Symbols and the Bifurcation between Procedural and Conceptual Thinking

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*Symbols occupy a pivotal position between processes to be carried out and concepts to be thought about. They allow us both to **do** mathematical problems and to **think** about mathematical relationships. In this presentation we consider the discontinuities that occur in the learning path taken by different students that lead to the divergence between conceptual and procedural thinking. Evidence will be given from several different contexts in the development of symbols through arithmetic, algebra and calculus, then on to the formalism of axiomatic mathematics. This is taken from a number of research studies recently performed for doctoral dissertations under my supervision at the University of Warwick. Those participating are all university staff in mathematics at institutions in the USA, Malaysia and Brazil, with data collected in the USA, Malaysia and the United Kingdom. All the studies form part of a broad investigation into why some students succeed yet others fail.*

Introduction: Building a theory

A necessary component of successful mathematics education is the understanding of mathematics itself. But this is not sufficient. To be able to analyse the development of mathematical thinking (including that of a wide range of students and professional mathematicians) requires a consideration of how we conceive of mathematics, and how we learn, use and create it.

Various theories have been proposed, some building from cognitive studies of children learning elementary mathematics, others based on diverse viewpoints such as the logical structure of propositional thinking or computer metaphors for brain activity. My own approach favours attempting to understand how the biological human species builds from activities in the environment to developing highly subtle abstract concepts. This begins with the ability to *perceive* things, to *act* on them and to *reflect* upon these actions to build theories (figure 1).

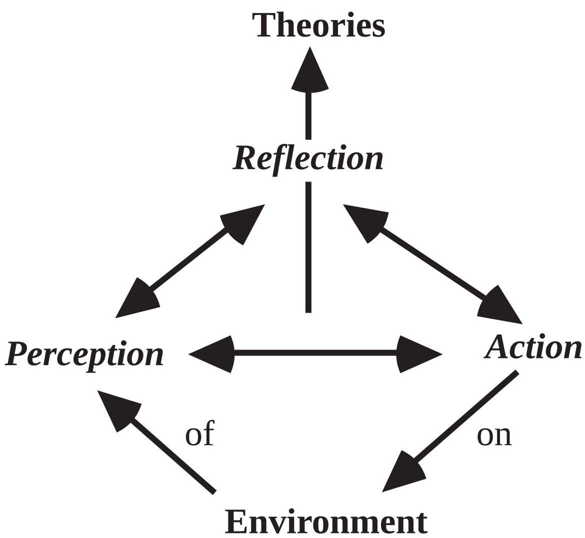

Figure 1: Combining Reflection, Perception & Action

Some authors see various activities occurring in specific sequences. For instance, Dubinsky and his colleagues propose a theory (e.g. Dubinsky,

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1991; Cottrill *et al.*, 1996 etc.) in which *actions* become routinized into *processes* that are then encapsulated as *objects*, later to be embedded into cognitive *schemas* (referred to by the acronym APOS). Such a sequence occurs widely in cognitive development, and will often occur in this paper. However, my own view is that perception, action and reflection occur in various combinations at a given time and a focus on one more than the others can lead to very different kinds of mathematics.

Perception of the world leads to the study of *shape and space*, eventually leading to *geometry*, where verbal formulations lead on to Euclidean proof. ***Actions on*** the world, such as counting, are represented by symbols and lead on to the *symbolic mathematics* of number, arithmetic and thence on to generalised arithmetic and algebra. ***Reflection on perception and action*** in mathematics leads eventually to the desire for a consistent axiomatic theory based on formal definitions and deductions (figure 2).

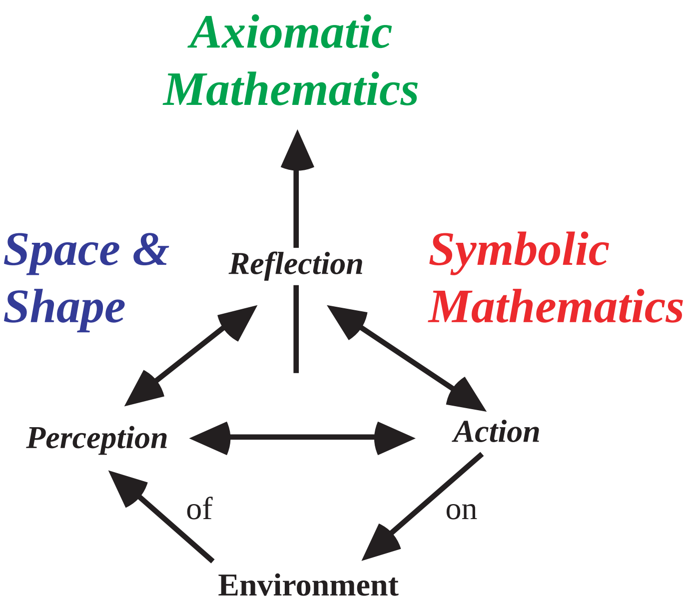


Figure 2: Various types of mathematics

Each of the three types of mathematics (space & shape, symbolic mathematics, axiomatic mathematics) involve different types of cognitive development (figure 3).

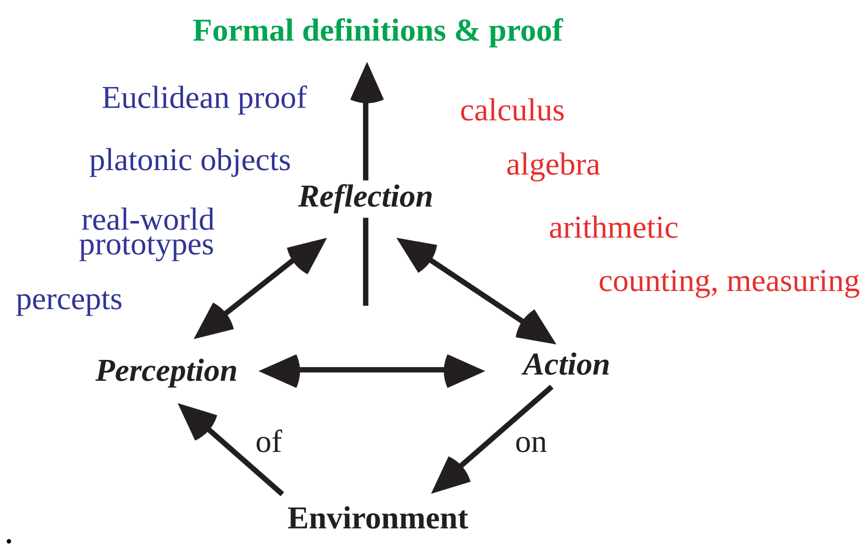


Figure 3: Conceptual development of selected mathematical concepts

Perceptions of objects in the world are initially recognised as whole gestalts. Some are individual, such as a child's mother, or the family pet, but more often they are perceived as *prototypes* that apply to a wide range of percepts. For instance, dog, cat, bird are prototypes for various kinds of living creatures. Some creatures are evidently birds (such as a robin), whereas others, such as an ostrich, are also classified as birds even though they fail to fly. It is interesting to note that these classifications do not begin from the bottom up, or from the top down, but in terms of centrally typical levels of recognition. For instance, children usually recognise *dog* before the more specific types of dog such as alsatian, poodle, or more general notions such as mammal, animal. Likewise in mathematics, the recognition of concepts such as square, rectangle, parallelogram, quadrilateral, polygon, take time to organise into a conceptual hierarchy which is done neither bottom up nor top down.

This development of geometric conceptions involves a number of cognitive re-constructions. For instance, when squares and rectangles are initially considered by young children, they are invariably seen as disjoint concepts (a square is *not* a rectangle, because a square has four equal sides whilst a rectangle has only *opposite* sides equal). Disjoint categories of geometric shapes must be reconstructed to give hierarchies of shapes (a square is a rectangle is a parallelogram is a quadrilateral). Further re-constructions are necessary to see a shape not as a physical object, but as a mental object with perfect properties, and then to imagine geometry not just in terms of two and three dimensional euclidean geometry, but as a variety of different geometries (affine, projective, elliptic, hyperbolic, differential, etc.) Such a cognitive development and its succession of cognitive stages has been documented in the work of van Hiele (1986).

Prototypical shapes, such as straight line, triangle, circle, are described verbally in increasingly subtle ways leading to the imagination of perfect platonic representations: a perfect straight line that has no width and can be extended arbitrarily in either direction, a perfect square, a perfect circle. Thus, paradoxically, perfect geometric entities depend on language to enhance their imagination.

Euclidean proof builds on this use of language to give verbal argument to support deductions based on visual concepts. Later still, the invention of new geometries sharing some, but not all, properties of Euclidean geometry leads to the need to formulate formal definitions and deductions to build a system that is not only coherent, but also *deductive* from the selected axioms.

Symbolic mathematics develops somewhat differently. Instead of the initial focus being on objects, it is upon *actions* on those objects: counting, sorting, ordering. The focus of attention starts with *counting*, then on to *measuring*,

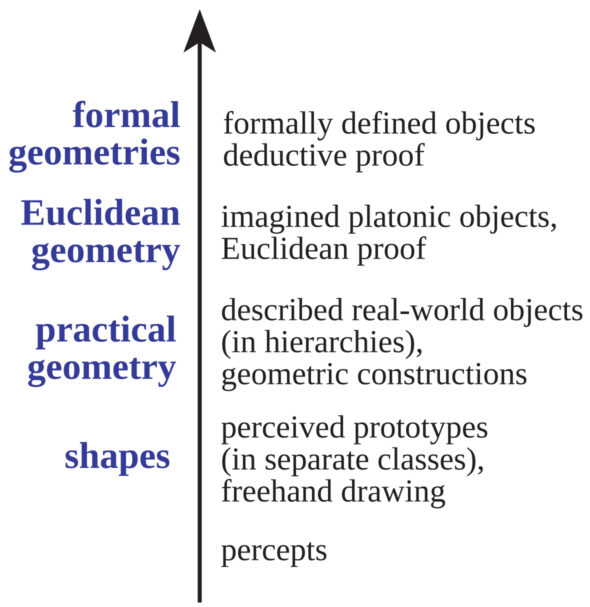


Figure 4: cognitive development of geometrical concepts

moving successively through practical *arithmetic*, *algebra* and c*alculus*. In each case the symbols involved have a very special property; they can be *manipulated* to give quantitative solutions to problems, enabling us to exercise control and the power of prediction over real world and hypothetical events.

The main focus of the rest of this presentation will be on the development of these symbols in elementary mathematics and the transition to the definitions and deductions of advanced mathematics.

Compression of process into concept using symbols

The development through arithmetic, algebra and calculus uses symbols in a very special way. Many of them evoke both a *process* to be carried out (e.g. the process of addition) and a *concept* which is output by that process. (Table 1).

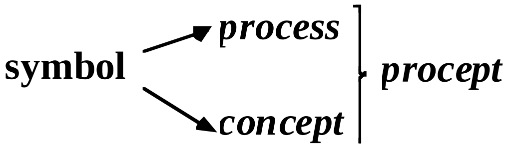
symbol	***process***	***concept***
3+2	addition	sum
–3	subtract 3 (3 steps left)	negative 3
3/4	division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$, $\sum_{n=1}^{\infty} 1/n^2$	tending to limit	value of limit
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n

Table 1: Symbols as process and concept

In many (but not all) instances[1], the dual use of symbol as process and concept usually begins by becoming familiar with the process and routinising it, to carry it out with less attention to specific details. At successive times the process is conceived in various ways. Counting, for instance, is a complex process of saying a sequence of number words at the same time as pointing in turn at objects in a collection once and once only. As we count a number of apples, we might say “there are one, two, *three* apples.” As this becomes more routine, the counting may be performed silently, “there are [one, two,] three apples”, then compressed into “there are … *three* apples” or just “there are three apples”. In this way the *process* of counting is compressed into the *concept* of number. The symbol 3 then evokes either a (counting) process or a (number) concept . Likewise the symbol 3+2 can evoke either the process of addition or the concept of sum.

[1] Typically, as the individual’s cognitive structure grows more complex, there will be alternative ways of building up conceptions. For instance, solution of differential equations can be *seen* visually, as well as performed numerically or symbolically, offering an alternative route fundamentally different from a procedure-process-object development.

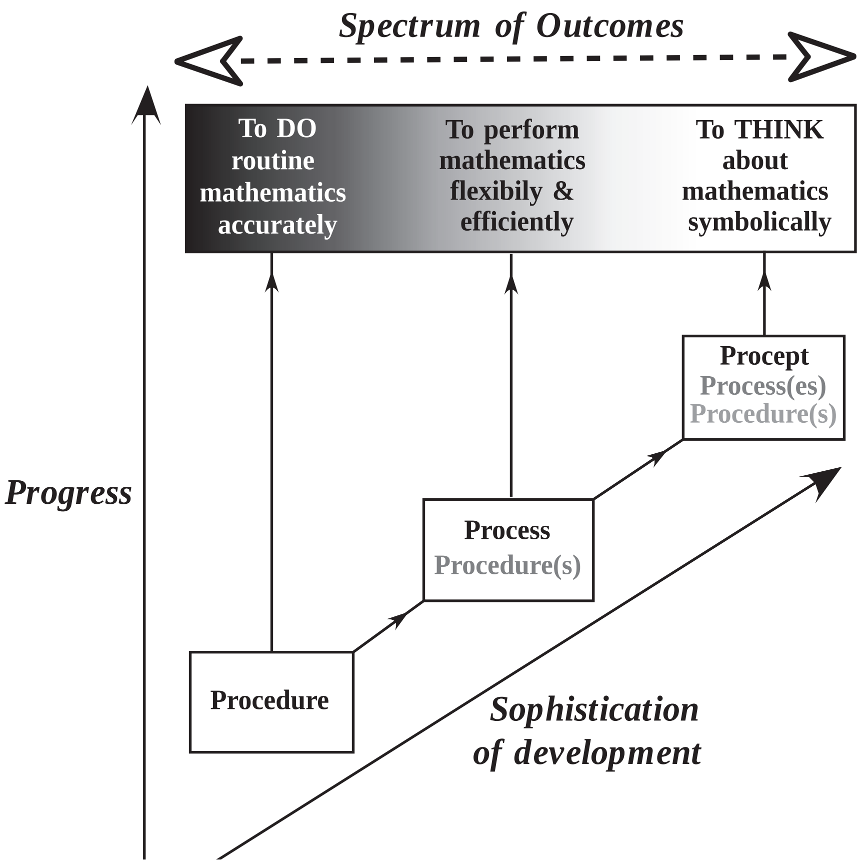
Gray & Tall (1994) refer to the combination of symbol representing both a process and the output of that process as a *procept*.



The procept notion has been given increasingly subtle meaning since its first formulation (Gray & Tall, 1991). It is now seen mainly as a *cognitive* construct, in which the symbol can act as a *pivot*, switching from a focus on process to compute or manipulate, to a concept that may be *thought* about as a manipulable entity. We believe that procepts are at the root of human ability to manipulate mathematical ideas in arithmetic, algebra and other theories involving manipulable symbols. They allow the biological brain to switch neatly from *doing* a process to *thinking* about it in a minimal way.

The process of compression from process to concept has been described in similar ways by various authors (e.g. Sfard, 1991 ; Dubinsky, 1991). Gray & Tall (1994) use the word *procedure* to mean a specific sequence of steps carried out a step at a time. The term *process* is used in a more general sense to include any number of procedures which essentially “have the same effect”. For instance, the process of differentiating the function $(1+x^2)/x^2$ can be done by various procedures such as the quotient rule, the product rule (for $1+x^2$ and $1/x^2$), or other strategies such as simplification to $x^{-2}+1$ prior to differentiation.

Knowing a specific procedure allows the individual to *do* a specific computation or manipulation. Having one or more alternatives available allows greater flexibility and efficiency to choose the most suitable route for a given purpose. But also being able to think about the symbolism as an entity allows it to be manipulated itself, to think about



mathematics in a compressed and manipulable way, moving easily between process and concept. This gives a spectrum of performance (figure 5) in which it is possible, at certain stages, for students with different capacities all to succeed with a given routine problem, yet the possible development for the future is very different. Those who are procedurally oriented are limited to a particular procedure, with attention focused on the steps themselves, whilst those who see symbolism as process or concept have a more efficient use of cognitive processing. Long-term, as students meet new tasks the same kind of spectrum occurs, with more and more tending to be coerced into procedural thinking. This means that those who are (or who become) procedural have a considerably greater burden to face in learning new mathematics than those who are able (in addition) to focus on the essential qualities of the symbolism as both process and concept.

Procedure, Process and Procept in College Algebra

The processing of linear forms in algebra is highly prone to the procedural-process-procept spectrum. DeMarois (1998) asked a class of college (pre-)algebra students to write down the output of the two following function boxes in algebraic form and asked if they were the same:

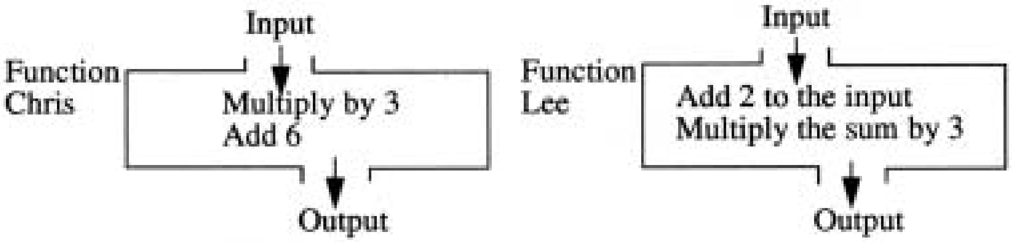


Figure 6: What are the outputs of these two function boxes and are they the same?

Three students were chosen, so that Student 1 was one of the highest achievers, Student 2 was in the middle and Student 3 was struggling. Their responses were as follows:

	Function Chris	Function Lee	*Are functions equal?*
Student 1	$3x+6$	$3(x+2)$	Yes, if I distribute the 3 in Lee, I get the same function as Chris.
Student 2	$x3+6$	$(x+2)3$	Yeah, but different processes.
Student 3	$3x+6$	$x+2(3\times)$	No, you come up with the same answer, but they are different processes.

Table 2: A spectrum of responses to functions as procept, process and procedure

It was difficult to read student 3's answer for the function Lee; he wrote the x in "$x+2$" and the "×" in "3×" both as a cross. A probable interpretation of this expression is as meaning "$x+2$, three times." Remarkably, Student 1 gives a proceptual answer, Student 2 a process answer and Student 3 a procedural answer. These three students exhibit the spectrum suggested in figure 5.

DeMarois also performed a study of the class as a whole, setting the detailed studies of these three students into a wider context. He found the spectrum spread through the class, that the more successful students were usually at an efficient process stage (meaning that they could often choose a more efficient solution method), others were procedural at best. He also tested links between the various representations (numeric, algebraic, graphic, function-machine, verbal, etc.) and found most students in his class were process-oriented or procedural at best in each of the various facets and that there were very different types of links between them. For instance, some facets, (such as algebraic and graphic) were often only one-way (e.g. algebraic formula to graphic picture on a calculator) in individual students.

Discontinuities in the development of symbols

There is a general perception amongst educators that curriculum design requires the construction of a sequence of lessons in which each builds smoothly and inexorably on the previous ones. This does not happen in mathematics. Working in a given context leads to beliefs that may need reconstructing at a later stage. For instance, in using numbers for counting, the "next" number after 3 is 4, so how can there be any numbers "in between"? For some individuals this causes great difficulties with fractions. Likewise, "you can't have less than nothing" when working with whole numbers and fractions, which requires a further reconstruction when introducing negative numbers. Handling the product of two negatives requires more conceptual reconstruction. Many just "accept" the result and begin the slippery slope to learning by rote to pass examinations. (It is interesting here to ask the reader if s/he really "understands" why minus one times minus one is plus one. A formal theorist will deduce it from axioms. However, the development of a formal perspective requires a radical reconstruction of knowledge which few undergraduates accomplish in any meaningful sense.)

Throughout elementary mathematics there are times when reconstruction becomes necessary and previously meaningful ideas no longer work according to their original meaning. For instance, the power 2^3, meaningfully means "three twos multiplied together". From this meaning, the properties of powers $2^3 \times 2^4 = 2^7$ easily follows because the left side has three lots of two times four lots of two, giving seven lots of two. But from this meaning, what does $2^{1/2}$ mean? How can one have "half a lot of twos multiplied together"? The use of the "power law" to justify it no longer uses the original meaning of multiplying together specific numbers of twos.

Duffin & Simpson (1993) suggest that different students have differing approaches to mathematics, some are *natural* learners who must make sense of the ideas from their own perspective before they can use them, others are *alien* learners in the sense that they are able to play mathematics as a game and the game makes sense in itself without reference to other meanings. Alien learners will have less difficulty with the general power law than natural learners. The alien learners learn *what* to do, but care little about external meaning. The natural learners find the general power law meaningless and are reduced to learning by rote, not meaning.

Different kinds of procepts

Procepts occur throughout arithmetic, algebra and calculus. However, they behave in different ways which may require students to reconstruct their understanding as they move to new contexts causing the discontinuities noted in the previous section. In whole number arithmetic the symbols have a built-in computational process which children learn to compute a specific answer. All the basic arithmetic symbols have this dual meaning of process and object but there are subtle nuances that are different. For instance the sum of two whole numbers is another whole number , the process of addition does not create a new type of object. But with fractions, the dividing of something into a whole number of pieces and the taking of a certain number of these pieces, *gives a new entity*: a fraction. These violate previous experiences of (counting) numbers. For instance, although five is the “next” number after “four”, fractions introduce many “numbers” between four and five and, more generally, no number has a “next” number.

Likewise, although in regular whole number arithmetic and the arithmetic of (positive) fractions, experience intimates that “you can’t have less than nothing”, in calculating temperature or bank balances, new numbers are introduced which now allow “less than nothing”. However, to construct a product of such negative numbers being positive usually involves a significant cognitive reconstruction. Some manage to cope procedurally with the rules without meaning, but many fail. The spectrum between procedural and conceptual is again widened.

The shift from arithmetic to algebra leads to a new kind of procept where the expression $2+3x$ has only a *potential* process of evaluation (when the numerical value of x is known). Thus the student again has to reconstruct experiences to give meaning to these new kinds of procept.

The shift from algebra to calculus poses even new problems. The limit symbols which occur such as

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right),\quad \sum_{n=1}^{\infty} 1/n^2,$$

all have *potentially infinite* processes. They seem to “go on forever”, perhaps never reaching the output limit concept. Again the difficulties of moving from finite algebra to potentially infinite limit processes have been widely documented (see Cornu, 1991 for a review). Although it seems simple to a mathematician who has attained the flexibility of symbol usage, almost all students I have ever met have initial difficulties. It is no wonder that so many are relieved to find that the rules of differentiation such as

$$\frac{d}{dx}(\sin x \cos x)$$

can be performed by a *finite* manipulative process. This returns the student to a kind of security reminiscent of the operational procepts in arithmetic. The rules for computing derivatives again give a definite answer, albeit in the form that an operation on a formula gives another formula. Few students cope with the limit concept and many seek the procedural security of the rules of differentiation. Thus, in the calculus, procedural students are happier with the rules of differentiation and may make no formal sense of the limit concept. Likewise, in the theory of limits of sequences and series, students often prefer the achievable computation offered by the tests for convergence of series (such as comparison test, ratio test, or the alternating sign test). All of these have the familiar operational notion of a built-in finite computation to give an answer.

At the formal level, there are still procepts (for instance, the elements of a transformation group can be thought of both as processes and concepts). However, the procept notion is now reduced to a more minor role. For instance, the notion of a group itself is not a procept. It is an altogether bigger mathematical structure which is constructed by deduction from formal definitions. The processes are now *logical* processes and the concepts are *formally* constructed. Again therefore, a new stage of development involves a new kind of procept and a new reconstruction of personal knowledge (figure 7).

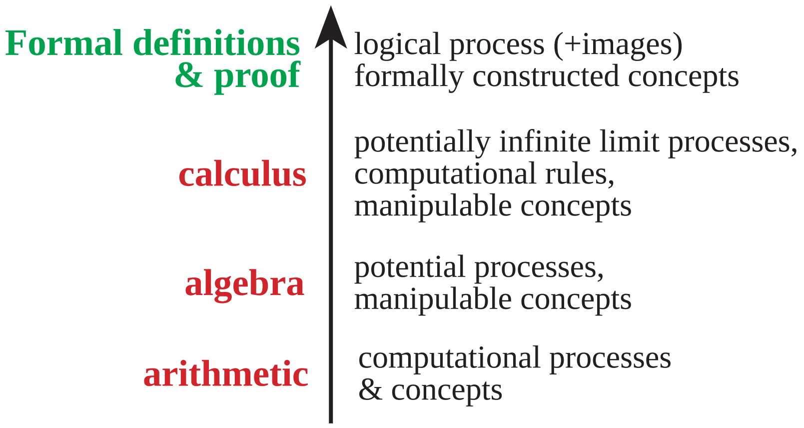


Figure 7: Different types of process & concept in mathematics

Conceptual and Procedural Links in College Algebra

Returning to the conceptual development in algebra, the various links available to a given student at a given time will vary according to the context, but they provide fascinating evidence of partial or incomplete cognitive structures. For example, the *x*- and *y*- intercepts for a given straight line equation may be found by several methods: reading the points off a graph, setting $y=0$ solving for x, and then $x=0$, solving for y. However, a student who tends to seek the security of procedures rather than having the ability to make appropriate links may not always be so fortunate. In a study by Crowley of students taking a preparatory course for college algebra (Crowley & Tall, in preparation), Kristi was asked:

> Find the *x*- and *y*- intercepts of $3y+x–12=0$.

When asked, "What would you do here? ", Kristi replied:

> *Divide everything by 3. . . . in my mind I'm visually moving everything, and dividing x by 3 is …. one third x plus …, so the y-intercept is 4.*

She put the equation into slope-intercept form $y=mx+b$ to find the *y*-intercept *b*. Had she had the conceptual link to do so, it would have been much simpler to set x to zero to find the *y*-intercept. She was then asked, "what's the point? What do you graph?" She immediately marked the point (0,4) on the *y*-axis. When then asked how to find the *x*-intercept, she replied:

> *On the calculator screen, where x is … if y is what, then hit intersect and try to find where the x is.*

She was able to find the *y*-intercept when she could put the equation in $y=mx+b$ form, even identifying the point each time with $x=0$ as $(0, b)$, but she did not seem to have the link for the x intercept in terms of putting $y=0$. There is, of course, an asymmetry between the treatment of x and y in the equation $y=mx+b$, and it is very possible that

Kristi's experience with this form, (which was needed to type it into the calculator to get the graph), predisposed her to see the roles of *x* and *y* differently. The data from the interviews was consistent with her having certain links that were stronger than others and yet others which may have been non-existent. Her activities involved carrying out procedures, subject to certain favoured strategies (e.g. "get it into $y=mx+b$ form") and even these were carried out by standard steps ("divide everything by *m*", "get the *x* over the other side", etc.).

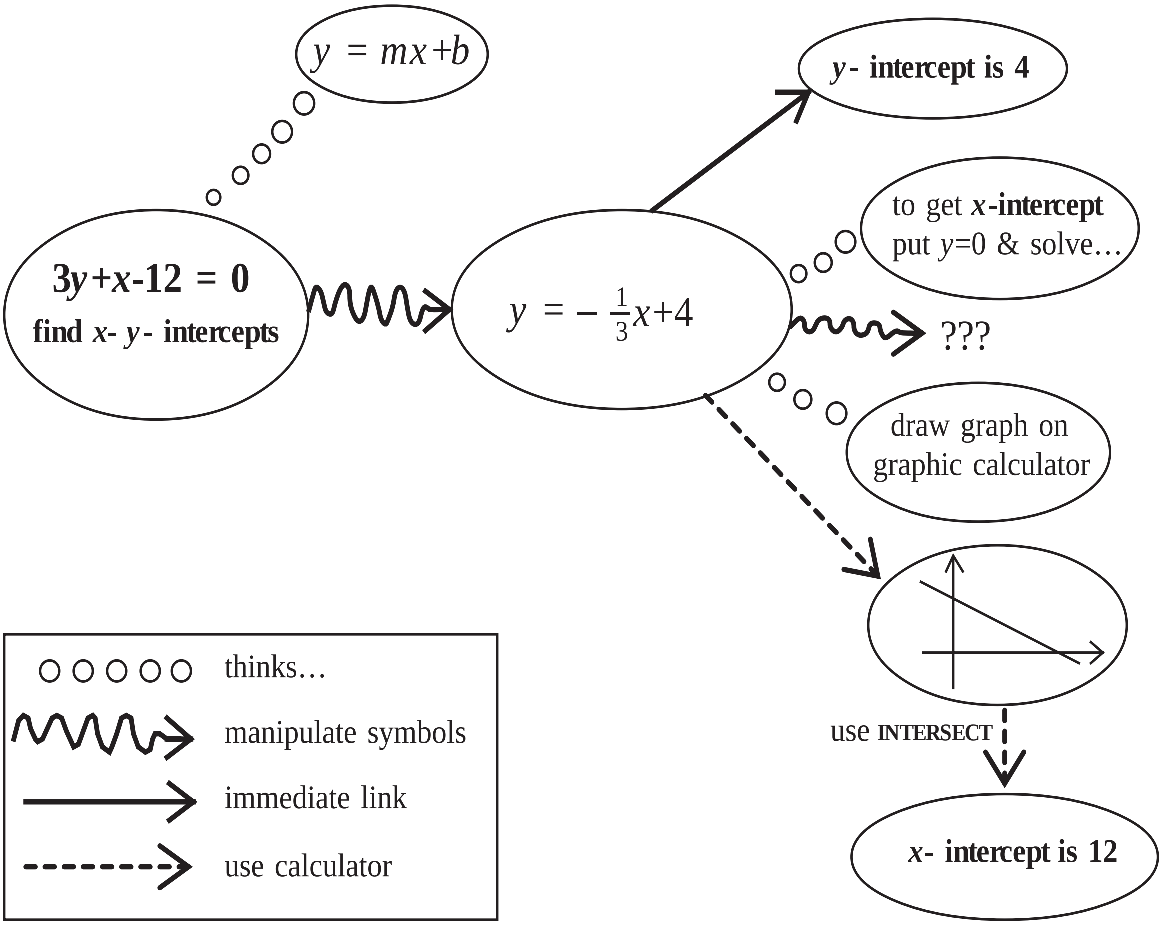


Figure 8: Kristi's strategies for finding *x*- and *y*- intercepts of $3y+x–12=0$.

Student concept maps

One method of investigating the growing conceptual structure of students is to ask them to build up their own concept maps of the material covered as the course progresses. McGowen (1998) did this with a class of college students studying a preliminary algebra course based on the function concept. Students were asked to build up a concept map using moveable "post-it" labels before making a permanent record.

Figure 9 shows the concept maps of Student SK after 4 and 9 and 15 weeks respectively. This should be compared with the concept maps of Student MC (figures 10, 11).

There are many ways in which these may be compared. Given the possible distinctions earlier with the procedure-process-procept spectrum, it is interesting to note that the final map of Student SK is described in terms of processes (simplify, evaluate, solve), whereas Student MC refers more to *concepts* rather than processes.

However, in the case of these two students, there is a remarkably simple way in which the schematic outlines of these concept maps can be considered. Here we remove all reference to the actual items in the maps and simply look at their successive growth, adding or removing items, or moving items elsewhere (figure 12).

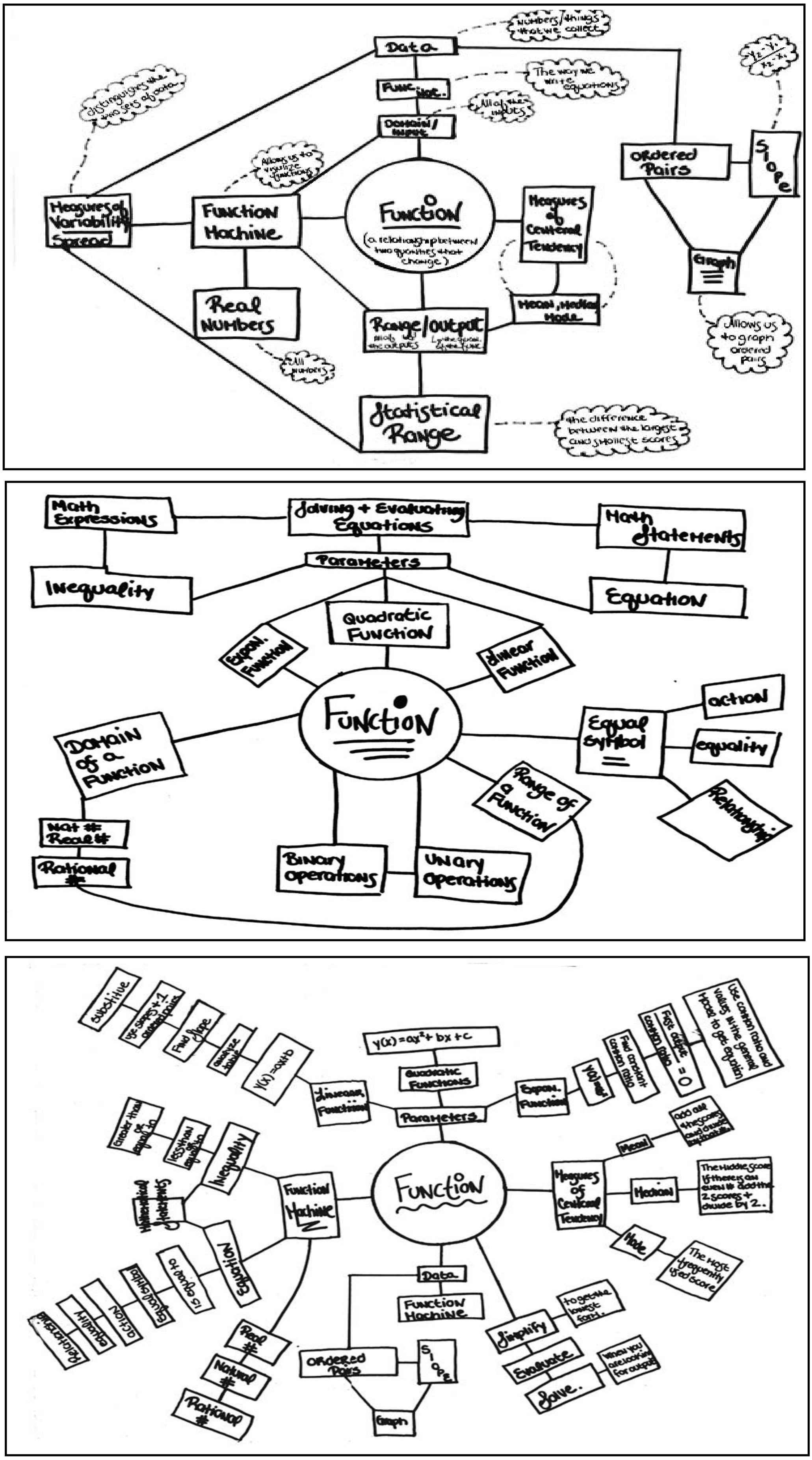


Figure 9: Student SK concept maps after 4, 9 and 15 weeks

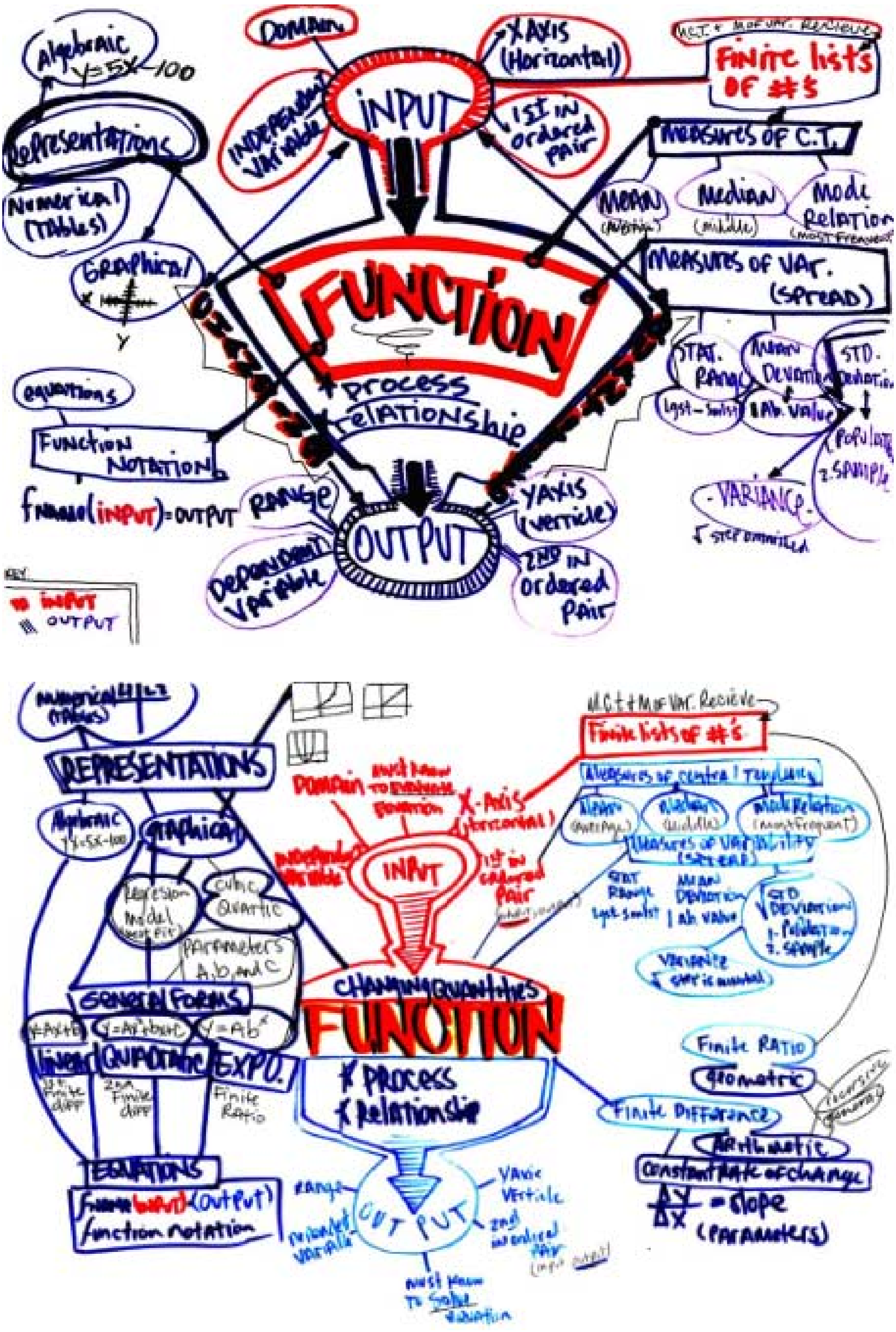
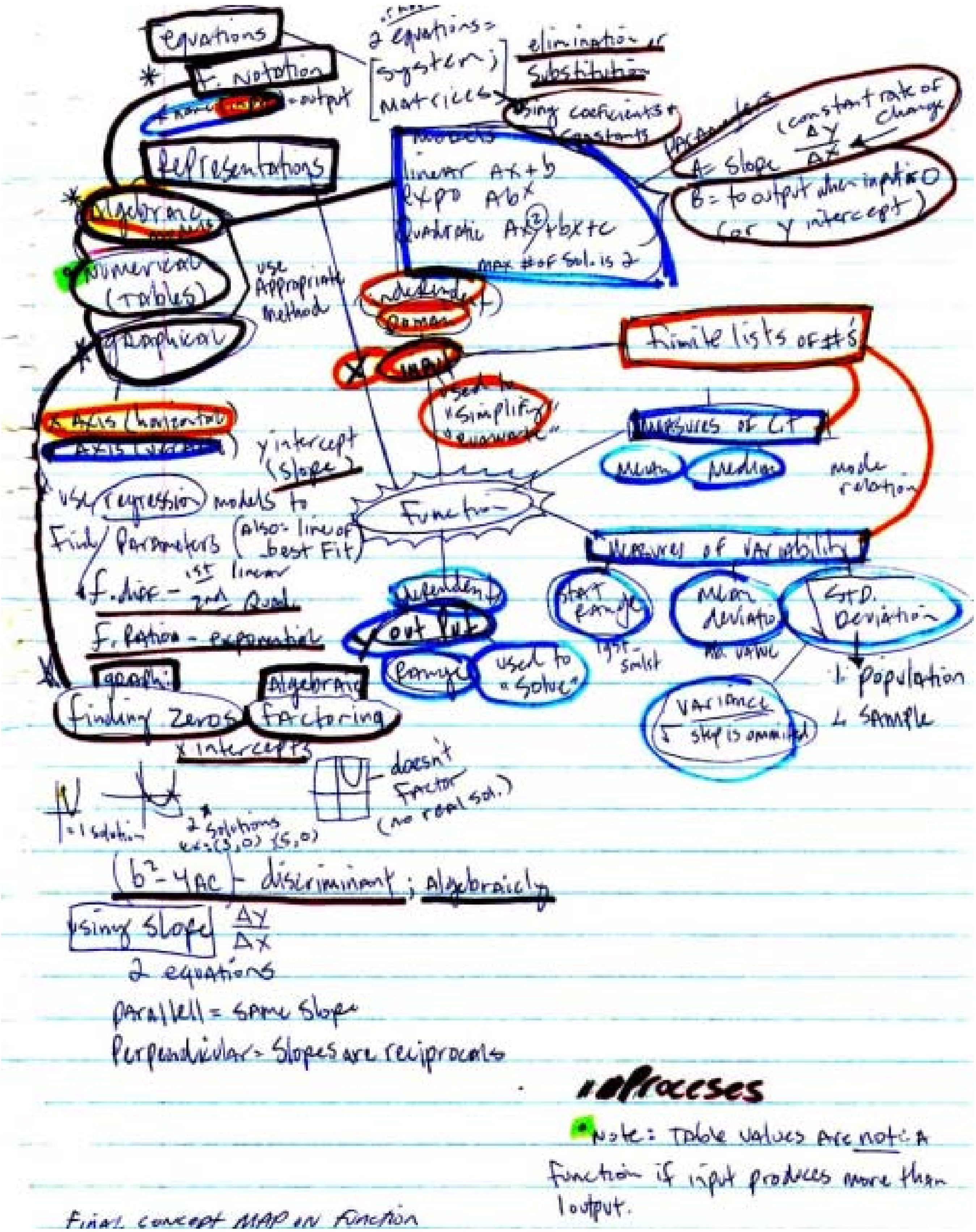


Figure 10: Student MC, concept maps after 4 and 9 weeks



Student MC : sketch for concept map after 15 weeks
(The final map is of the same form as his others, but too large to reproduce here)

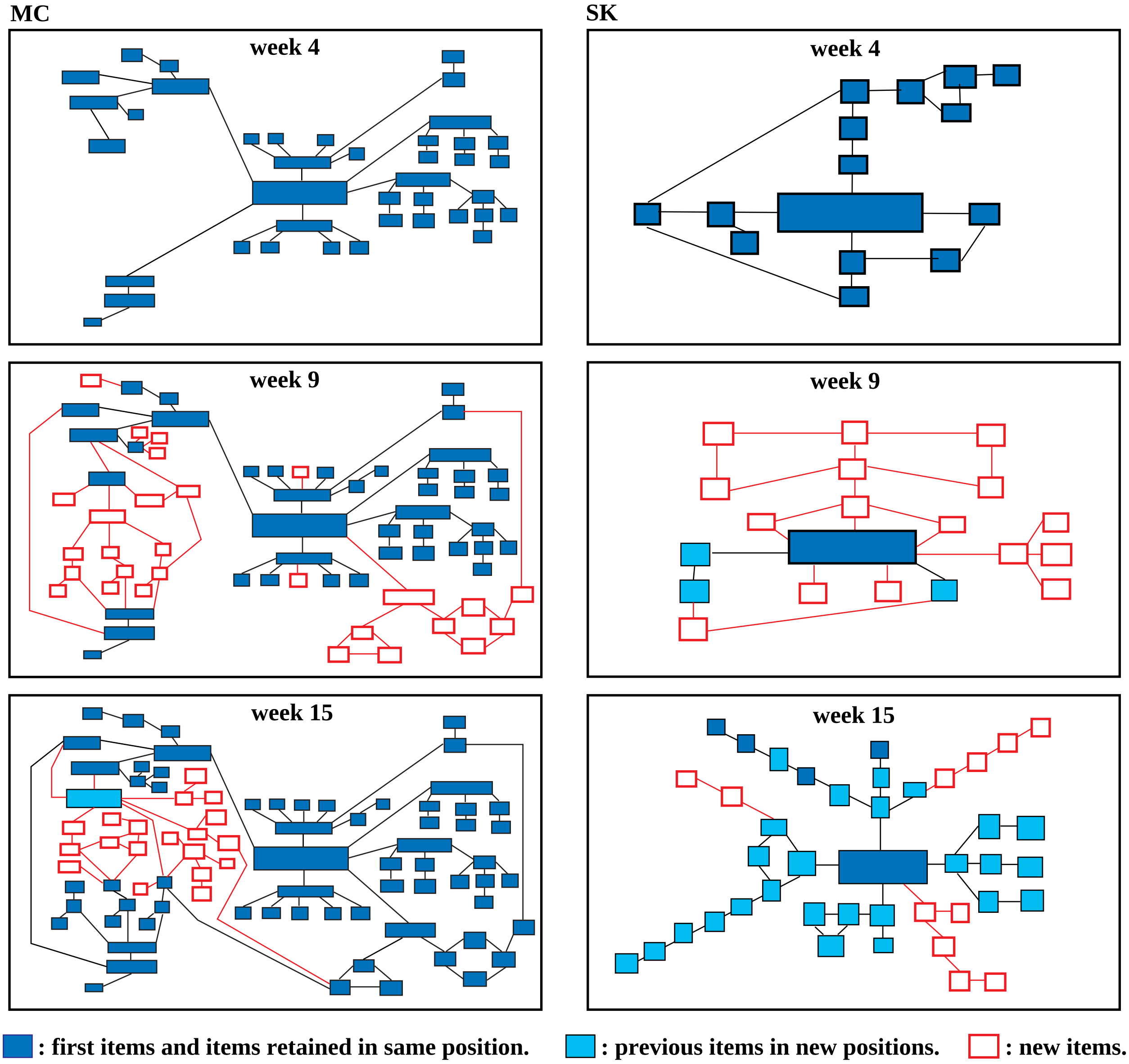


Figure 12: Schematic concept maps for students MC and SK

Note that MC has a more detailed and complex concept map than SK. More interesting, while MC builds new material on older foundations, only moving one item to a new position in week 15, student SK starts each map almost anew with only the central item (function) remaining constant. In this way, MC (who proves to be far more successful) builds a more comprehensive and more detailed structure of a growing complex of related concepts whilst SK (who is far less successful) builds a new map every time. SK appears to lack the stable connections that give MC the conceptual power, leaving SK mainly with procedures or processes to carry out. Whilst SK stays procedural, MC builds a more conceptual structure.

Conceptual Preparation for Calculus Procedures

As some students develop from single procedures to use alternative more efficient solutions, it is interesting to see how flexible they become in solving calculus problems that essentially only require the selection and operation of an appropriate procedure. The rules of calculus, such as the derivative of a product or quotient may benefit from a little conceptual simplification before carrying out the algorithm. For instance, the problem:

Determine the derivative of $\frac{1+x^2}{x^2}$

becomes quite complicated if it is treated immediately as a quotient:

$$y = \frac{1+x^2}{x^2},$$

$$\frac{dy}{dx} = \frac{(2x)(x^2)-(2x)(1+x^2)}{(x^2)^2} = \frac{2x^3-2x-2x^3}{x^4} = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

However, if the expression is first simplified as $x^{-2}+1$, then its derivative is immediately seen to be $-2x^{-3}$, affording a considerable reduction in processing. More successful students may be able to “see” the symbol $\frac{1+x^2}{x^2}$ as two fractions like this:

$$\frac{1}{x^2} + \frac{x^2}{x^2}.$$

By seeing $\frac{1}{x^2}$ as x^{-2}, and $\frac{x^2}{x^2}$ as +1, the solution can be written in a single step.

Maselan Bin Ali (1996), chose 36 students in three groups of 12, who were high (grade A), medium (grade B) and low (grade C) achievers respectively. The students in the various grades performed as follows:

Students’ grade	Conceptually prepared	Post-algorithmic simplification	No further simplification
A	10	2	0
B	6	6	0
C	4	7	1
Total	20	15	1

Table 3: Student responses to a differentiation problem

The difference between the A and C grade students is significant at the 5% level using a χ^2 test (with Yates correction). The more successful students are more likely to use conceptual preparation to minimise their work in carrying out the algorithm.

Students were asked how many different ways they could do this example, (e.g. by product rule, quotient rule, simplification first, or implicitly differentiating $yx^2 = x^2+1$). The number of students offering different (correct) methods were as follows:

Students’ grade	0 or 1 methods **[procedure]**	2 or 3 methods **[process]**
A	3	9
B	7	5
C	9	3
Total	19	17

Table 4: Flexibility of student solution processes

Of those giving two or three methods, the number of A students (9 out of 12) is significantly better than the number of C students (3 out of 12) at the 5% level using a χ^2 test (with Yates correction). However, the spectrum in this example is not an all-or-nothing phenomenon. Some A students use single procedures just as some C students show some flexibility. The A students, therefore have a greater tendency to be (at least) process-oriented than the more procedurally-oriented C students (Ali & Tall, 1996).

The Transition: proceptual/perceptual to formal

As students move through arithmetic and algebra, they develop a spectrum of ways to cope with procedures and concepts. Those who move on to study mathematics at university may be expected to be at the more flexible part of the procedural-proceptual spectrum of symbol manipulation (although this is not universally true, some are just extremely competent at mathematical procedures). Some may also have become increasingly sophisticated in dealing with perceptual objects, moving through real world prototypes and platonic objects on to Euclidean proof. They now move on to encounter the use of formal definitions and proof (Figure 13)

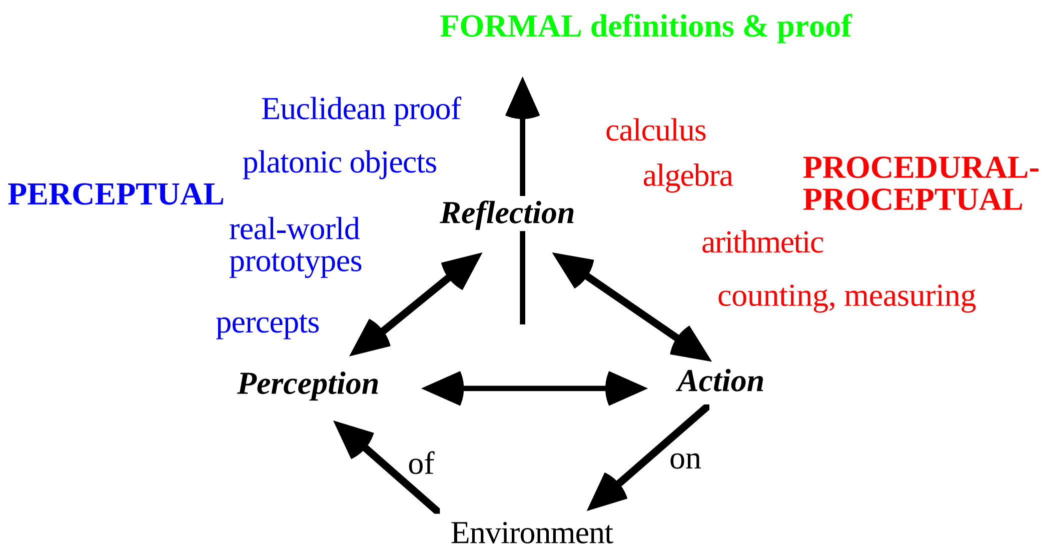


Figure 13: From perceptual & procedural to formal mathematics

The introduction of formal mathematics presents a serious new discontinuity. In almost all previous experience, students have encountered objects that possess properties and symbols that can be manipulated. In both cases, the meaning of the objects and symbols comes from the experience of playing with them and finding out their properties. In formal mathematics this development is *reversed*. The student is now presented with *definitions* in words and symbols that give rise to new mathematical entities through deduction, building up their properties through a sequence of theorems and proofs.

Students have *enormous* difficulty coping with this new view of definitions. Their current rich conceptual structure where they already "know" a great deal of mathematics is not entirely consistent with a formal theory where everything must now be deduced from definitions by logical inference. The fact is that *mathematicians do not use only logic*. They have an interplay between imagery (to suggest) and deduction (to prove). Likewise there are different tendencies noticeable among students. Marcia

Pinto (1998) studied a spectrum of students working through a beginning analysis course to see how they handle definition and deduction. She found two widely differing strategies, often used separately by different students, occasionally used in tandem by others:

- ***giving meaning*** to a definition from a range of personal images, percepts, processes, etc.,
- ***extracting meaning*** from the definition through routinising the definition through use and deduction.

Both intend to construct a *formal concept* (consisting only of those things that can be deduced from the definition). Pinto found that *students can be* ***successful*** *with either route*, but that both routes were very difficult and were often not completed during the course.

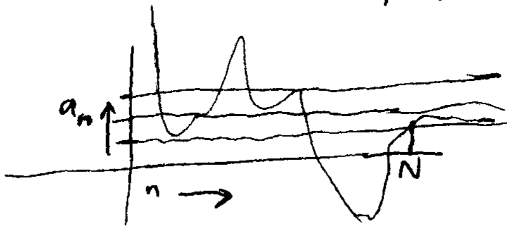
Each strategy has different places where difficulties occur. For instance, the student who *gives* meaning is involved in continual reconstruction of ideas as (s)he expands mental images to take account of new phenomena. The student who *extracts* meaning first must routinise the definition to be able to use it, and then use this definition to build up a repertoire of properties that have been proved from the definitions. More often students fail. Those who attempt to "give meaning" from perceptual images, so they can "see what happens", often find they are trying to prove something that is "obvious" for which the far more obscure proof has no meaning. These can have a "sense" of what is going on but fail to do any more than rote-learn proofs for exams. Alternatively those who attempt to "extract meaning" from a definition that they often cannot remember, let alone understand, are in even greater difficulties. They may not lack mental pictures, but these are not generative in the same way as those who "give meaning". Instead they often represent a single instance (such as a monotonically increasing sequence that does not 'reach' a limit). They are therefore inflexible and intimate properties that are not implied by the formal definition. Such students have only confused images and weak grasp of formalism, so that little progress is possible beyond minimal rote-learning. (Pinto, 1998).

Examples of successful students giving and extracting meaning

Chris was a remarkable student who built his definitions by giving meaning from his imagery:

> "I don't memorise that [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down."

~~If $a_n \to L$ then there~~ exists
For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$
such that $|a_n - L| < \varepsilon$ for all $n \geq N$



He explained as he drew the picture, gesturing with his hands to show that first he imagined how close he required the values to be (either side of the limit), then how far he would need to go along to get all successive values of the sequence inside the required range:

> ”I think of it graphically ... you got a graph there and the function there, and I think that it’s got the limit there ... and then e once like that, and you can draw along and then all the ... points after *N* are inside of those bounds. ... When I first thought of this, it was hard to understand, so I thought of it like that’s the n going across there and that’s an. ... Err this shouldn’t really be a graph, it should be points.” (Chris, first interview)

Ross, on the other hand, took a formal approach, extracting meaning from the formal definition. He explained that he learns the definition:

> “Just memorising it, well it’s mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it gets imprinted and then I remember it.” (Ross, first interview)

The difference between *giving* and *extracting* meaning is shown strikingly when students are asked to say what it means for a sequence *not* to converge. Chris was an exceptional student. From his long experience working mentally with the concept, he was able to perform a direct thought experiment and write the definition of non-convergence straight out:

A sequence (a_n) does not tend to a limit if
for any L, there exists $\varepsilon > 0$ such that
$|a_n - L| \geq \varepsilon$ ~~whenever n ≥ N, for~~
for some $n \geq N$ for all $N \in \mathbb{N}$

(Chris, second interview)

Ross, an extractor of meaning, first wrote the definition of convergence and then negated successive quantifiers symbolically. By negating the definition:

$$\forall\, \varepsilon > 0,\ \exists\, N(\varepsilon) \in \mathbb{N} \text{ s.t. } \forall\, n \geq N$$

$$|a_n - L| < \varepsilon$$

he obtained his version of non-convergence as:

$$\forall L,\ \exists\ \varepsilon>0\ \text{s.t.}\ \forall\ N(\varepsilon)\in\mathbb{N}\ \exists\ n\geq N,\ \text{s.t.}$$
$$|a_n - L| \geq \varepsilon$$

(Ross, second interview)

Notice that both students made errors. In his explanation of convergence, Chris drew a continuous curve when he should have drawn discrete points. However, he was not focusing on the precise nature of the drawing, rather on the *behaviour* of the sequence as it moved about up and down, getting within a prescribed range either side of the limit. He experimented with these ideas over several weeks so that he had an integrated meaning for convergence and non-convergence as part of a highly connected conceptual structure. Ross made a subtle error in his negation by writing "for all $N(\varepsilon)$" where N no longer depends on ε. This arose because he had written "$\exists\ N(\varepsilon)$" in the definition to give it greater meaning than simply "$\exists\ N$". Had he written the latter, the error would not have occurred (or would not have been noticed). Even *successful* students make errors as they attempt to come to terms with complex ideas. They become more successful because they overcome these setbacks.

Giving meaning unsuccessfully

Many students were unable to cope with the limit concept. Laura evoked many personal images for the idea:

> "The number where the sequence gets to, but never quite reaches."
>
> Let a_n be the sequence and L is the limit which it tends to. Then when some initial values are placed into the formula of the sequence the answers will never reach the value of L (negative or positive).
>
> "... oh, yes, I put 'never reach', and it *can* reach, and that will be the limit of it. ..."
>
> "... But it won't never get bigger than the limit. The limit is like the top number it can possibly reach . And I put never reach."

(Laura, various quotations, first interview)

However, she was unable to write down the definition in any formal sense, although she had mental pictures which gave her meaning for some of the theorems. She could "see" things in an idiosyncratic manner, but not prove them.

Extracting meaning unsuccessfully

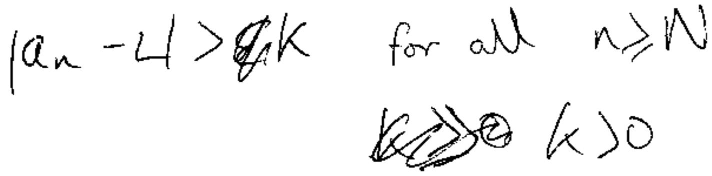
Rolf attempted to learn the definition. He thought he had learned it properly, but was mistaken:

> "Umm ... I wrote it many times because we use it all the time, every time we are asked a question we have to use and that's how I remembered it. (??) *I don't think I will ever forget it now*. We have done it so many times."

$$(a_n) \to L \quad \text{if}$$
$$|a_n - L| < \varepsilon \qquad \begin{matrix}\varepsilon\in\mathbb{Z}\\ \varepsilon>0\end{matrix} \qquad \text{for } n\geq N$$

(Rolf, first interview)

He was consequently unable to write a correct definition of non-convergence, writing:



(Rolf, second interview)

As with many students, Rolf could not handle the quantifiers in a satisfactory way. Many students worked from the inner quantifier out, some negating the definition simply by changing round the inequality in the inner statement from “<” to “>”or “≥”, others coping with one or two quantifiers (see Dubinsky *et al.*, 1988).

We thus see the move to formal mathematics occurring in different ways. Some students build up strong imagery which is flexible and capable of being used to suggest and support formal proofs, others attempt to build a consistent theory from the definition in a formal way. Neither of these routes is easy, and many fail.

Reflections: Considering the broader picture

Now we have seen that the development through symbolic and axiomatic mathematics involves a number of discontinuities involving changes in meaning of concepts and the use of symbols to compress different kinds of processes into new types of concept. At the formal level these are compounded by coping with the use of definitions which students respond to in a variety of ways. Figure 14 shows an outline of the development, with a number of discontinuities marked.

Some of these discontinuities have figured in earlier discussion. The diagram is also liberally sprinkled with various possible lines of discontinuity to underline the fact that that reconstruction of mental concepts is a vital part of mathematical learning. As mathematicians, we may not be aware of the precise nature of students' difficulties. This suggests that mathematics *cannot* be structured as a simple curriculum steadily expanding the concepts building on old foundations in established ways. It requires

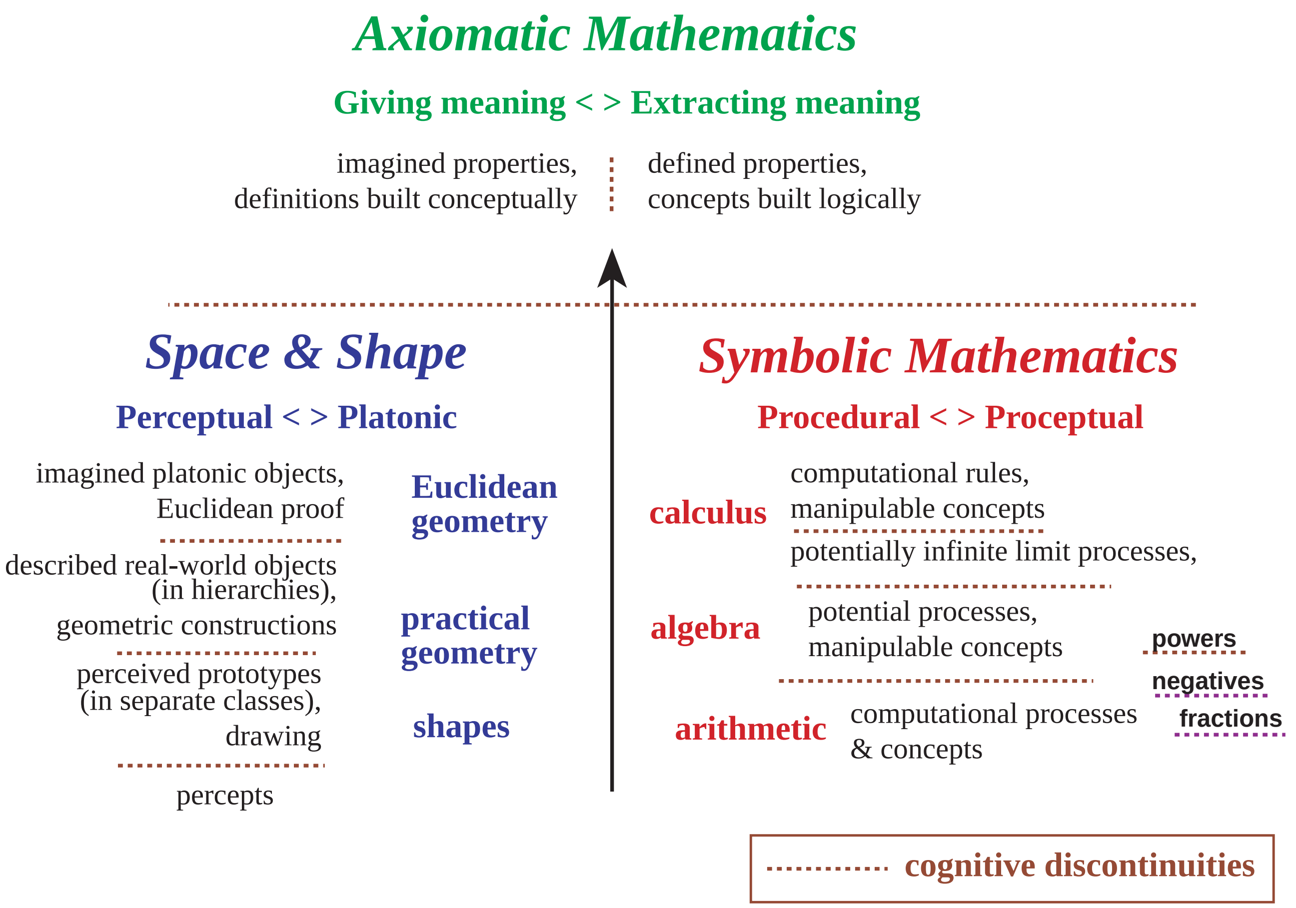


Figure 14: Cognitive growth in selected topics in mathematics, with associated *discontinuities*

constant re-thinking of concepts which proves possible, even invigorating, for some, but forces others into rote-learnt rules to cope in new contexts where the old ideas no longer hold true. In our examples of student development in proof, Chris positively enjoyed the struggle of making sense of ideas that confused him. He had a long experience of the satisfying pleasure of success and now sought the excitement of the struggle to maintain his high state of mental awareness. Laura, on the other hand, had learnt to fail. She took no pleasure out of failing again and could not begin to make sense of formal proof in a context which she would never meet again in her life as a primary school teacher.

The whole of the curriculum, from elementary school through university mathematics is a fascinating journey of reconstruction and conquest which appeals to those who develop a taste for the struggle through successive victories and overcoming defeats, but it is a minefield for others, who may genuinely attempt to understand mathematics at one level, yet are thrown off course by a discontinuity in learning which renders many new ideas incomprehensible.

Is there a moral to this tale? Certainly I do not claim that *all* students can learn mathematics if it is "presented right" and they are willing to "work hard". The "American dream" that *anyone* can do *anything*, I believe is causing a nightmare for a vast number of students studying mathematics. The evidence shown here is that some failing students are faced with a much greater cognitive load to achieve a lesser, more pragmatic, procedural goal. On the other hand, I do not claim that it is *impossible* for a specific student to conquer specific difficulties. Certainly there will be individuals who have at one time failed who succeed despite the initial complexity of their view of the task in hand.

However, all of this must be seen in the wider context of the processes of development involved in learning mathematics through arithmetic, algebra, calculus and analysis. It is a challenge that leads to a bifurcation between those who succeed in compressing knowledge into a flexible form and those who tend to seek security in learned procedures. Whilst the flexible knowledge compressors have a more powerful system at their disposal, the procedural learners may be able to solve routine problems but have a cognitive structure which makes it more difficult to build up further sophisticated knowledge.

I have personally addressed this problem by using a variety of tactics. These include attacking the concepts of calculus using *visual imagery* to underpin the ideas of rate of change and cumulative growth (e.g. Tall, 1985; Tall & Blokland, 1990), giving a physical *enactive* basis to the solution of differential equations using computer software (Tall, 1991, 1993), or using *programming* to allow the student to focus either on the steps in a *process* of evaluation of an expression or on the *concept* of equivalent expressions that for given input always return the same output (Tall & Thomas, 1991).

Given the way in which students' development diverges into a spectrum of qualitatively differently thinking, I do not believe there is a single way of *teaching* mathematics without taking into account different ways of student *learning*. The human interface between teaching and learning is a constant source of renewal and frustration which will encourage imaginative teachers to keep seeking for a pragmatic solution that respects individual student needs in complex learning situations.

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