COMPUTERS AND THE LINK BETWEEN INTUITION AND FORMALISM

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We consider the wide spectrum of meanings which individuals can give to a figure. It may be conceived as being ***passive,*** *merely being associated with a given concept or* ***organisational,*** *allowing the individual to represent several pieces of relevant information compactly in a single diagram. Alternatively, it may be* ***generative,*** *in the sense that a learner uses it to guide their thinking Such generative imagery may be* ***conceptually generative*** *suggesting intuitive insights into mathematical relationships. It may also be* ***formally generative,*** *linking to the formal arguments required to develop a coherent formal theory. In this paper we consider how visual computer software may be used in generative ways, with particular reference to how different kinds of visual magnification can be used to offer generative ideas about continuity and differentiability.*

Introduction

Visualisations may be used in mathematics in a broad spectrum of ways. In particular, we may contrast the use students make of their visual imagery into *generative* and *nongenerative* – whether or not they are guided to mathematical ideas by their images.

For instance, the forces in a mechanical problem might be represented in a drawing and, by resolving and taking moments appropriately, equations may be derived which can then be manipulated symbolically to lead to **solve the problem.** In this *organisational* use, a picture represents sufficient information to be able to solve the problem. It may also represent conceptual links with verbal, symbolic and numerical data. But for many students it may not figure again in the solution process. In this sense, the use of a picture is nongenerative. Similarly Stubbs (1996) highlights instances of students drawing pictures when they begin to solve a problem, but which are never referred to, either in writing or verbally, during the solution and, in some cases, bear no obvious resemblance to the form of the mathematics used in the solution. We might call these pictures *passive* – they are clearly associated in

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the minds of the students, but play no part in their attempts at a symbolic solution to the problem.

In contrast to this, a picture may also be used to give some kind of visual insights into a mathematical situation, emphasising various visual, dynamic and spatial facets without necessarily linking directly to the solution of a problem or the proof of a theorem. For instance, in gaining insight into the mean value theorem, it may be helpful to draw pictures which begin to build a sense of what conditions are necessary and how they effect the result of the theorem, without necessarily giving any insight into an appropriate deductive proof of the theorem. Figure l (taken from Kowalczyk & Hausknecht, 1994) shows how the theorem would fail if the function were not differentiable at every point in the interval concerned.

We say that such a picture is *conceptually generative*. It can succeed in its particular purpose and other pictures may be drawn to show what might happen if the function were differentiable in the open interval but not continuous at one or other endpoint.

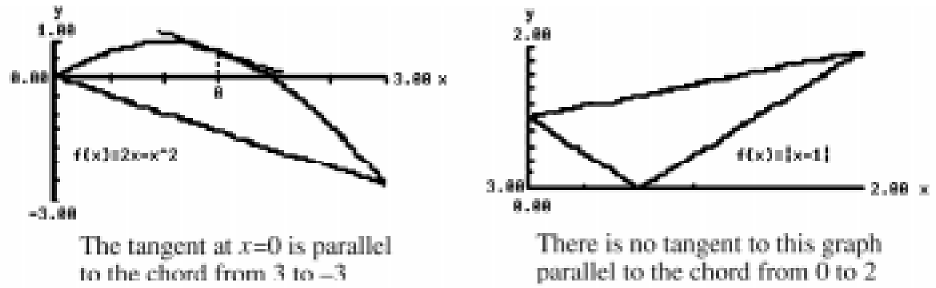


Figure 1: Illustrating an example which fails (adapted from Kowalczyk & Hausknecht, 1994)

It may even be possible to engineer the dynamic visualisation of a line moving parallel to the chord from A to B in figure 2. As the line is moved further away from the chord, “at the very last moment”, as it is tangential to the curve at a point c, then the tangent at C has the same gradient as the chord AB, giving the result of the mean-value theorem. This gives a gestalt underpinning

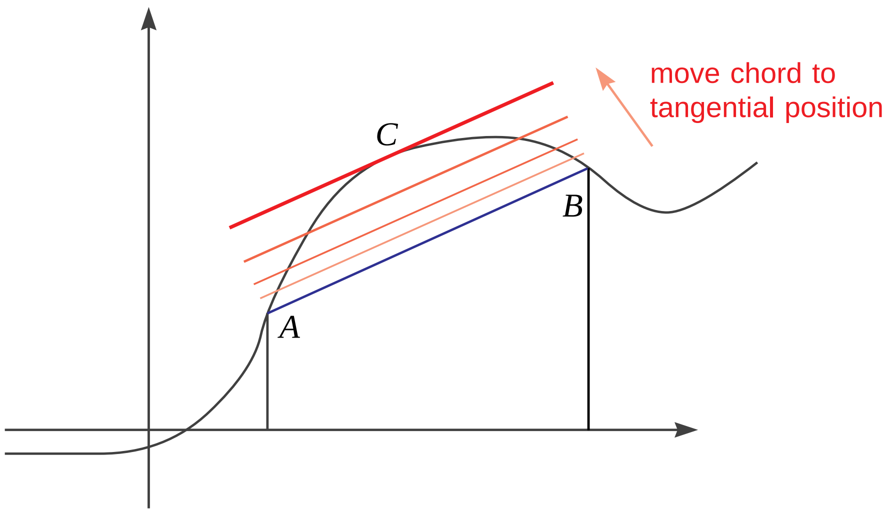


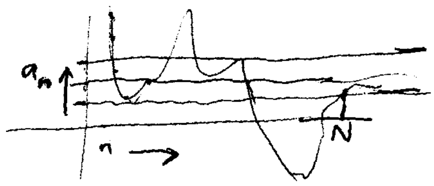
Figure 2: “Continuously” moving the chord into tangential position

to “see” that the theorem is “true”. But for most students this has no generative power to suggest a formal logical proof. Indeed, for many, *such a demonstration renders a formal proof unnecessary*, because it seems to convey the truth of the statement *with absolute certainty*. It also appeals not to the definition of continuity, but to an enactive physical movement, and the idea of a tangent “just touching” the curve with its implicit conflicts in meaning from the formal definition (Vinner, 1983).

Natural and Formal Students Use of Visualisation

Amongst our students at university there is a wide spectrum of different approaches to pictures. Pinto (1996) reports how different individuals can be “successful” in mathematical analysis either with or without using pictures in a generative way. Some (whom she terms ***natural*** learners, after Duffin & Simpson (1993)) can “see” a picture in a formally generative way, using the picture to build up formal definitions and to construct formal proofs as intimated here:

> “I don’t memorise [the definition of limit]. I think of this [picture] every time I work it out, and then you just get used to it. I can nearly write that straight down.”



> “I think of it graphically ... you got a graph there and the function there, and I think that it’s got the limit there ... and then e once like that, and you can draw along and then all the ... points after N are inside of those bounds,....”

Others (termed ***formal*** learners) internalise the definitions by repetition and use:

> “Just memorising it, well it’s mostly that we have written it down quite a few times in lectures and then whenever I do a question I try to write down the definition and just by writing it down over and over again it get imprinted and then I remember it.”

In this particular case the student found great difficulty in using pictures to suggest how to prove things. He obtained meaning from going through symbolic proofs and reflecting on how they were built up logically.

It should be noted, however, that the drawings produced by the natural learners building on their generative visualisation may have many similarities to those drawn) by the formal learners from the passive visualisation (used either to organise the symbols or merely associated with the limit concept). It may also happen that specific elements in a picture (such as a sequence drawn as monotonically decreasing) may interfere with the general formal concepts (such as the notion of an arbitrary convergent sequence satisfying the formal definition) so that some formal learners distrust visual imagery and are not helped by pictures at all.

How can we, as teachers, hope to cope with such a wide range of perceptions and uses of visualisation? The simple answer is that we cannot. But by speaking from our own viewpoint we will only carry with us those students who think in a similar way and may fail others, who are not necessarily less talented, but because they think differently.

A common pedagogical error is to take the mathematics in a mathematical form and simply present it in a way which grows from the formal idea. For instance, the formal definition of continuity is often translated into a picture with a rectangular box that trains the student to take any vertical interval from $f(x_0)\neq\delta$ to $f(x_0)+\delta$ and to seek a horizontal interval from $x_0–\varepsilon$ to $x_0–\varepsilon$ so that the graph of f over the interval is contained within the rectangular box (figure 3).

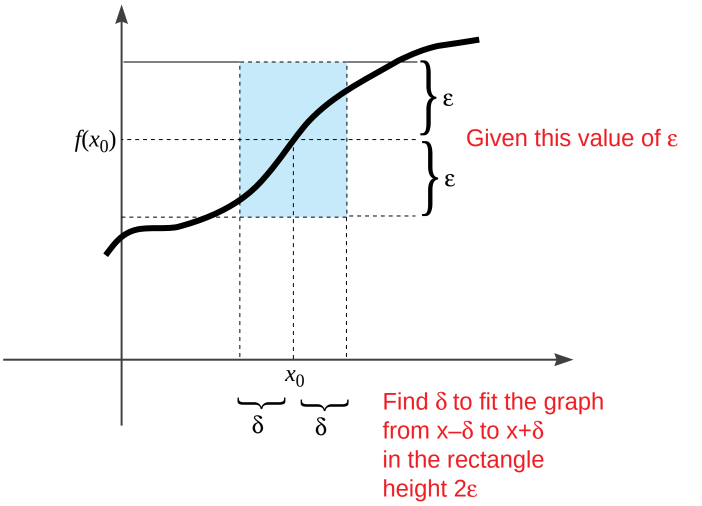


Figure 3: “mathematical” picture of continuity

Others have seen that this approach does not attempt to build on the student’s current knowledge structure and have taken a radically different approach - taking the student’s notion of a “continuous curve”, drawn “without taking the pencil off the paper” and showing how horizontal stretching of such a curve on a computer screen eventually pulls the curve flat (Tall, 1980, Kawski, 1997). This approach is visually attractive and economical. one just *looks* at a graph and imagines it being dynamically pulled out flat to see how the idea works (even with a function that is nowhere differentiable, in this case the *blancmange function (Tall,* 1982) which has left and right gradients $–\infty$ and $+\infty$ at the point concerned!

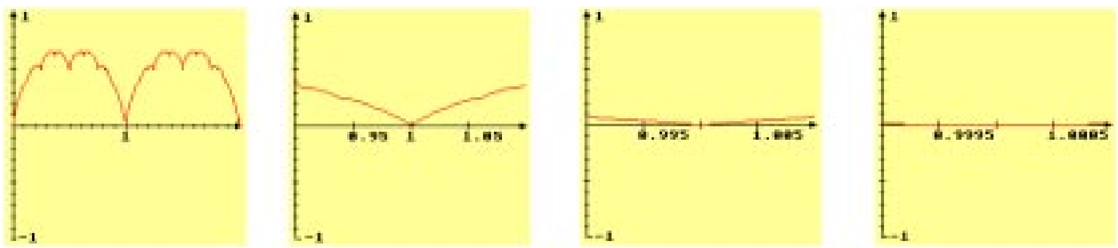


Figure 4: Successively stretching a continuous function flat

Continuity and Differentiability

The concept imagery which students imagine to relate to the notions of continuity and differentiability are often ill-formed, in the sense that a "continuous function" carries with it ideas of being given by "a single formula" or having a "smoothly turning tangent" (Tall & Vinner 1981). By seeing a "continuous function" as one that can be 'pulled flat", and a differentiable function as one that "magnifies to look locally straight" it is easy to gain a visual insight into a continuous function that may be very wrinkled as nowhere differentiable. These ideas can easily be studied using a graphic calculator with different zoom ratios on the horizontal and vertical axis. By setting the horizontal stretch to be k (where k might be 2 or 4 or 10, say) and the vertical stretch to be k^n, then a "horizontal stretch" occurs when n=0, an equal magnification occurs when n=1 and, more generally, a stretch of nth order occurs for other values of n (which may be any real number). This useful notion is discussed in greater detail in Tall (1981) and Kawski, (1997) (where the former defines the notion of an astigmatic lens with n=1, and the latter deals with magnifications of the nth order in higher dimensional vector calculus).

Here we consider the case of continuity in one dimension, which allows a formally generative image valuable for proving theorems in real analysis. Note first, that the notion of "locally flat" means

> Given pixels of height ε then there is a δ such that in a window width $x_0 \pm \delta$ the graph is contained within the pixels at the height representing $f(x_0)$.

That is, zooming in on the point $(x_0, f(x_0))$ with the astigmatic lens of order 0, the function is represented by a horizontal line of pixels (figure 5). Notice, in particular, that this gives the definition *of pointwise* continuity.

This carries with it the property that if $f(x_0)>0$, then the function is positive in an interval $x_0 \pm \delta$. To "see" this, choose a scale with ε sufficiently small that the pixels at the height representing $f(x_0)$ lie visibly above the x-axis (for instance, take e= $^1/_2\, f(x_0)$, this amounts to zooming in with a lens of order 1), then select a width d to stretch the graph horizontally so that the graph around x_0 is all represented by the horizontal line of height $f(x_0)+\varepsilon$. Thus $f(x)>0$ in the interval in the picture.

Summary

By using suitable visual interpretations of mathematics it may be possible to draw one or more pictures which in total are formally generative, in the sense that they may be interpreted appropriately by some students to lead to corresponding formal arguments. For some students (successful natural learners), the pictures may allow them to construct a personal meaning for the definitions which allows them to build a rich conceptual structure to support the formal mathematics. For others (successful formal learners), working by interiorising the definitions and reflecting on the formal proofs may also lead to successful understanding of theorems and proofs, although further cognitive reconstruction is likely to be needed if the student wishes to integrate these new formal ideas with older intuitions.

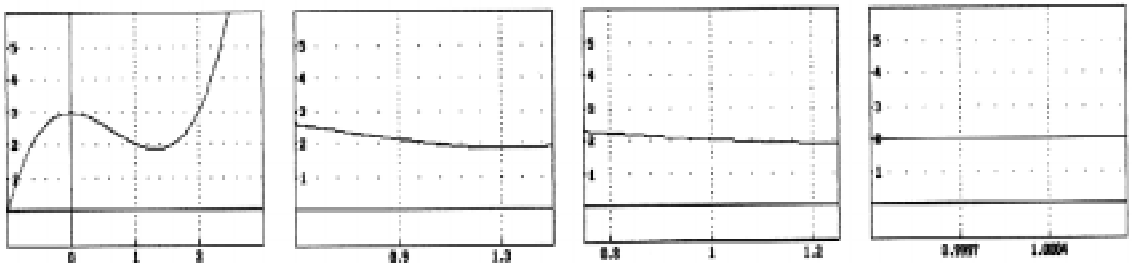


Figure 5: Zooming in with an astigmatic lens of order 0 around x_0=1

Failure to comprehend the definitions may occur for various reasons. Formal learners who fail will have an inadequate formal cognitive structure available. on the other hand, natural learners who use visualisations may fail in the formal sense and yet still have global intuitions which give a conceptual structure richer in intuitive connections but failing with formal deductions. It is the role of the good teacher to attempt to help students make sense of the mathematics shared by the mathematics community. It may be that the best way in which this can be done is through providing images which students may adopt as both formally and conceptually generative.

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