

Information Technology and Mathematics Education: Enthusiasms, Possibilities and Realities

David Tall

Mathematics Education Research Centre
University of Warwick
Coventry, CV4 7AL, UK

This presentation addresses critical issues in the use of information technology in Mathematics Education. By reflecting on human thinking processes, it will consider developments of enthusiastic researchers using technology to teach mathematics as new facilities develop, the possible gains shown by this research and the realities that may be achieved in the classroom.

1. A time of great change

We live in changing times. Noble species which have been on the earth for millions of years such as the whale and the elephant are threatened with possible extinction by mankind and now a human sub-species, the mathematician, may be under threat from the competition of information technology.



Will Homo Mathematicus become extinct?

As the President of the Royal Society, Sir Michael Atiyah, has said:

Whereas the eighteenth and nineteenth centuries witnessed the gradual replacement of manual labour by machines, the late twentieth-century is seeing the mechanisation of intellectual activities. It is the brain rather than the hand that is being made redundant. (Atiyah, 1986, p. 43.)

The performance of routine tasks traditionally taught in mathematics education has been taken over by technology in a spectacular way. The supermarket

checkout assistant no longer adds up the cost of the items and calculates the change. Software using a machine bar-code reader not only does the arithmetic, it also prints out an itemised bill for the customer and automates stock-control for the trader. Does this mean that traditional mathematical skills are becoming less important?

Information technology highlights the difference between being able to perform standard skills and being able to “think mathematically”. Current technology is no match for a creative mathematical mind. As Edward de Bono observed on a recent BBC TV “Brains Trust”, the “poor engineering” of the human brain gives it the ability to make associative links and leaps of insight. Because of its logical and orderly design, today’s technology is incapable of musing, as Einstein could imagine, “*what would happen if I were sitting on a train travelling at nearly the speed of light?*”

This imaginative strength is the product of the complex way in which the brain works, and in turn is linked to the way in which human learning occurs. Whereas the computer can be reprogrammed by replacing its software, erasing all previous data from the memory, the human mind is built up through a life-time of experience and evolves by building the new upon the old, subtly retaining elements of the old alongside the new. The corporate beliefs of the mathematical community therefore serve as a stabilising factor, preserving the familiar and taking time to adjust to new possibilities.

Meanwhile, technology changes at an extremely fast pace that predicting the next stage is a hazardous business:

If you take the way the Internet is changing month by month – if somebody can predict what’s going to happen three months from now, nine months from now, even today, my hat’s off to them. I think we’ve got a phenomenon here that is moving so rapidly that nobody knows exactly where it will go.

(Bill Gates, 1996).

The result is that enthusiasts are forever chasing the cutting edge of technology, often moving on to the next innovation before the wider community has absorbed the last one, and operating at a speed which means that the long-term effects are often not known until long after the changes have already been made.

So how can we attempt to make sense of the impact of information technology? My own chosen route is to be aware of technological changes and possibilities, but to see how they interact with the nature of human learning. As mathematics *educators* we need to know the realities as well as the possibilities for human learning in an age of information technology.



But can a computer *think*?

2 Different forms of mathematical knowledge

The first step is to consider the nature of mathematical knowledge, to see how different parts of this knowledge structure are effected by technology.

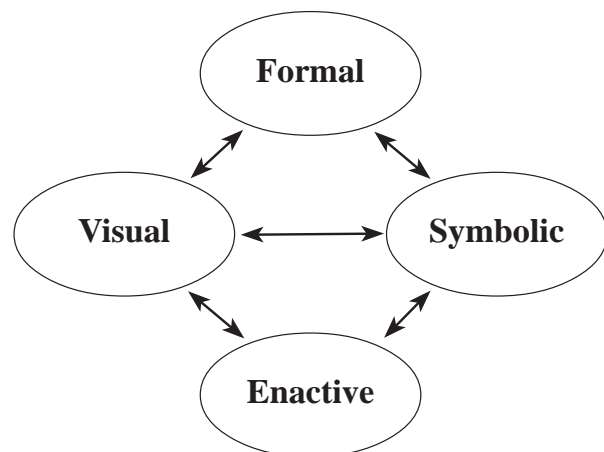
Human evolution passed through several million years before the development of speech. The first form of mathematics was therefore *enactive*, involving physical manipulation of objects. This remains the first form to be encountered by the developing child and forms the initial stages of mathematics education. *Pictorial* representations in the form of cave paintings are 30,000 or 40,000 years old and *written language* developed in Phoenicia some 5,300 years ago, by which time arithmetic notation was already being used in trade and exchange. *Arithmetic* symbolism of various forms, for counting and measuring, developed in ancient civilisations such as those of Mesopotamia and Egypt, then two and a half thousand years ago the Greeks developed the abstract theory of *geometry* expressed *verbally* through Euclidean proof.

Manipulable *algebraic* symbols were introduced comparatively recently in the sixteenth century and the flowering of calculus occurred in the seventeenth. It is the ability to *calculate* with symbols that has contributed to the vast acceleration of human achievement in the last three hundred years and it is this which has become the focus of mathematics education in schools.

It is only in the last century that the attempt has been made to reorganise the whole of mathematical knowledge into a *formal* theory, founded on verbal definitions and logical deductions.

Before the development of the computer, we therefore had various forms of mathematics, including:

- (1) *Enactive mathematics*, with physical actions on actual objects,
- (2) *Visual mathematics described verbally*, with physical properties of objects verbalised and built into a systematic deductive theory as in Euclidean geometry,
- (3) *Symbolic mathematics*, (arithmetic, algebra, calculus etc.), arising from actions on real-world objects (such as counting) and developing through computation and symbol manipulation,
- (4) A combination of (2) and (3) linking symbolism and graphical representation,
- (5) *Formal mathematics*, with concepts *defined* by verbal-symbolic axioms and further properties *deduced* by formal proof.



These different forms of mathematics interrelate in a complex way, but they do have different characteristics which can give insight into the learning process.

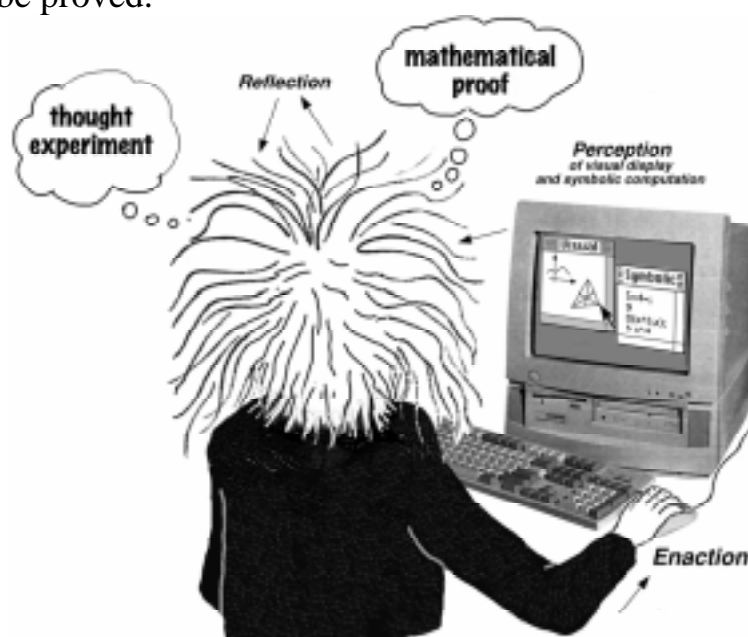
3 New computer facilities

Computer technology developed in a sequence which contributed to different parts of this knowledge structure, setting successive agendas for mathematics education. The arrival of the computer first focused on the elementary *symbolism of numeric computation*, then had a *graphical display* added, followed quickly by an *enactive* interface allowing selection and manipulation of objects drawn onscreen. Software to enable *symbolic manipulation* required more sophisticated programming and has gone through several reincarnations to produce a more human user interface.

So far we have used the computer less in handling formal proof in mathematics education (with the honourable exception of the use of a language such as ISETL with the formal structure of set theory complete with quantifiers and logical implication.) “Theorem proving” and “theorem checking” software exist in certain contexts, and computers have been used to carry out lengthy checking procedures beyond the capacity of the individual, such as in the celebrated computer proof of the Four Colour Theorem (Appel & Haken, 1976). But standard computer technology still has the Achilles heel noted of the pioneering design of Charles Babbage in the nineteenth century:

The Analytical Engine has no pretensions whatever to originate anything. It can do whatever we know how to order it to perform. It can follow analysis; but it has no power of anticipating any analytical relations or truths. Its province is to assist us in making available what we are already acquainted with. (Ada Lovelace, *Observations on Mr Babbage’s Analytical Engine*, quoted in Evans, C, 1983, p. 31.)

Although modern computers provide an enactive human interface with manipulable visual display and symbolic facilities, it still needs the mind of a mathematician to perform thought experiments to decide what is important and what needs to be proved.



Creative human thought and algorithmic computer processing

4 Computers in mathematics education

Before computers became widely available, there was scepticism about their value in education:

It is unlikely that the majority of pupils in this age range will find [a computer] so efficient, useful and convenient a calculating aid as a slide rule or book of tables.
(Mathematical Association, *Mathematics 11 to 16*, 1974.)

Such illusions were soon shattered and slide rules and books of tables lingered for only a short time before they became obsolete.

4.1 Numerical algorithms

The first microcomputers (for instance, the Apple, in 1976) were sold with the BASIC language available for programming. So the first enthusiasms were for mathematicians and their students to program their own numerical methods. The enthusiasts believed that by *programming* the students could learn to understand the processes of mathematics. In reality there were too few computers for wide-spread student programming at the time and so the practice did not spread far. Research was produced to show that children programming in BASIC had a better insight into the use of letters as variables in algebra (e.g. Tall & Thomas, 1991). But BASIC had a bad press as a poorly structured language and by the time more computers became available the agenda changed and programming in BASIC was widely regarded as ancient history.

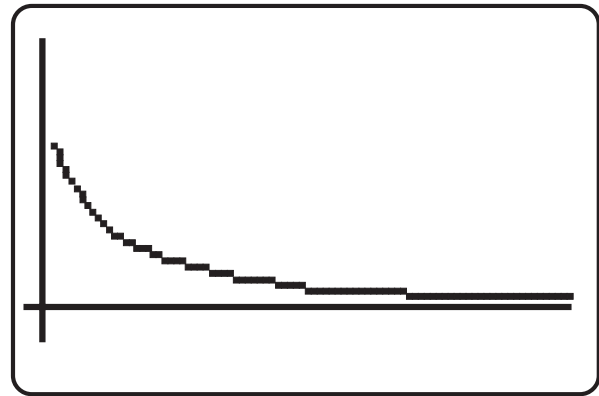
4.2 Graphic visualisations

In the early eighties high resolution graphics brought the next stage, including such things as graph plotters to represent functions and programming in Logo for children of all ages.

The visual possibilities also brought the experimental study of chaos and fractals by mathematicians and introduced new *graphical* approaches to the teaching of such things as geometry, statistics, calculus and differential equations. The student could now be helped by *visualising* mathematical ideas. This was a time of great creativity with mathematics educators writing little pieces of software to visualise mathematical concepts.

There was soon considerable evidence that a visual approach to graphs helped students to gain a wider conceptual understanding without necessarily affecting their ability to cope with the corresponding symbolisation.(e.g. Heid 1988, Palmiter 1991). But on the debit side there was also evidence that students who lacked the sophistication to interpret the meanings of the graphs could develop serious misconceptions.

A classic instance was the case of young children watching the cooling curve of a liquid on a display with fairly large pixels, seeing the move from one pixel to another as a sudden drop in temperature (Linn & Nachmias, 1987).



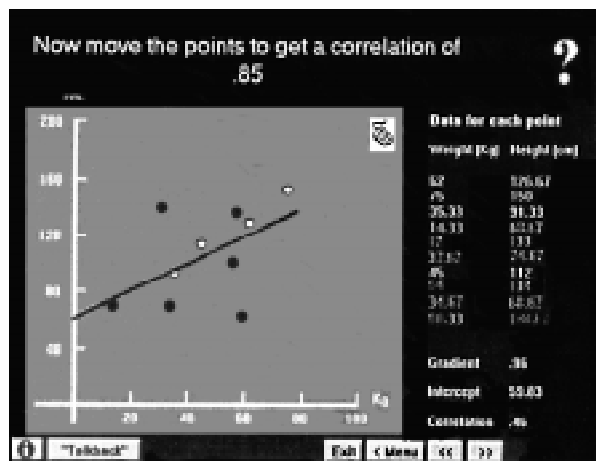
Apparent sudden drops in temperature

In the same way, when drawing graphs of functions, the choice of range to give a suitable picture becomes crucial and it is possible to misinterpret the meaning of the graph on the screen. (Goldenberg, 1988).

4.3 Enactive control

In 1984 the “mouse” was introduced to give the computer an enactive interface. Instead of having to type in a line of symbols, the user could now select, and control the display by intuitive hand-movements. This allowed a completely different approach to learning which encouraged active exploration rather than first learning to do procedural computations.

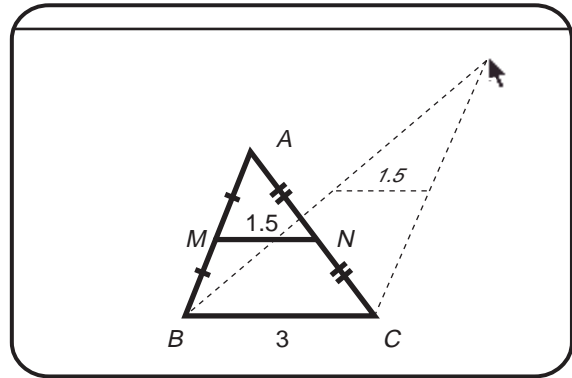
For instance, statistics is often taught by procedural “cook-book” methods because few teachers, let alone students, have the experience to understand the underlying formalities. Yet software allowing an enactive exploratory environment can be used to give a “sense” of the nature of statistical data and to see how robust interpretations are when they are subject to variation. For example, in giving a line of “best fit” to data, and computing various rules such as “least squares fit” it is possible to use computer software to enactively move the line of fit until it looks good by eye, or to move data points around and to see how this causes variations in the various fitness measures.



Visual statistics under enactive control

In this way sophisticated mathematical concepts can be given an intuitive visuospatial meaning without (or before) the need to study the procedures that the computer was using for internal computation.

Interactive programs in geometry offer enactive exploratory environments giving new dynamic conceptualisations of geometric figures. For instance, a triangle ABC with the midpoints M , N of AB , AC joined could be pulled around by holding on, say, to the vertex A to see that the length of MN is always half that of BC . The figure takes on a new meaning which holds whatever position the triangle is moved to,



Manipulating a triangle with midpoints joined

subject to the given construction and provides a rich environment for exploration and hypothesising. However, note that the available actions involve selecting a point and pulling it round. There is no move that lifts up one triangle and, retaining its size and shape, allows it to be moved onto another (congruent) triangle. The environment does not contain the seeds of Euclidean transformations and leads to a different kind of mathematical knowledge from that required in the systematic building of theorems and Euclidean proofs.

4.4 Computer algebra systems

Computer algebra systems had been around in various guises before the *American Mathematical Monthly* carried a full page advertisement for the computer algebra system *MACSYMA*, in the fateful year 1984. It asserted that the software

... can simplify, factor or expand expressions, solve equations analytically or numerically, differentiate, compute definite and indefinite integrals, expand functions in Taylor or Laurent series.

In less than a decade, computers had successively developed *numerical*, *graphical* and *symbolic* facilities each offering new methods of conceptualising mathematical ideas and these came to be conceived as the three major representations in college calculus:

One of the guiding principles is the ‘Rule of Three,’ which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and where students see each major idea from several angles. (Hughes Hallett 1991, p. 121)

The American calculus reform is based on a wide range of software that uses various representations. There is evidence that students learn to use the computer algebra systems to “think with”, by formulating the solution of problems in a way that can be carried out by computer algorithms. (Davis et al, 1992). However, there is also evidence that many students using computer algebra systems do not understand what is going on internally and do not link the mathematical ideas in the same way as those with a more traditional experience. For instance, students may use graph-plotters to “see” solutions of

equations, but not necessarily relate them to the symbolic meaning of the problem. Caldwell (1995) expected students to find the roots and asymptotes of the rational function

$$f(x) = \frac{x(x-4)}{(x+2)(x-2)}$$

by algebraic means, only to be given a substantial number of approximate solutions such as 0.01 and 3.98 read from the graph. Hunter et al. (1993) found that a third of the students using a computer algebra system could answer the following question before the course, but not after:

‘What can you say about u if $u=v+3$, and $v=1$?’

As they had no practice in substituting values into expressions during the course, the skill seems to have atrophied.

The reality of the classroom can prove to be different from the possibilities envisioned by enthusiasts.

4.5 Personal portable tools

The technology migrated from desk-top machines to portable calculators and computers for personal use. Four function calculators progressed to include scientific functions, then programming facilities, then graphical representations.

In 1996 we now have hand-held computers which will do all the numeric and



symbolic algorithms which were the staple diet of mathematics exams and includes an implementation of *Cabri Géometre* to explore geometric ideas. This offers most of the facilities discussed so far, with the added advantage that it can be used at will by the student at any time in any place, though it retains an input-line of commands and lacks the freedom of fully enactive computer environment.

4.6 Multi-media

The last two or three years have seen the development of multi-media interactive software to use for individual study. This, as yet only partially realised, facility promises to allow the learner to have a variety of materials giving explanations in text, words, video, within a software environment that offers interactive facilities to explore mathematical processes and concepts. It allows the possibilities of the return of smaller interactive units from the eighties to be embedded in a more coherent overall environment.

4.7 The World Wide Web

More recently the world-wide web has become a reality, allowing information and software to be passed from one individual to another around the globe. World-wide mathematical courses for multi-media interaction are becoming available. Students increasingly have freedom to access software at any time to suit their own timetable, offering yet new promises for the future. Currently the promise is often different from what happens in reality, as the internet gets clogged up with huge numbers of users and the bandwidth available is often too narrow to transfer large amounts of data for sound and pictures in real-time.

People say the Internet is carrying multimedia today, but then dogs can walk on their hind legs. (Robert X. Cringely, *Accidental Empires*, 1996, p. 344.)

Yet change comes quickly and greater carrying capacity is around the corner so that the world-wide information superhighway seems to be inevitable.

5 Developing a theory to consider the evidence

So how do we make sense of this change? It is evident that information technology is here to stay and we as mathematics educators need to come to terms with its use. Many teachers at this moment are suspicious about technology which carries out processes that they have devoted a life-time to teach to their pupils. It is easy enough to express Luddite opinions and to fear the practicalities that will change our livelihood. At the same time we should attempt to develop some kind of understanding of the processes involved that enables us to make coherent judgements as to the best use of the new facilities.

In my own work I took a route dictated by the sequence of technological development. I had done some empirical work into students' understanding of limits and happened to enter the computer world as graphics arrived, and developed a graphical approach to the calculus. At the same time I was working on a Mathematical Association Committee with many colleagues who were devoted to programming numerical algorithms, and as time passed, we attempted to take in the new ideas of symbol manipulation. To put together the diverse threads of visualisation and symbolisation, I thought about how the human being operates, *perceiving* the real world, *acting* upon it for survival and *reflecting* on personal thoughts to maximise their effectiveness.

This combination of *perception*, *action* and *reflection* fitted together to help me formulate my own views on cognitive development. I saw the contrast between what I term "object-based mathematics" typified by geometry and "action-based mathematics" typified by the actions of counting and measuring in arithmetic. I also saw that reflecting on these experiences allow mature experts to develop a "property-based mathematics" with axioms and formal deductions. In the 1960s the "new math" tried to develop a "property-based" set-theoretic approach to the curriculum. It did not work. For learning at earlier

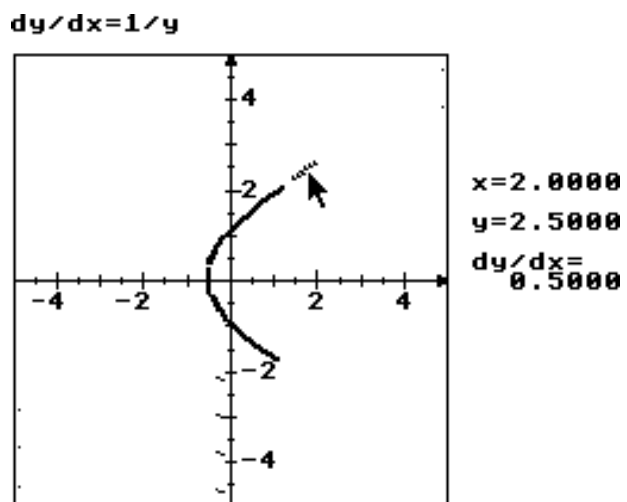
stages a combination of enactive, visual and symbolic may offer a more practical solution. It happens that these may be well-served by a computer.

5.1 Enactive and visual mathematics

The computer can provide an enactive way to manipulate visual mathematical objects. This allows powerful “sense making” of subtle concepts at a primitive enactive level. It can provide what I have termed a “cognitive root” from which a progressively sophisticated theory can grow (Tall, 1989). This can happen not only in geometry, but in other areas of mathematics. For instance it can be illustrated by the notion of a solution of a first order differential equation, embedded in wider experiences of visualising and manipulating graphs.

At the root of this idea I see the formal notion of derivative in a primitive visual way as the gradient of the graph. I do not talk about tangents, or locally linear approximations, or any formal notions in the initial stages. Simply by magnifying graphs on the computer screen, many can be seen to be “locally straight”, that is, under high magnification, they are *perceived* as being straight. This can then be linked to numeric and symbolic approaches to give the notion of derivative a computable meaning. However, the root idea of local straightness can be used to visualise the solution of the reverse problem—to construct a graph given its gradient.

In this context, computer software can use the knowledge of the gradient to draw a small line segment of that gradient. If this is under the control of the user, say by moving around with the mouse or with the cursor keys, it becomes possible to stick together short lines end to end to build up a solution of the equation. The solution is “locally straight”, in fact the picture is built up with approximate straight line segments, with its gradient given by the differential equation.



Enactively building a solution of a differential equation

In this way the computer can provide an environment in which the learner can physically experience the ideas of the mathematics at a fundamental human level. This involves vision and bodily movement without the need at the time to concentrate on the symbolism and the computations required to produce a solution.

Having obtained such human insights, it is still necessary to be able to construct a solution in a more accurate quantitative manner. The symbolic solution of such a problem involves quite different mental activities.

5.2 Symbolic mathematics

Inspired by a succession of thinkers in the cognitive development of mathematical processes and concepts, including Dubinsky (1991) and Sfard (1991), I was fortunate to collaborate with Eddie Gray to develop a viewpoint that proved useful for analysing not only how individuals use symbolism, but also how we interact with the symbolism manipulated by a computer.

We noted, as had others before us, that symbols in arithmetic, algebra, calculus, and a wide range of other mathematical contexts had a certain characteristic. The following symbols illustrate this:

$$5+4, 3 \times 4, 3a+2b, \\ \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a}, \frac{d}{dx} \left(\frac{\sin x + \cos x}{x^2 + 3x + 1} \right), \int_0^{2\pi} e^{2x} \cos x \, dx, \sum_{n=1}^{\infty} \frac{1}{n^2},$$

These all play a dual role representing both a mathematical *process* to be carried out and the *result* of that process. For instance $5+4$ evokes the *process* of addition to produce the *concept* of sum $5+4$, which is 9, $3a+2b$ is a both process of evaluation and a concept of algebraic expression, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the process of evaluating an infinite sum to find the limit value (which happens to be $\pi^2/6$).

The name *procept* was introduced for the combination of symbol, process and concept which occurs when a symbol evokes a process to give the resulting concept (Gray & Tall, 1994). We were interested in the way in which individuals interpret symbols in arithmetic, algebra and calculus, causing some students to find mathematics essentially easy yet others finding it increasingly difficult.

We emphasise that the cognitive notion of procept carries with it no implication as to how that cognitive structure is built. Indeed one of our purposes was to investigate the concept-building of such symbols. However, in pre-computer contexts, we often found that the meaning of symbols developed through a sequence of activities:

- (a) *procedure*, where a finite succession of decisions and actions is built up into a coherent sequence,
- (b) *process*, where increasingly efficient ways become available to achieve the same result, now seen as a whole,
- (c) *procept*, where the symbols are conceived flexibly as processes to *do* and concepts to *think about*.

Initially the individual builds an “action schema” (in the sense of Piaget) as a coordinated sequence of actions. At the *procedural* level, the focus of attention concentrates on how to do each step and how this leads to the next. We use the term “procedure” for a specific finite sequence of decisions and actions. In contrast the term “process” is used in a more general sense, such as “the process of addition” or the “process of solving a linear equation”. A process may have

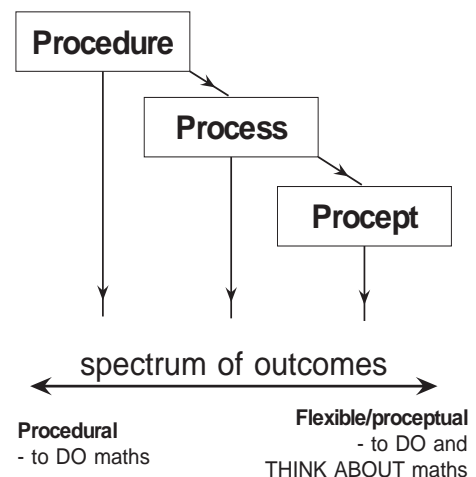
several different procedures which give the same result. For instance, the symbols $2(x+3)$ and $2x+6$ involve two different sequences of computation, but represent what we consider to be the same process. In this way the function $f(x)=2(x+3)$ is the same function as $g(x)=2x+6$ because they have identical input and output.

In the case of an addition such as $2+7$, it might be performed in a variety of ways, say by counting two sets, then both together, or starting at 2 and counting on 7, or counting on 2 starting at 7, or simply knowing that $2+7$ makes 9. Now the symbol $2+7$ may be seen not only as a *process* (of addition), but also as a *concept* (of sum), so that $2+7$ not only *makes* 9, but $2+7$ *is* 9. This can lead to a rich web of relationships, so that, if “ $2+\text{something}$ ” is 9, then the “something” is 7, and on to other facts involving place value, such as $32+7=39$ or $70+20=90$. Meanwhile the child who sees addition only as a “counting-on” procedure is likely to see subtraction as a “counting-back” procedure, counting back $9-2$ in two steps as “8, 7”, or $9-7$ as count-back seven steps “8, 7, 6, 5, 4, 3, 2” incorporating lengthy counting procedures that prove to be increasingly more difficult to carry out correctly.

Procedures allow individuals to *do* mathematics, but learning lots of separate procedures and selecting the appropriate one for a given purpose becomes increasingly burdensome. *Procepts* allow the individual not only to carry out procedures, but to regard symbols as mental objects, so they can not only *do* mathematics, they can also *think* about the concepts. For such a student with powerful mental connections, greater abstraction gives greater simplicity, whilst the less successful student is left with ever increasing complexity and the greater likelihood of failure.

A consequence of this is that those students who do *not* make enough appropriate mental connections have a far greater mental burden and fall back on the need to routinise mathematics to be able to “do” the procedure to get an answer. They can therefore “do” a problem in a limited context and see this as “success” but are not developing the long-term connections to be able to think about more sophisticated ideas.

I conjecture that this is a major reason that many students are “damaged” by their experiences in school, apparently learning how to “do” mathematics but unable to link together ideas which are, for them, either meaningless or too complex. Such students who require remedial help at college may benefit from a visual/graphical approach, which can increase their confidence as they are, at



last, able to make sense of *something*. Yet such students may find it continually difficult to make sense of the symbolism and link it to the visual ideas. Meanwhile, more successful students who have some conceptions of the mathematical connections may benefit enormously by extending their powers, using computer software as a tool to think with.

Essentially I conjecture that our role as mathematics educators is not just to teach procedures (to “do” mathematics) but also flexible relationships between various ways of considering process and concept (to “think mathematically”).

5.3 Long-term difficulties with symbols

As the mathematical curriculum develops through arithmetic, algebra and calculus, the symbols operate in subtly different ways:

- (i) *arithmetic procepts*, such as $5+4$, 3×4 , $\frac{1}{2} + \frac{2}{3}$, or $1.54 \div 2.3$, have explicit algorithms to obtain an answer, but become increasingly difficult for the procedural learner,
- (ii) *algebraic procepts*, such as $2a+3b$, do not have an “answer” (except by numerical substitution), but they can be manipulated using more general strategies, which again coerces the procedural learner into rote-learning of isolated techniques,
- (iii) *limit procepts*, such as

$$\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a}, \frac{d}{dx} \left(\frac{\sin x + \cos x}{x^2 + 3x + 1} \right), \int_0^{2\pi} e^{2x} \cos x \, dx, \sum_{n=1}^{\infty} \frac{1}{n^2}$$

involve a potentially infinite process of “getting close” to a limit value, which may be computed by a numerical approximation and sometimes by a symbolic algorithm.

Each of these requires new ways of thinking about the symbolism, a change of conceptualisation that proves difficult for many. A child who thinks of a sum $4+3=7$ as a counting procedure in which “4 plus 3 *makes* 7” may find it difficult to cope with a symbol such as $4+3x$ which does not “make” anything, except perhaps to “do the bit $4+3$ that makes sense” and get $7x$. This leads to great confusions for many students starting algebra.

Likewise, a student who is used to “doing” mathematics in a finite number of steps may find it difficult to cope with the potential infinity of the limit process and seek the security of the symbolic algorithms in calculus which at least operationally give an “answer”.

Instead of being a comfortable sequence of successive logical steps, the mathematics curriculum is actually littered with subtle hurdles that are not always apparent to the expert.

Therefore there continues to be a role for the mathematics educator, to not only “teach” mathematics, but to be aware of the ways in which children *learn* mathematics and the pitfalls of the routine teaching of how to *do* procedures without also considering how to organise and *think* about the resultant concepts.

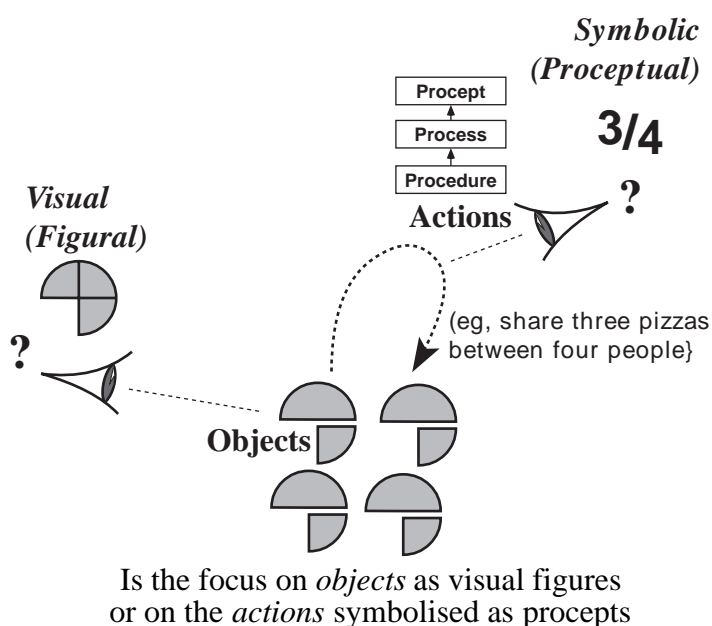
The Mutual Roles of the Visual and the Symbolic

The individual makes sense of the environment by *perception* to receive information, *reflection* to think about it, and *action* to manipulate it. When acting on objects, it is possible to focus either on the objects themselves and the results of the action, or on the actions. For instance, one way to share three pizzas between four people is to cut two in half, give one half to each, then cut the remaining pizza into four, and give a quarter to each. Visually one can see each person having three quarters of a pizza. Alternatively, the action of dividing three by four can be expressed symbolically as a fraction.

The visual conception gives a real-world, practical view of the task, the symbolic conception only begins to make sense after a long sequence of mental compression through counting numbers, sharing and equivalent fractions. The former is more generally available to children, perhaps even an end in itself for real-world mathematics, yet the latter is a basis for long-term mathematical growth.

These two aspects of the same idea typify how the visual can provide a global, holistic idea in mathematics whilst the symbolic produces a sequential, operational method capable of great computational power. However, the two do not always fit together easily (think, for instance, of visual models appropriate for the sum or product of two fractions, or the extension of these ideas to negative rationals). A concentration mainly on symbols may lead to a rote procedural approach which grows in complexity as the number of unconnected rules increases. A concentration only on the visual may give an insight into what is going on in a restricted context perhaps with limited power to generalise.

It is here that the computer can be of vital assistance, suitably supported by guidance from the teacher as mentor. Because the computer is able to carry out the algorithms to enable visual manipulation and symbolic manipulation, it is possible to allow the learner to focus on specific aspects of importance whilst the computer carries out the algorithms implicitly. This provides what I have termed, somewhat grandiosely, as the *principle of selective construction* (Tall, 1993). It allows the learner to obtain an overall holistic grasp of ideas either before, or at the same time as studying the related symbolic procedures that were traditionally the first things to be studied and practised by the learner,



enabling the growing individual to gain a new equilibrium with mathematical ideas in a new technological age. It is not a universal panacea, for different individuals have different ways of coping with the mathematical world, but it offers differing kinds of experiences which can be supportive to a wide spectrum of approaches.

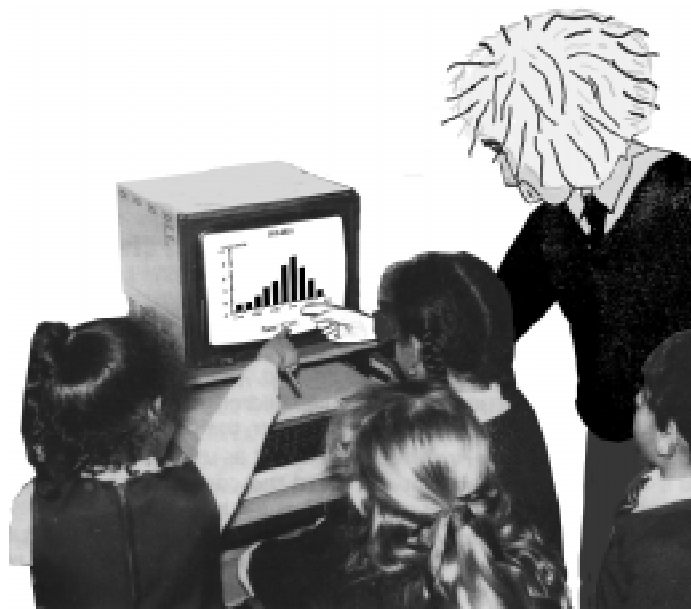
6. The continued need for mathematics educators

The volatile nature of the development of information technology continues to defy prediction, both in general, and in mathematics:

Anyone who presumes to describe the roles of technology in mathematics education faces challenges akin to describing a newly active volcano – the mathematical mountain is changing before our eyes... (Kaput, 1992, p. 515.)

We may no longer need to prepare children to use regular mathematical routines as a central feature of their future employment, but they will need to grow in a way that enables them to survive in a new technological world. The evidence we have suggests that it is insufficient just to give individuals tools to carry out procedures if they are not properly integrated into a cognitive structure that can make sense of the relationship between the various processes, concepts and representations.

In this new world, the creative mathematician still has a full role to play with the cutting edge enthusiast pressing on with innovative possibilities. The reality of the learning process continues to require the reflective guidance of the good teacher.



The reflective guidance of a mathematics teacher

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