Cognitive Units, Connections and Mathematical Proof

Tony Barnard

Mathematics Department
Kings College
Strand, London WC2 2LS
e-mail: tony.barnard@kcl.ac.uk

David Tall

Mathematics Education Research Centre
Warwick University
Coventry CV4 7AL
e-mail: david.tall@warwick.ac.uk

Mathematical proof seems attractive to some, yet impenetrable to others. In this paper a theory is suggested involving "cognitive units" which can be the conscious focus of attention at a given time and connections in the individual's cognitive structure that allow deductive proof to be formulated. Whilst elementary mathematics often involves sequential algorithms where each step cues the next, proof also requires a selection and synthesis of alternative paths to make a deduction. The theory is illustrated by considering the standard proof of the irrationality of $\sqrt{2}$ and its generalisation to the irrationality of $\sqrt{3}$.

Cognitive units and connections

The logic of proof is handled by the biological structure of the human brain. As a multi-processing system, complex decision-making is reduced to manageable levels by suppressing inessential detail and focusing attention on important information. A piece of cognitive structure that can be held in the focus of attention all at one time will be called a ***cognitive unit***. This might be a symbol, a specific fact such as "3+4 is 7", a general fact such as "the sum of two even numbers is even", a relationship, a step in an argument, a theorem such as "a continuous function on a closed interval is bounded and attains its bounds", and so on. What is a cognitive unit for one individual may not be a cognitive unit for another; the ability to conceive and manipulate cognitive units is a vital facility for mathematical thinking. We hypothesise that two complementary factors are important in building a powerful thinking structure:

1) the ability to *compress* information to fit into cognitive units,
2) the ability to *make connections* between cognitive units so that relevant information can be pulled in and out of the focus of attention at will.

Compression is performed in various ways, including the use of words and symbols as tokens for complex ideas ("signifiers" for something "signified"). These may sometimes be "chunked" by grouping into sub-units using internal connections. A more powerful method uses symbols such as 2+3 as a pivot to cue either a mental process (in this case addition) or a concept (the sum). This has become a seminal construct in process-object theories (Dubinsky, 1991; Sfard, 1991). Gray & Tall (1994) coined the term *procept* for the amalgam of *pro*cess, con*cept* and symbol which could evoke either. However, the notion of procept is not the only instance of compression in mathematics:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston 1990, p. 847)

Published in *Proceedings of PME 21*, Finland, (1997), vol. 2, pp. 41–48.

Compressibility of mathematical ideas relies on the nature of the connections from the focus of attention to other parts of the cognitive structure. The cognitive structure readily linked to cognitive unit(s) in the focus of attention will be termed the *intermediate working memory*. As different items are brought into the focus of attention, the intermediate working memory changes dynamically, opening up new connections and shutting off others. As a consequence, different external prompts may lead to very different connections being made.

To maximise the power of the cognitive structure, dynamic sequences of links are routinised to be performed in the background, taking up little focus of attention. This may produce procedural ability to carry out familiar processes, but more is required to give the flexible synthesis of logical deductions. A powerful aspect of reflective thinking is the ability to compress a collection of connected cognitive units—which may be processes, sentences, objects, properties, sequences of logical deduction etc—into a single entity that can be both manipulated as a concept and unpacked as a cognitive schema. This idea has been formulated many times in different ways (eg the "varifocal theory" of Skemp (1979) in which a concept may be unpacked as a schema and a schema viewed as a concept, or the encapsulation of a schema as an object (Cotrill *et al*; in press). More than just saving mental space by being a shorthand in place of a collection of items, it *carries with it*, just beneath the surface, the structure of the collection and is *operative* in the sense that the live connections within the structure are able to guide the manipulation of the compressed entity. These may then become new units in new cognitive structures, building a hierarchical network spanning several layers. Used successfully, this offers a manageable level of complexity in which the thought processes can concentrate on a small number of powerful cognitive units at a time, yet link them or unpack them in supportive ways whenever necessary.

Mathematical proof introduces a form of linkage different from the familiar routines of elementary arithmetic and algebra. In addition to carrying out *sequential* procedures in which each mathematical action cues the next, mathematical proof often requires the synthesis of several cognitive links to derive a new *synthetic* connection. In the proof of the irrationality of $\sqrt{2}$, for instance, having written $\sqrt{2}=(a/b)$ as a fraction in lowest terms, the step from "$\sqrt{2}=(a/b)$" to "$a^2= 2b^2$" is a sequential link of algebraic operations, but the step from this to the deduction that "a is even" requires a synthesis of other cognitive units. for instance "a is either even or odd" and "if a were odd, then a^2 would be odd." We hypothesise that synthetic links constitute an essential difference between procedural computations and manipulations in arithmetic and algebra and the more sophisticated thinking processes in mathematical proof.

Data collection

To investigate the role of synthetic links in proof, clinical interviews were carried out focusing on the proof of the irrationality of $\sqrt{2}$ and $\sqrt{3}$. Eighteen students were selected at three different stages in the mathematics curriculum: 15/16 year old students in a mixed comprehensive school taking mathematics GCSE, 16/17 year olds in the sixth form of a boys' independent school taking A-level mathematics, and first year university mathematics students. It would be unlikely that students would be able to produce a proof of the irrationality of $\sqrt{2}$ without prior experience, so each was invited to participate in a two-person dialogue, attempting to make sense of a proof presented as a sequence of steps. At each stage he or she was asked to explain the given step and perhaps suggest a strategy for moving on:

(i) *Suppose $\sqrt{2}$ is not irrational.*
(ii) *Then $\sqrt{2}$ is of the form a/b, where a, b are whole numbers with no common factors.*
(iii) *This implies that $a^2 = 2b^2$,*
(iv) *and hence that a^2 is even.*
(v) *Therefore a is even.*
(vi) *Thus $a=2c$, for some integer c,*
(vii) *It follows that $b^2 = 2c^2$,*
(viii) *giving that b^2,*
(ix) *and hence also b, is even.*
(x) *The conclusion that a and b are both even contradicts the initial assumption that a and b have no common factors.*
(xi) *Therefore $\sqrt{2}$ is irrational.*

After this each student was asked to suggest a proof for the irrationality of $\sqrt{3}$.

Analysis of responses

(i) the notion of proof by contradiction

Before being shown the proof, the idea of supposing that $\sqrt{2}$ was *not* irrational and looking for a contradiction was not suggested by *any* students who had not met the proof before. At this stage they are used to manipulating symbols through sequential action schemas to produce a "solution". They are unfamiliar with the possibility of proving something true by initially supposing it to be false—a conflict likely to provoke cognitive tension and insecurity.

(ii) translation from verbal to algebraic

Students with no previous experience of the proof found the idea of writing a fraction in its lowest terms a familiar concept, but the idea of writing this in the algebraic form "$\sqrt{2}=a/b$" proved less obvious, but acceptable.

(iii) a routinised algebraic manipulation

Having agreed to suppose that $\sqrt{2}$ is equal to the fraction *a/b*, where *a* and *b* are whole numbers, students were usually successful in showing that this implies $a^2=2b^2$ using routine algebraic manipulation. However, some students who had seen the proof before and resorted to attempting to memorise it did not always handle the algebra securely. For instance, university student S began by stating the general strategy for the proof by contradiction, yet could not deal with many details. Instead of constructing the proof himself, he recalled that the lecturer "… did some fancy algebra which I couldn't actually reproduce.” When asked to do so, he wrote “$\left(\frac{a}{b}\right)^2 = 2$”, followed by “$a^2=4b^2$”, saying, “I think that’s what he did, but he did it in one step whereas normally I would’ve taken two.” When asked to fill in the details, he obtained the correct result $a^2=2b^2$. Similarly, student M said, “I remember him saying to prove that a is even” but could not remember how. In contrast, Student L compressed the whole operation in a single step giving further details on request. The majority were able to complete the step routinely.

(iv) a link from algebra to verbal representation

None of the students new to the proof spontaneously linked “$a^2=2b^2$” to “a is even”, although they all readily accepted its truth. (The link loses information, saying “a^2 is twice a whole number” rather than “a^2 is twice *the square of* a whole number”. Students may feel instinctively uneasy losing information, without articulating their concern.)

(v) Synthesising a non-procedural step

The step from “a^2 is even” to “a is even” requires a more subtle synthesis of links with other cognitive units. Students offered a number of different strategies, including:

(a) *Correct justification* involving a sequence of appropriate connections, usually along the lines “a is either even or odd”, but “a odd implies a^2 is odd”, and as “a^2 is *not* odd”, this implies “a must be even.”

(b) *Strong conviction but without justification*, such as, “an even number square has got to have a square root that is even” and “well, it just sort of *is* [even].”

(c) *Empirical verification*, trying some numeric cases and asserting that there are no exceptions.

(d) *Inconclusive reasoning*, offering related statements, justified or otherwise, which did not help further the argument, such as, “If you could say that a^2 had a factor of 4, then that [a even] would definitely be true.”

(e) *False reasoning*, using incorrect links. An example which occurred more than once was the claim that if a^2 was an integer multiple of 2, then a was an integer multiple of $\sqrt{2}$.

(f) *Unable to respond without help*.

The correct justification was not evoked initially by most students new to the proof or by some of those who sought to remember the proof by rote. The cognitive units “a is even” and “a^2 is even” coexist in the focus of attention so they may be seen as happening at the same time rather than one implying the other. “a^2 is even” seems to have a stronger natural link to “a is even” than to “a is odd”, thus failing to evoke the alternative hypothesis.

Some students responded in several of the given categories. For instance, Student S began with response (b) asserting the authority of the lecturer, saying, “the root of an even number is even—he just assumed it.” When challenged, he went on to reason inconclusively and then to try specific cases:

Interviewer: So the root of six is even.
Student S: Good point. [five seconds pause]
Interviewer: If a number is not even, what is it?
Student S: It's odd.
Interviewer: So you've got a choice of odd or even, does that help you?
Student S: Yeh, I see, it's got to be rational, I think, so … a rational root is either … odd or even and if the square is even, then the rational root is even. Is that clear?
Interviewer: Uh, well …
Student S: So what I'm thinking is the root of 4, 4's even and 2's even, root of 16 equals 4, … 's even. I can't remember any other simple squares in my head that are even …

Students who could not proceed (category (f)) were given a prompt referring to the odd-even dichotomy. This often led to a response of type (a), (d) or (e) above. For example, the prompt "Every integer is even or odd" was often followed by the response, "An odd number squared is odd". The thought of looking at concrete examples (category (c)) was not something brought to mind by this kind of cue.

(vi) From "a is even" to "a=2c for a whole number c"

The translation from the verbal statement "a is even" to the algebraic statement "$a=2c$" was usually straightforward, but again students such as university student M—who admitted trying to memorise proofs—had a faulty recollection of what to do:

Interviewer: If you know that a is even, how can you write a? How do you write down that a is an even number ?
Student M: If you put a 2, … you put an a in front of it, like $4a$ ... I don't know, I'm sorry. I can't remember.

(vii)–(ix) The chance to repeat earlier arguments

Having concluded that "$a=2c$ for a whole number c", the next steps of the proof often evoked earlier ideas. No student had any difficulty with the procedural steps substituting "$a=2c$" into "$a^2=2b^2$" and simplifying "$4c^2 = 2b^2$" to get $b^2= 2c^2$. Students invariably saw that this situation was similar to the earlier case for a, and asserted that b is also even.

(x)–(xi) establishing the contradiction

Some students new to the proof did not recall that a/b was assumed in lowest terms, and so did not see that "a and b both even" gives a contradiction. Student C remained silent for 45 seconds until reminded: "we cancelled out until we had no common factors," then immediately responded:

> "Oh, right, … that can't be the case because if they are both even numbers, then they will have common factors, like two."

Those who had seen the proof before in school or at university immediately grasped the contradiction, including those who had misremembered the detail of earlier steps.

Generalising the proof to the irrationality of √3

When proving the irrationality of √3, all students began by supposing that √3 was equal to a fraction a/b in its lowest terms, a typical remark being, "I presume you start in the same way." On translating this to $a^2=3b^2$, *all* of them evoked the link with a being "even or odd" and were unable to proceed further. (Just one student wondered whether the "eveness" might relate to the 2 under the square root sign.) A suggestion that the equation $a^2=3b^2$ tells more than "the oddness of *a*" usually evoked divisibility by 3, but this led to a further sticking point in attempting to prove that "a^2 is divisible by 3" implies "*a* is divisible by 3". *None* of the students could do this unaided. In particular none considered the algebraic argument squaring the three cases $a=3n$, $3n+1$ or $3n+2$ (a synthetic connection requiring the coordination of three different possibilities).

A further suggestion focusing on factorisation into primes was sufficient to help all the university students and some sixth formers to produce suitable arguments although often expressed in an idiosyncratic manner. Student T, for instance, said:

> " ... the (square) root of a^2, I mean *a*, that doesn't involve the factor 3. Therefore you've still got a factor 3 which you can divide into *a*."

She seems to be saying that if 3 does not divide one of the *a*-factors of $a \times a$, then it must divide the other *a*. Student J in the youngest group also imagined a^2 as a product of two *a* factors saying:

> " that has got repeated factors of that, so you can't get [ten seconds pause] … just imagining how many factors of things. … They're going to have the same factors. So yes, 3 would have to divide *a*."

Discussion

Figure 1 is a representation of some of the typical linkages that may occur in an initial proof that √2 is irrational, omitting idiosyncratic links (which occur widely in individual cases). It is a collage of difficulties encountered by students where links denoted by $\rightsquigarrow$ often prove more difficult than those denoted by $\longrightarrow$ and those in grey scale are intermediate links which may or may not be evoked in detail.

Figure 2 displays a compressed proof structure available as an overall strategy to many students who had experienced the proof before. Even Student S, who remembered little detail and used loose terminology to describe his ideas was able to say,

> I'd take the case where I assumed it was a rational and fiddle around with the numbers, squaring, and try to show that … if it was rational then you'd get the two ratios *a* and *b* both being even so they could be subdivided further, which we'd assumed earlier on couldn't be true so our assumption it was rational can't be true.

(a) The overall notion of proof by contradiction (which becomes less problematic with familiarity)

(b) translation between familiar terms :"odd and even" and algebraic representations are acceptable, but not always initially evoked,

(c) The step "a^2 even implies *a* even" is initially not easy to synthesise and remains so for those in the sample who attempted to remember the proof by rote. For some the cognitive units "a^2 even" and "*a* even" coexist and the direction of implication is not relevant; for others the idea "a^2 *even*" is more strongly linked to "*a even*" rather than to the operative alternative "*a odd*".

(d) The repetition of the argument to show *b* is also even was readily evoked by most students.

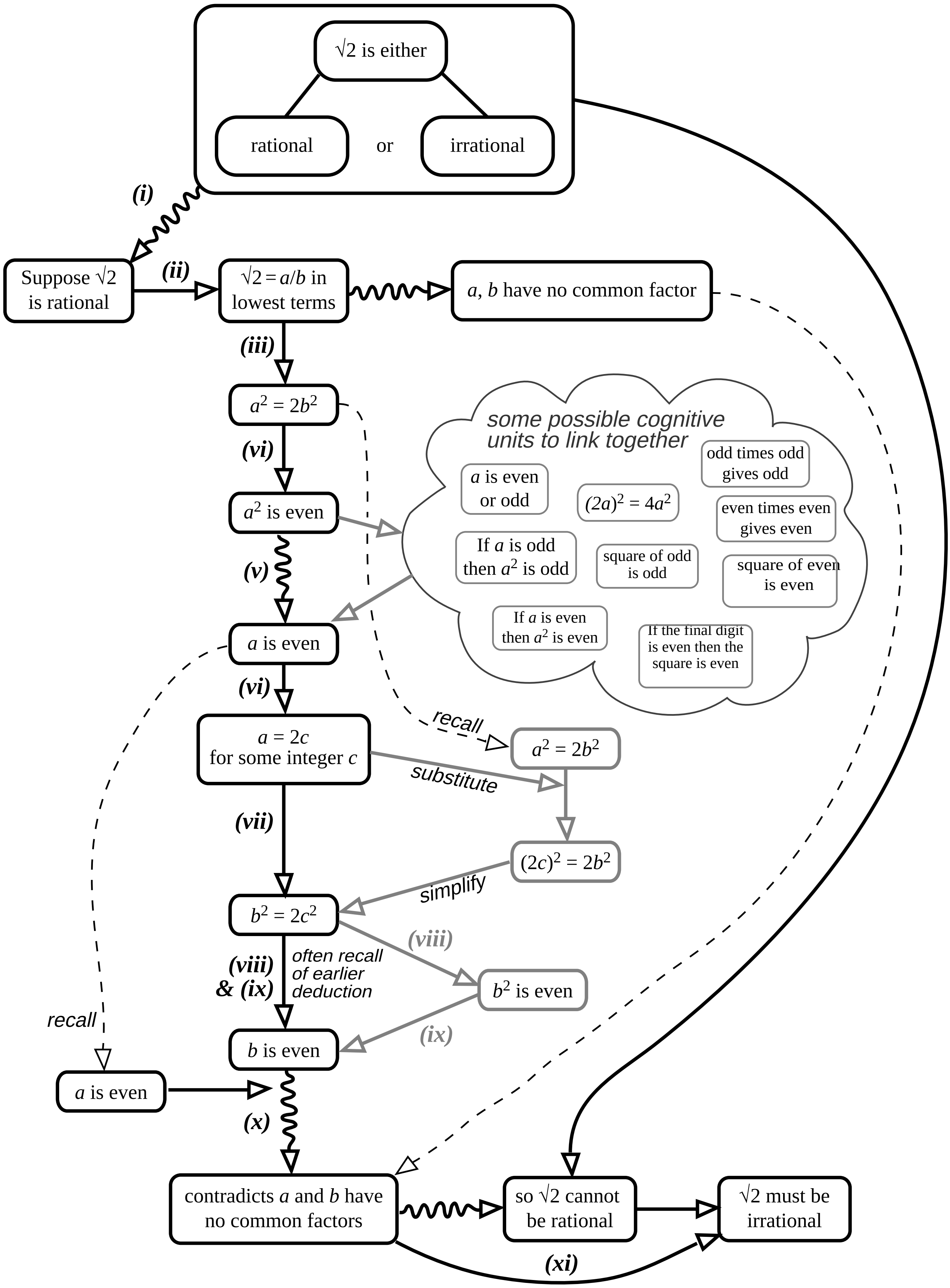

Figure 1: Observed cognitive units and connections in a typical initial proof of $\sqrt{2}$ irrational

(e) The assumption that a/b is *in lowest terms*, was not easily recalled by students meeting the proof for the first time, but became part of the global strategy in the long-term.

(f) The link in the proof of "√2 irrational" to the colloquial terms "even-odd" was more powerful than the link to "divisible or not by 2", thus blocking the natural extension to the corresponding proof for irrationality of √3.

Summarising the broad development of the proof of the irrationality of √2 and √3 we see that there are several initial difficulties that make it a formidable challenge for the uninitiated. Some become less problematic with familiarity, but there is sufficient difficulty to cause a bifurcation in understanding. Some students make meaningful links that allow them to compress the information into richly connected cognitive units. Others remember some of the ideas they were told—even the overall strategy of the proof—yet rely on the authority of their teacher rather than building their own meaningful links which might enable them to reconstruct the subtle detail.

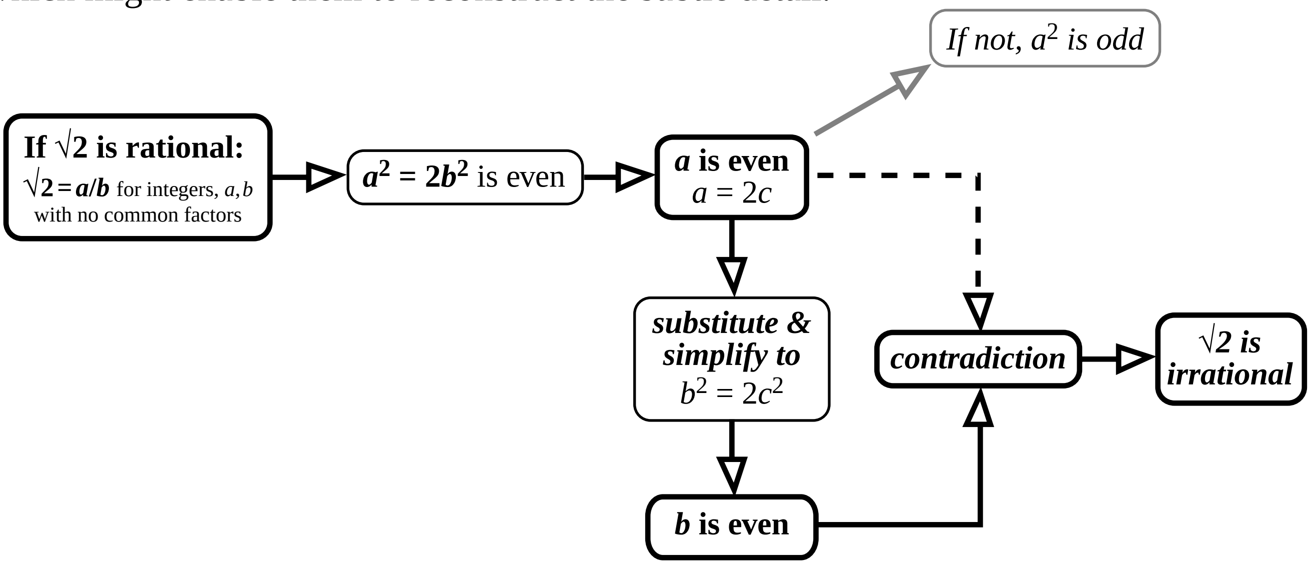

Figure 2: a compressed proof that √2 is irrational by deriving a contradiction

Figure 3: A compressed strategy for the proof

References

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (in press). 'Understanding the limit concept: beginning with a coordinated process schema', *Journal of Mathematical Behavior*.

Dubinsky, E. (1991), 'Reflective Abstraction in Advanced Mathematical Thinking'. In D. O. Tall (ed.) *Advanced Mathematical Thinking*, Kluwer: Dordrecht, 95-123.

Gray, E. M. & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *The Journal for Research in Mathematics Education*, **26** 2, 115–141.

Sfard, A.(1991). 'On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin', *Educational Studies in Mathematics* **22**, 1-36.

Skemp R.R. (1979). *Intelligence, Learning, and Action*, Wiley.

Thurston, W. P. (1990). Mathematical Education, *Notices of the American Mathematical Society*, 37 7, 844-850.