. I contend therefore that geometry starts as an *object-based theory*. This is not to say that there are no processes—of course there are (drawing, measuring, constructing etc). However the focus of these processes is to gain knowledge about the objects themselves.

Concepts in geometry occur with many parallel activities involving physical interaction with the real world, but also, in a very real sense, they depend on the growing sophistication of language. Rosch's theory of prototypes (Rosch et al, 1976) shows that children first recognize "basic categories" such as 'dog', or 'car', only later moving to super-ordinate or sub-ordinate categories, such as 'poodle-dog-animal' or 'Ford-car-transport'. Such basic categories have properties that relate naturally to immediate perception. A basic category can be represented by a prototypical mental image; it is the highest level at which category members have similarly perceived overall shapes and the highest level at which a person uses similar motor actions for interaction with the members. The focus on a category of basic *objects* occurs naturally through a coherent combination of perceptions. I would contend here that it is the *object* that is the focus of attention, with the actions being the agents of that perception. It is only later, in a Van Hiele type growth (using a dialectic back and forth rather than a strict sequence of stages) that language used for description enables the conscious mind to build platonic objects such as lines with "no width" and "infinite extensibility".

On the other hand, the growth of knowledge in arithmetic and algebra begins with *pseudo-empirica*l abstraction and hence more closely follows an APOS sequence. In counting, there is the action of repeating the number words and beginning to accompany this by pointing at objects in turn. Later various learning sequences set up neuronal connections in the brain, routinizing the procedure, seeing it as a process when it is realised that different orders of counting the same set give the same number, and then "encapsulating" the process into the concept of number. In fact the encapsulation follows a sequence of counting "one, two three, four", then silently counting all but the last number, then saying just the last number without counting at all. Number names and number symbols play an essential role in this development.

I was utterly flabbergasted to see that *nowhere* in Czarnocha et al, (1999) is there a single mention of the word *symbol*. Symbols are at the heart of cognitive development of arithmetic (and later in algebra). They can be spoken, heard, written, read, used in action games and songs. They are the stuff that children work with. Indeed, those who focus more on the objects being counted than on the symbols for the counting prove to have much greater difficulties in later development (Pitta & Gray, 1997).

To develop a more elaborated theory to describe these phenomena, Gray & Tall (1994) formulated the notion of procept as follows:

> An *elementary procept* is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object.
>
> A *procept* consists of a collection of elementary procepts which have the same object.

The procept notion has strong links with APOS theory, but there are significant differences. We have always insisted on focusing on the cognitive structure itself and do not imply that the mathematical process involved must first be given and "encapsulated" before any understanding of the concept can be derived. For instance, in introducing the notion of solving a (first-order) differential equation, I have designed software to show a small line whose gradient is defined by the equation, encouraging the learner to stick the pieces end to end to construct a visual solution through sensori-motor activity. This builds an embodied notion of the existence of a unique solution through every point, with difficulties only occurring at singularities where the differential equation does not give the value of the gradient. It provides a skeletal cognitive schema for the solution process before it need be filled out with the specific methods of constructing solutions through numeric and symbolic processes. It uses the available power of the brain to construct the whole theory at a schema level rather than follow through a rigid sequence of strictly mathematical action-process-object.

Figure 2 shows a succession of uses of processes and concepts in symbolic mathematics (Tall, 1998). Arithmetic has computational processes, algebra has potential evaluation processes but manipulable concepts, the dynamic limit concept at the beginning of calculus involves potentially infinite computational processes that lead to the mental imagination of "arbitrarily small", "arbitrarily close" and "arbitrarily large" quantities. It is no wonder that so many students cling to the comfort of rote-learned finite rules of the calculus.

In Advanced Mathematical Thinking, students meet an entirely new construction: the axiomatic object in which the *properties* (expressed as axioms) are the starting point and the *concepts* must be constructed by logical deduction. Although (strict) APOS again can

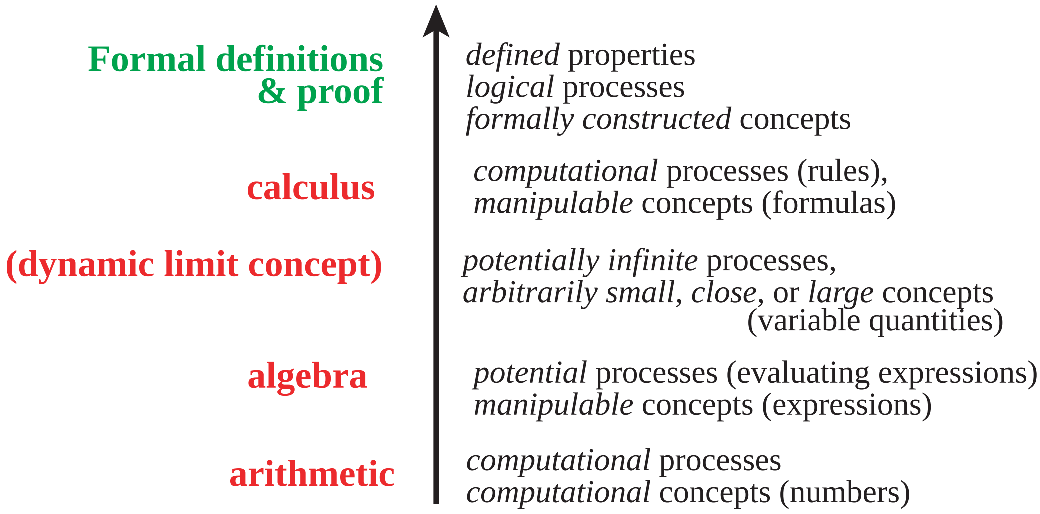

Figure 2: Development of process/concept in symbolic mathematics (Tall, 1998)

describe the learning processes (as it always will), there is a far more serious area of study in the relationship between embodied knowledge and formal deduction (Alcock and Simpson, 1999). Procepts are only of value here in certain aspects (for instance, in the element of a transformation group can be both process and an object), but the notion of group is not as it does not have a symbol dually representing process and concept. Is this a failure of the notion of procept compared with the broader application of APOS theory? Superficially, of course. However, the very fact that there is a serious cognitive reconstruction using symbols in formal mathematics in an entirely different way from the procepts of elementary mathematics suggests a chasm that many students have difficulty in crossing.

Dubinsky and his colleagues have a *brilliant* way of looking at the group axioms: formulate it as a function which outputs whether a set and its operation is a group or not. This to me is a fantastic solution (both in terms of computers and cognition). However, I still have serious concerns about the cognitive constructions made in such a sequence.

Different styles of Advanced Mathematical Thinking

One of my favourite quotations, which I have used often before, is the following:

> It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban who pushes on his trenches against the place besieged, leaving nothing to chance. The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advanced guard. (Poincaré, 1913, p. 210)

The approach to the development of APOS is closer to Vauban than the bold cavalryman. As implemented it often begins with symbolic procedures or programming activities with visual activities coming later, if at all. Such a curriculum clearly can be of great value but I cannot help sensing we need to develop bold cavalrymen with insight—even if it is flawed—for these leaps can be the stuff of future solid conquest.

There is such a difference between analysts who fear the fallibility of pictures and geometers or topologists that live by them. But this does not mean that either style is necessarily always superior:

> In the fall of 1982, Riyadh, Saudi Arabia … we all mounted to the roof … to sit at ease in the starlight. Atiyah and MacLane fell into a discussion, as suited the occasion, about how mathematical research is done. For MacLane it meant getting and understanding the needed definitions, working with them to see what could be calculated and what might be true, to finally come up with new "structure" theorems. For Atiyah, it meant thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs. This story indicates the ways of doing mathematics can vary sharply, as in this case between the fields of algebra and geometry, while at the end there was full agreement on the final goal: theorems with proofs. Thus differently oriented mathematicians have sharply different ways of thought, but also common standards as to the result. (Maclane, 1994, p. 190–191.)

Even though different approaches to research both end up with formal proof, the mathematical insights gained are very different. In a time of fast technological development, it is clear that we need our cavalrymen to make precarious advances as well as those who carefully operate safely step-by-step.

Pinto and Tall (1999) reveal a wide spectrum of thinking processes in undergraduate mathematics students including some who build meaning for definitions from their own experiences and others who take the definitions given by others and build meaning mainly by deducing theorems. The latter students seem more amenable to an action-based APOS course than the former, who build on a whole range of embodied cognitive constructs. Whilst cognitive actions are always necessary to construct cognitive concepts, is it providing a service to necessary diversity in human thought by restricting the learning sequence to one format of building mathematical actions, mathematical processes and mathematical objects?

Summary

Strict APOS can be used to formulate the idea that cognitive concepts must be preceded by cognitive operations. In this sense APOS theory is a "ToE" (Theory Of Everything). However, given that the learner has a wide range of embodied constructs in need of reflection and reconstruction, I contend that the accent on sequences solely built on action-process-object-schema distorts the wider enterprise. Figure 3 shows my own vision of the development of major themes in mathematics. APOS theory has many applications in the elementary mathematics of arithmetic, algebra, and calculus, but is of less relevance in the study of space and shape. In RUME the evidence shows its power in designing certain types of highly successful curricula. However, it is not the whole world. The varieties of thinking in professional mathematicians need an expression beyond that of the measured action-process-object-schema development. For instance, in my own approach to calculus, I begin not with the new process of programming functions in a computer language, but with the embodied visuo-spatial notion of "local straightness" which is then explored in parallel with the symbolic operations of differentiation. I welcome APOS as a major contribution to the understanding of mathematical cognition, but as a valued tool, not a global template.

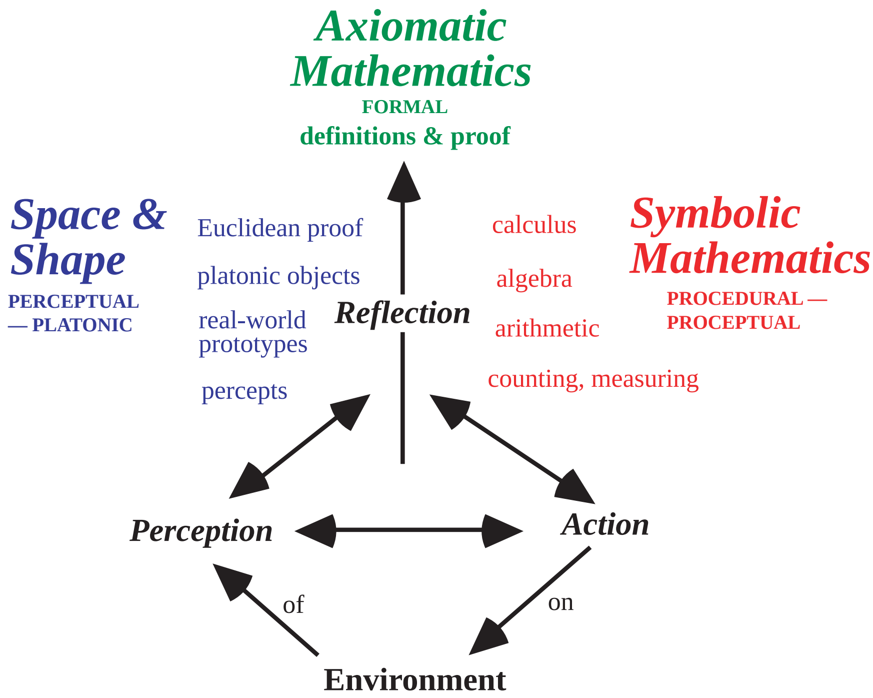

Figure 3: cognitive themes in the development of mathematics (Tall, 1995, 1998)

References

Alcock L. & Simpson, A. D. (1999). The rigour prefix. *Proceedings of PME23* (this conference).

Carter, R. (1999). *Mapping the Mind*. Weidenfeld & Nicolson: London.

Czarnocha, B., Dubinsky, E., Prabhu, V., Vidakovic, D., (1999). One theoretical perspective in undergraduate mathematics education research. *Proceedings of PME* 23this conference).

Dehaene, S. (1997). *The Number Sense*. New York: Oxford University Press.

Dubinsky, E., Elterman, F. & Gong, C. (1988). The Student's Construction of Quantification. *For the Learning of Mathematics 8*, 44–51.

Gray, E. & Tall, D. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *26*, 2, 115-141.

Hubel, D. H., & Wiesel, T.N. (1959). Receptive fields of single neurons in the cat's striate cortex, *Journal of Physiology*, 148, 574–579.

Lakoff & Johnson, (1999). *Philosophy in the Flesh*. New York: Basic Books.

MacLane, S, (1994). Responses to Theoretical Mathematics, *Bulletin (new series) of the American Mathematical Society*, 30, 2, 190–191.

Piaget, J. & Garcia, R. (1983). Psychogenèse et Histoire des Sciences. Paris: Flammarion.

Pinto, M. & Tall, D. (1999), Student constructions of formal theory: giving and extracting meaning, *Proceedings of PME 23*, (this conference).

Pitta, D. & Gray, E., (1997). In the Mind. What can imagery tell us about success and failure in arithmetic?, In G. A. Makrides (Ed.), *Proceedings of the First Mediterranean Conference on Mathematics*, Nicosia: Cyprus, pp. 29–41.

Poincaré H. 1913: *The Foundations of Science*, (translated by Halsted G. B.) The Science Press.].

Rosch, E., Mervis, C., Gray W., Johnson D. & Boyes-Braem, P. 1978. Basic objects in natural categories. *Cognitive Psychology* 8, 382-439.

Tall, D. O. (1998). Symbols and the Bifurcation between Procedural and Conceptual Thinking, Plenary presentation at *the International Conference on the Teaching of Mathematics, Samos*.

Tall, D. O.: 1995, Cognitive growth in elementary and advanced mathematical thinking', in D. Carraher and L. Miera (Eds.), *Proceedings of X1X International Conference for the Psychology of Mathematics Education*, Recife: Brazil. Vol. 1, 61–75.