Understanding the Processes of Advanced Mathematical Thinking[1]

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Introduction

In preparing successive generations of mathematicians to think in a creative mathematical way, it is difficult to convey the personal thought processes which mathematicians use themselves. So many students unable to cope with the complexity resort to rote-learning to pass examinations. In this presentation I shall consider the growth of mathematical knowledge and the problems faced by students at university. If they are given opportunities to develop mathematical thinking processes, albeit with initially easier mathematics, they may develop attitudes to mathematics more in line with those preferred by mathematicians while standard mathematics lectures designed to "get through the material" may force them into the very kind of rote-learning that mathematicians abhor.

I didn't understand all that stuff between "Good morning, class" and "That concludes my lecture for today."

The development of mathematical thinking

Mathematicians struggle with ideas in research, but the ideas taught to undergraduates have been organised in a clear and logical sequence. Why is it that, when presented with these well-organised theories, students struggle too? Is it just students' lack of effort or intellect, or are there other reasons?

Axiom I: All mathematicians are born at age 0.

Axiom II: to reach the age M of mathematical maturity, the mathematician must pass through ages 0, 1, 2, …, M–1.

Theorem: A cognitive development is necessary to become a mathematician.

Proof: Since no child aged 0 has produced any important mathematical theorem, something happens between ages 0 and M that makes mathematical thinking possible. □

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This “proof” which caricatures a mathematical style is perhaps amusing but certainly mathematically flawed. The non-existence of a known counter-example is clearly insufficient to prove something. But if we think in *mathematical* terms about how humans think, our arguments are also liable to fail. This happened, for example, in the set-theoretic approach to school mathematics in the sixties when the apparently obvious route of introducing modern mathematics into schools failed to produce the understanding that was expected. It is therefore clear that we must take the nature of *cognitive* growth much more seriously if we are to understand the development of mathematical thinking. I propose to do this by hypothesising fundamental cognitive principles and considering the consequences.

Cognitive Principle I: For survival in a Darwinian sense, the individual must maximise the use of his/her cognitive structure by focusing on concepts and methods that *work*, discarding earlier intermediate stages that no longer have value.

Corollary: The individual is likely to *forget* much of the learning passed through in years 0, 1, ..., M–1 and the mathematician is likely to attempt to teach current methods that work *for him/her*, not methods that will work for the student.

> One finally masters an activity so perfectly that the question of how and why students don’t understand them is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question. (Freudenthal, 1983, p. 469)

> After mastering mathematical concepts, even after great effort, it becomes very hard to put oneself back into the frame of mind of someone to whom they are mysterious.
> (Thurston, 1994, p. 947)

This is not something that should cause embarrassment to mathematicians, for it is sensible for a professional to do everything to climb to the summit of his or her profession. But it does suggest that there is need for professionals of a possibly different kind to devote attention to the cognitive growth of mathematical thinking to help the next generation to scale similar heights.

To understand cognitive growth it is useful to consider a second principle, which may seem initially to have little to do with mathematics, but proves in practice to have *everything* to do with its underlying power of mathematical thinking:

Cognitive Principle II: The brain has a small focus of attention and a huge space for storage and therefore cognitive growth needs to develop:

(a) a mechanism for compression of ideas to fit in the focus of attention.

(b) a mechanism for linking with relevant stored information and bringing it to the focus of attention in an appropriate sequence.

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.
> (Thurston, 1990, p. 847)

But how do we help growing mathematicians to achieve these levels of compression? Simply *telling* them the theory proves sadly to be insufficient:

> … in their university lectures they had been given formal lectures that had not conveyed any intuitive meaning; they had passed their examinations by last-minute revision and by rote. (W. W. Sawyer 1987, p. 61)

To help students become mathematicians I hypothesise we need to provide them with an environment in which they can construct their own knowledge from experience and learn to *think mathematically*:

Cognitive Principle III: A powerful agent in learning with understanding is by going through mathematical constructions for oneself and then *reflecting* on one's own knowledge – *thinking about thinking*.

> We believe that people learn best by *doing* and *thinking* about what they do. The abstract and the formal should be firmly based on *experience*. (Dubinsky & Leron, 1994, p. xiv)

This principle will help students to become autonomous thinkers, and to become responsible for their own learning. Dubinsky & Leron use the programming language ISETL (Interactive SET Language) to get the students to engage in programming mathematical constructs in group theory and ring theory. Because the programming language is close to mathematical notation, it enables the students to construct abstract concepts like cosets and Lagrange's theorem in a concrete manner, showing considerable success in what is traditionally a difficult area.

A possible difference between this learning and the thinking of formal mathematicians is intimated by Thurston (1994, p. 167) who suggests that

> … as new batches of mathematicians learn about the subject they tend to interpret what they read and hear more literally, so that the more easily recorded and communicated formalism and machinery tend to gradually take over from other modes of thinking.

Reflective thinking on these matters is an indispensable part of research mathematics. But it is rarely taught to undergraduates, where the focus is on content of lecture courses. At the school level problem-solving is a central part of the NCTM standards in the USA, and mathematical investigations are part of the British mathematics curriculum. Perhaps now is the time to introduce the study of mathematical thinking itself into university courses.

Of the three cognitive principles mentioned, the first essentially warns that those who have reached a greater level of maturity may have forgotten how they learnt. We therefore consider the other two principles in detail, first the nature of mathematical compression, and then move on to the process of how to teach reflective mathematical thinking.

The compression of knowledge in mathematics

There are various methods of compression of knowledge in mathematics, including:

(1) representing information visually (a picture is worth a thousand words),

(2) using symbols to represent information compactly,

(3) if a process is too long to fit in the focus of attention, practise can make it routine so that it no longer requires much conscious thought.

Method (1) is used by many (but not all) mathematicians. In his classic study of how mathematicians do research, Hadamard explained that, with certain exceptions:

> … mathematicians born or resident in America, whom I asked, … practically all … – contrary to what occasional inquiries had suggested to Galton as to the man in the street – avoid not only the use of mental words, but also, just as I do, the mental use of algebraic or any other precise signs; also as in my case, they use vague images.
> (Hadamard, 1945, 83–84)

Einstein reported that visual, kinetic and other imagery proved useful in his research:

> The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be "voluntarily" reproduced and combined … The above mentioned elements are in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will. (Albert Einstein, in a letter to Hadamard, 1945, 142–3)

In recent interviews with research mathematicians, Sfard (1994) found exactly the same phenomena. One mathematician reported to her:

> 'To understand a new concept I must create an appropriate metaphor. A personification. Or a spatial metaphor. A metaphor of structure. Only then I can answer questions, solve problems. I may even be able then to perform some manipulations on the concept. Only when I have the metaphor. Without the metaphor I just can`t do it.'
>
> 'In the structure [which he created in his mind in the attempt to understand], there are spatial elements. Many of them. It`s strange, but the truth is that my student also has noticed it... a great many spatial elements. And we are dealing here with the most abstract things one can think about! Things that have nothing to do with geometry, [that are] devoid of anything physical... The way we think is always by means of something spatial... Like in `This concept is above this one` or `Let`s move along this axis or along the other one`. There are no axes in the problem, and still...' (Sfard, 1994)

Mathematicians may use images in this way to relate ideas in their highly developed cognitive structure. Such *thought experiments* are highly advantageous in contemplating possible relationships before the question of logical proof arises. But it is necessary, as Hadamard said, to be "*guided by images without being enslaved by them*" (ibid, p. 88).

Students do not have such a developed cognitive structure and instead they may be deceived by their imagery. They already have their own concept images developed through previous experience (Tall & Vinner, 1981). Such imagery is often in conflict with the formal theory (see Tall, 1991a, 1992 for surveys). Even though concepts are given formal definitions in university mathematics, students may appeal to this imagery and infer theorems through the use of their own thought experiments. For instance, "continuous" might carry the inference of something "going on without a break", so a continuous function must clearly pass through all intermediate values, and must also be bounded and attain its bounds. For a proof by thought experiment, just imagine a picture and see.

Visualising Mathematical Concepts

Although the private images of mathematicians may be difficult to communicate, public images, such as diagrams and graphs enable a great deal of information to be embodied in a single figure. Software which allows visual representations to be controlled by the user, to see dynamic relationships make even more powerful use of visualisation. Having been fascinated by the non-standard idea that a differentiable function infinitely magnified looks like a straight line (within infinitesimals), I wrote computer programs to look at computer drawn graphs under high magnification (figure 1). This allows a visual approach to the notion of differentiability. By using fractals such as the Takagi function (Takagi, 1903) – rechristened the "blancmange" function because of its similarity to a wobbly English milk jelly – functions could be drawn which *never* magnified to look straight (figure 2), hence intimating the notion of a *nowhere differentiable function.* Indeed, a visual proof of this argument is easy to give (Tall, 1982). By taking a small version of the blancmange function $bl(x)$, say $w(x) = bl(1000x)/1000$, for any differentiable function $f(x)$, consider the graph of $f(x)+w(x)$. This looks the same on the computer screen to a normal magnification, but under high magnification (say times 1000), wrinkles appear. This shows visually that, for every differentiable function $f(x)$ there is a non-differentiable function $f(x)+w(x)$, so there are at least as many non-differentiable functions as diff-erentiable ones (figure 3).

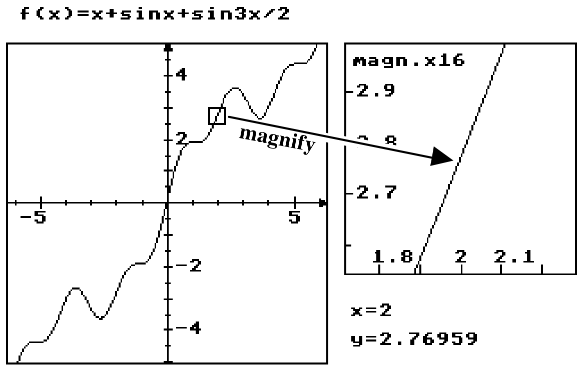

Figure 1 : magnifying a locally straight graph

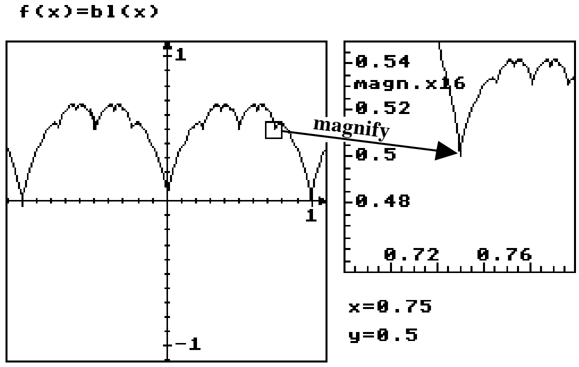

Figure 2 : magnifying the nowhere differentiable blancmange

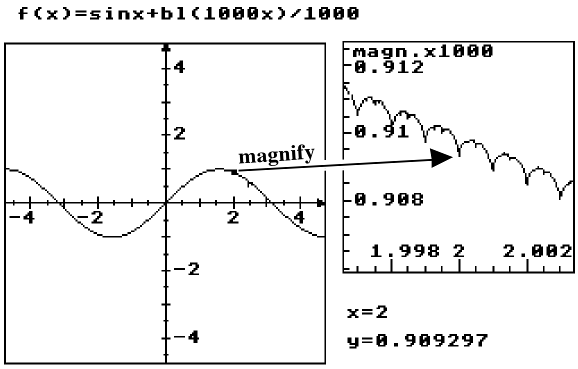

Figure 3 : magnifying an interesting graph

A problem with visualisation is that the human mind picks up implicit properties of the imagery and the individual builds up a concept image that incorporates these properties. Graph-plotters tend to draw graphs that consist of continuous parts. So I designed a graph plotter to simulate functions that are different on the rationals and irrationals (Tall 1991b, 1993). (The routine uses a continued fraction technique to compute a sequence of rationals approaching a given number and, when a term of the sequence is within ε of the number, it is said to be (ε–N)-pseudo-irrational if the denominator of the fraction exceeds N. By suitably fixing the size of ε and N, computer numbers can be divided into two subsets, (pseudo)-rationals and (pseudo)-irrationals that model various properties of rationals and irrationals.)

This allows visual insight into more subtle notions. For instance, just as differentiability can be handled visually by magnification maintaining the same relative scales on the axes, continuity can be visualised by maintaining the vertical scale and stretching the horizontal scale to show less and less of the graph within the same window. A continuous function is one such that any picture of the graph will pull out flat. Figure 4 shows a picture of a graph of a function $f(x)$ which takes the value 1 if x is rational and x^2 if x is irrational. By pulling it horizontal, it is visually continuous at x=1 and x=–1, but this clearly fails elsewhere.

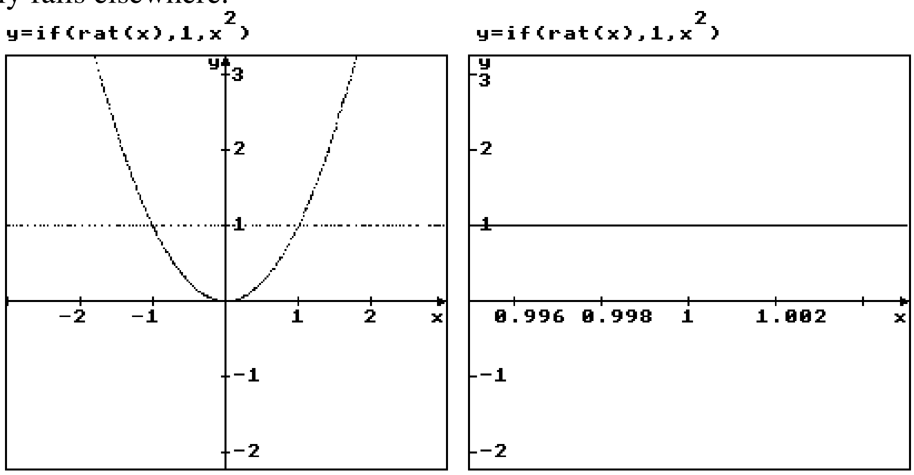

Figure 4 : Stretching a graph horizontally to “see” if it is continuous

Visual software has been developed in a wide variety of ways, such as Koçak (1986), Hubbard and West (1990) for visualising the solution of differential equations, and a growing mountain of software resources presented each year at the annual *Technology in College Mathematics Teaching* conferences. Such software can give students powerful gestalts to enable them to imagine sophisticated mathematical ideas as simpler visual images. For instance, suppose that a student knows that a differentiable function is locally straight and that a first order differential equation such as $dy/dx=-y$ simply tells the gradient dy/dx of that graph through a point (x,y). Then it is visually clear that a good approximation to the solution can be made by sticking together short straight-line segments with the appropriate gradient. Drawing a picture shows how good this approximation is and visually confirms the existence of a solution, motivating theorems of existence and uniqueness of solutions provided that the gradient is defined along the solution path. This can be valuable both for students who will become mathematics majors and those who will use mathematics in other subjects. I have found such techniques of enormous value teaching science students who have little time for the formal niceties. It proves a good foundation for mathematics majors too, but one must not underestimate the difficulties of linking the visual imagery – which comes as a simultaneous whole – and the logical proofs which involve a different kind of sequential thinking.

Using symbolism to compress process into concept

Symbols such as $Ax=c$ for a system of linear equations express a relationship in a far more compact form than any corresponding use of natural language. But there is a common use of symbols in mathematics which introduces compression in a subtle way rarely used in ordinary language. It is a method of compression that mathematicians are aware of intuitively but do not articulate in any formal sense, yet it becomes of vital importance in cognitive development. Let me illustrate this with the concept of number and the difference between a mathematician's definition and the cognitive development of the concept.

According to the set-theoretic view of Bourbaki, (cardinal) number concepts are about equivalences between sets. But a set-theoretic approach to number was tried in the "new math" of the sixties and it failed. Why? Almost certainly because the set-theoretic approach is a natural systematisation when everything has been constructed and organised but it is less suitable as the beginning of a *cognitive* development. In essence it is a formulation which is likely to be suggested by experts who have forgotten their earlier development (cognitive principle I) but it proves unsuitable as an approach for the growing individual.

Even though small numbers of two or three objects can be recognised in a glance, cardinal numbers for these and larger numbers begin cognitively in young children as a *process*: the process of counting. Only later do the number symbols become recognised as manipulable number *concepts*.

It often happens that a mathematical process (such as counting) is symbolised, then the symbol is treated as a mathematical concept and itself manipulated as a mental object. Here are just a few examples:

symbol	***process***	***concept***
$3+2$	addition	sum
-3	subtract 3, 3 steps left	negative 3
$3/4$	division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$\sin A = \frac{opposite}{hypotenuse}$	trigonometric ratio	trigonometric function

$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\left.\begin{array}{l}\lim_{x\to 2}\frac{x^2-4}{x-2}\\ \sum_{n=1}^{\infty} 1/n^2\end{array}\right\}$	tending to limit	value of limit
$\sigma \in S_n$	permuting $\{1, 2, \ldots, n\}$	element of S_n
solve(f(x)=0,x)	solving an equation	solution of equation

Given the wide distribution of this phenomenon of symbols representing both process and concept, it is useful to provide terminology to enable it to be considered further.

Cognitive Definition: An ***elementary procept*** is the amalgam of a ***pro*cess**, a related **con*cept*** produced by that process and a **symbol** which represents both the process and the concept.

Cognitive Definition: A ***procept*** consists of a collection of elementary procepts which have the same object (Gray & Tall, 1994).

Caveat: This is a *cognitive notion*, not a mathematical one. Anyone with a mathematical background might be tempted to define an elementary procept as an ordered triple (process, concept, symbol) and a procept as an equivalence class of ordered triples having the same object. Such an approach is of little cognitive value in that the purpose of the procept notion is to echo the cognitive reality of how mathematical processes are compressed mentally into manipulable mental objects. This has been the focus of attention of many researchers in mathematics education both at school and university level, including for example, Piaget (1972), Greeno (1983), Davis (1984), Dubinsky (1991), Sfard (1991), Harel & Kaput (1991). The cognitive process by which processes become conceived as manipulable objects is called *encapsulation* by Dubinsky, following Piaget.

Had the definition of *procept* been a *mathematical definition*, doubtless some mathematician would have made it before. But the procept notion implies a cognitive ambiguity – the symbol can be thought of *either* as a process, *or* as a concept. This gives a great *flexibility* in thinking – using the *process* to *do* mathematics and get answers, or using the *concept* as a compressed mental object to *think about* mathematics – in the sense of Thurston:

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a*/*b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation. (Thurston, 1990, p. 847)

I claim that the reason why mathematicians haven't made this definition is that they *think* in such a flexible ambiguous way often without consciously realising it, but their desire for final precision is such that they write in a manner which attempts to use unambiguous definitions. This leads to the modern set-theoretic basis of mathematics in which concepts are defined as *objects*. It is a superb way to systematise mathematics but is cognitively in conflict with developmental growth in which mathematical processes *become* mathematical objects through the form of compression called encapsulation.

Sequential and procedural compression

A mathematician puts together a number of ideas in sequence to carry out a computation or a sequence of deductions in a proof using method (3). Hadamard performs such mental actions successively focusing on images before arguments are formulated logically:

> It could be supposed a priori that the links of the argument exist in full consciousness, the corresponding images being thought of by the subconscious. My personal introspection undoubtedly leads me to the contrary conclusion: my consciousness is focused on the successive images, or more exactly, on the global image; the arguments themselves wait, so to speak, in the antechamber to be introduced at the beginning of the "precising" phase. (Hadamard, 1945, 80–81)

Students who have little of this internal structure see in a proof just a sequence of steps which they feel forced to commit to memory for an examination:

> Maths courses, having a habit of losing every student by the end of the first lecture, definitely create a certain amount of negative feeling (as well as a considerable amount of apathy) and the aim for the exam becomes the anti-goal of 'aiming to get through so I don't have to retake' rather than the goal of 'working hard to do well because I enjoy the subject'. (Female mathematics undergraduate, 2nd Year)

This use of memory for routinizing sequential procedures is a valuable human tool when the mental objects to be manipulated will not all fit in the focus of attention at the same time. The memory scratch-pad available is small – about 7±2 items according to Miller (1956).

When individuals fail to perform the compression satisfactorily they do not have mental objects which can be held simultaneously in memory (Linchevski & Sfard, 1991). They are then forced into using method (3) as a *defence* mechanism – remembering routine procedures and internalising them so that they need less conscious memory to process. The problem is that such procedures can only be performed in time one after another, leading to an inflexible *procedural* view of mathematics. Such procedural learning may work at one level in routine examples, but it produces an escalating degree of difficulty at successive stages because it is more difficult to co-ordinate processes than manipulate concepts. *The failing student fails because he or she is doing a different kind of mathematics which is harder than the flexible thinking of the successful mathematician.*

The transition to formal mathematics

Students usually find formal mathematics in conflict with their experience. It is no longer about procepts – symbols representing a process to be computed or manipulated to give a result. The concepts in formal mathematics are no longer related so directly to objects in the real world. Instead the mathematics has been systematised (à la Bourbaki) and presented as a polished theory in which mathematical concepts are *defined* as mental objects having certain minimal fundamental properties and all other properties are *deduced* from this. The definitions are often complex linguistic statements involving several quantifiers.

This formal meaning is difficult to attain. For instance, of a group of mathematics education students studying analysis as "an essential part of their education", *none* could give the definition of the convergence of a sequence after two weeks of using the idea in lectures. Asked to complete the definition of limit, a sample of responses included:

(a) Given $\varepsilon > 0$, $\exists$ a δ s.t. ·····

(b) given $\varepsilon > 0$ $\exists l \in \mathbb{R}$ s.t. $|l - x_n| \leq \varepsilon$

(c) Given $\varepsilon > 0$ $\lim_{n \to \infty} S_n = l$

~~[illegible]~~ series —— tends to a limit

$\varepsilon + \lambda$ $\quad \varepsilon - \lambda$.

$S_n \to$ limit s.t. $\varepsilon + \lambda$, $\varepsilon - \lambda$

where λ

(d) $\lim_{n \to \infty} s_n = l$

as $n \to \infty$ the terms approach and get closer + closer to l (or may reach it) but l is not exceeded

ie $S_n - S_{n+1} \to 0$ as $n \to \infty$

Of course these students are not the "best" students studying analysis, but their failure is typical of a spectrum of levels of failure in understanding mathematical analysis. Even distinguished mathematicians remember their struggles with the subject:

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother me. I could juggle them quite well... [but] I was stumped by the infinitesimal subtlety of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on.
>
> (Halmos, 1985, p.47)

Halmos was fortunate enough to eventually find out what the 'real knowing' was all about:

> ... one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilon. I understood what limits were, and all of that stuff that people were drilling in me became clear. I sat down that afternoon with the calculus textbook by Granville, Smith, and Longley. All of that stuff that previously had not made any sense became obvious...
>
> (Halmos in Albers & Alexanderson, 1985, p. 123)

Regrettably many students never reach enlightenment. Although visual images may suggest theorems, the use of definitions demands a new form of compression of knowledge. The definitions used in mathematics must be *written* so that the information may be scanned to allow different parts to become the focus of attention at different levels. For instance, the definition of continuity is heard as:

> For any *ay* in the domain of the function *eff*, given an epsilon greater than zero, there exists a delta greater than zero such that if *ex* lies in the domain of *eff* and the absolute value of *ex* minus *ay* is less than delta then the absolute value of *eff of ex* minus *eff of ay* is less than epsilon.

It is far too long to be held meaningfully in the focus of attention through hearing alone. It only begins to make sense when compressed in symbolic writing concentrating first on continuity at a point $a \in D$:

> A function $f: D \to \mathbf{R}$ is continuous at $a \in D$ if:
> $\forall\ \varepsilon>0, \exists\ \delta>0$ such that $x \in D$, $|x-a|<\delta$ implies $|f(x)-f(a)|<\varepsilon$.

Then various parts can be scanned and chunked together:

$$\forall\ \varepsilon>0, \exists\ \delta>0 \text{ such that } \boxed{\boxed{x \in D, |x-a| < \delta} \text{ implies } \boxed{|f(x)-f(a)| < \varepsilon}}.$$

This may be focused at one level as

$$\text{For all } \varepsilon>0, \text{ there is a } \delta>0 \text{ such that } \boxed{\text{an implication is satisfied}} ,$$

or at another as

$$\text{For all } \varepsilon>0, \text{ there is a } \delta>0 \text{ such that } \boxed{\boxed{\text{one condition}} \text{ implies } \boxed{\text{another}}} .$$

It is possible to concentrate on *part* of the sentence, such as $\boxed{x \in D, |x-a| < \varepsilon}$ or $\boxed{\boxed{x \in D, |x-a| < \delta} \text{ implies } \boxed{|f(x)-f(a)| < \varepsilon}}$.

In a pilot study I interviewed mathematics majors at a university with a high reputation for pure mathematics, and found a wide difference in performance between the unsuccessful for whom the theory made no sense at all and the successful who understood the logical necessity of proof. *But even the most able student interviewed did not always internalise the definition and operate with its full meaning several weeks after it had been given and used continually in the lectures.* Others who were failing to use the definition went back to their visual images of a continuous function as a "graph drawn without taking the pencil off the paper" and performed *thought experiments* based on these images. They considered the statement of the intermediate value theorem to be simple and "obvious" but found the formal proof impossible to follow. Students such as these resort to damage limitation using rote-learning of procedures as reported in another investigation:

> … everyone is faced with courses whose purpose they have failed to grasp, let alone their finer details. Faced with this problem, most people set about finding typical questions and memorising the typical answers. Many gain excellent marks in courses of which they have no knowledge. (Second year university mathematics student)

What else can the failing student do? As Freudenthal said succinctly:

> …the only thing the pupil can do with the ready-made mathematics which he is offered is to reproduce it. (Freudenthal, 1973, p.117)

Can we teach students to "think mathematically"?

Can we encourage students to think like mathematicians? Even though we may not make every student a budding research mathematician, can we not alter attitudes and methods of doing mathematics that fosters a *creative* way of learning?

If students are given a suitable environment to relax and think about problems of an appropriate level, then such aspirations prove to be easy to attain. Typical problems (to be found in *Thinking Mathematically*, Mason *et al*, 1982) include:

- If a square is cut into regions by straight lines, how many colours are needed so that no two adjoining regions are painted the same colour?
- Into how many squares can one cut a square?

These problems, on the face of it fairly easy, prove to be challenging, especially when *proof* is required – for instance proving that it is *not* possible to cut a square into two, three, or five squares. The latter statement proves to be true under certain circumstances, but false under others. I will not spoil it by refining the conditions on the problem, except to say that the alternative solution was given by a thirteen year old girl in a master class, when it had not occurred to me or to several hundred mathematics undergraduates over a decade of problem-solving classes.

Reflective thinking in mathematics is built up by Mason *et al* following the *How To Solve It* approach of Polya (1945), but made more student-friendly by breaking problem-solving into three phases. The first is an *entry* phase in which the student must focus on the nature of the problem by asking "what do I want", reflect on any knowledge that may be available to begin the attack ("what do I know") and then think "what can I introduce" to move from what is known to what is wanted. The second phase is an *attack* which occurs when sufficient information is at hand to start to make the connections, and leads either to a dead-end, or to an insight which moves the problem on. The "dead-end" is seen as a valuable state because at least one method tried has not worked and by returning to the entry phase and re-assessing the position a new attack may ensue. If an insight occurs which may appear to solve the problem, then the third *review* phase needs to be undertaken, checking the method carefully, reflecting on how it was achieved and storing away strategies for future problems, then considering how to extend the problem in new ways. Whatever level of student participating in such a course, be it with children in school or final year mathematics students, the result appears always to be the same – a release from the routines of learning mathematics to pass examinations and a new spirit of adventure and confidence bred from success.

Yudariah Binte Mohammad Yusof worked with me as I taught the course on one occasion and she developed attitudinal questionnaires to ask students their opinions about mathematics and problem-solving. She then taught the course herself at another university and questioned 44 of her students before and after the course and then six months later during which time they took regular mathematics courses again (Yusof &

Tall, 1994). She also showed the questionnaires to 22 lecturers who taught the students various courses and asked them "How do you *expect* a typical student to respond?", then "How would you *prefer* the students to respond?"

Tables 1 & 2 shows the lecturers' expectations and preferences and, above these, the students' responses before problem-solving, after it, and after regular mathematics, the "aftermath" so to speak. Note that in almost every case, the change in response from what the lecturers expected and what they preferred is in the same direction as the change from before problem-solving to after. Thus the *problem-solving caused an attitudinal change in the students in the direction desired by the lecturers*. However, in almost every case, *during the regular mathematics the students' attitudes turned back again towards what the lecturers expected and away from what they desired.*

Some students appreciated that their knowledge in problem solving helped them to learn mathematics and solve problems more effectively:

> The problem solving techniques help me come to terms with the abstract nature of the maths I am doing. I try to connect the [mathematical] ideas together and talk about it with my friends. It is not that easy though. But I felt all the effort worth it when I am able to do so. (Male industrial science student, majoring in mathematics)

But a considerable minority (14 out of 44, that is 32%) reported that the mathematics they were being taught did not allow them to think in a problem-solving manner:

> Since following the course I know mathematics is about solving problems. But whatever mathematics I am doing now doesn't allow me to do all those things. They are just more things to be remembered. (Male computer education student)

> I believed mathematics is useful in that it helps me to think. Having said that it is hard to say how I can do this with the maths I am doing. Most of the questions given can be solved by applying directly the procedures we had just learned. There is nothing to think about. (Female industrial science student majoring in mathematics)

So what does this tell us? One interpretation may be that the problem-solving had relatively easy problems that allowed the students to "think mathematically" but that "serious" mathematics is demanding.

Evidence from another sources suggests that more open methods can work in analysis courses. In an experiment in Grenoble, large classes of analysis students were encouraged to work in groups in the lecture hall to propose theorems which they and other students subjected to a process of either proof or refutation by counter-example. A small minority said they preferred being told how to do mathematics in lecture classes, but 80% said they preferred the exploratory form of learning (Alibert, 1988; Alibert & Thomas, 1991).

It seems to me more likely that, because we fear failure in our students, we resort to the methods that "seem" necessary throughout mathematics. When students are likely to fail, we lack the faith in their ability to think for themselves and *tell* them how to do the mathematics in an organised way. The result is that they behave as we expect, rather than as we might prefer – they learn the material to pass the exam.

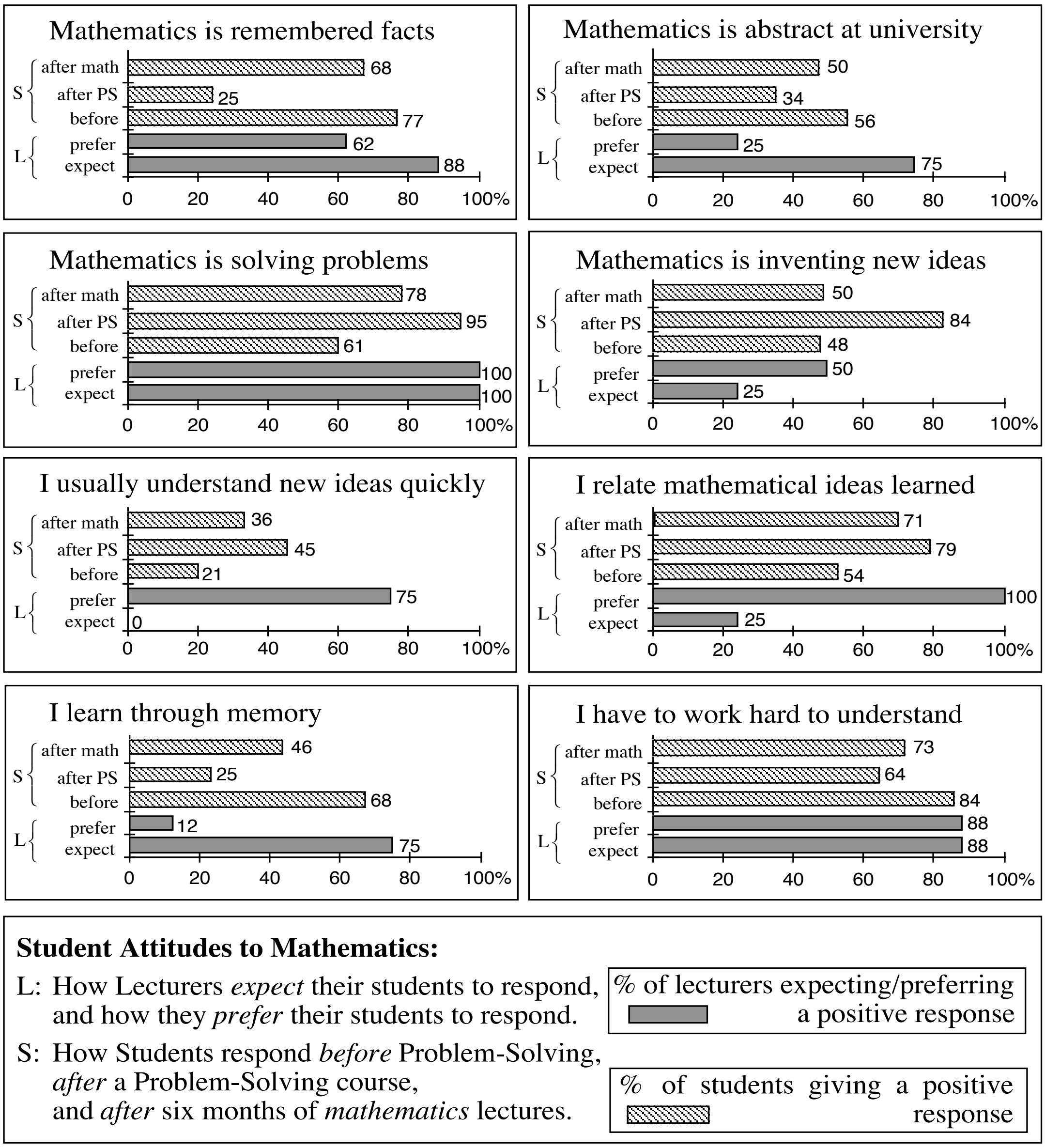

Table 1 : Student attitudes to mathematics after doing problem-solving (data from Yusof, to appear)

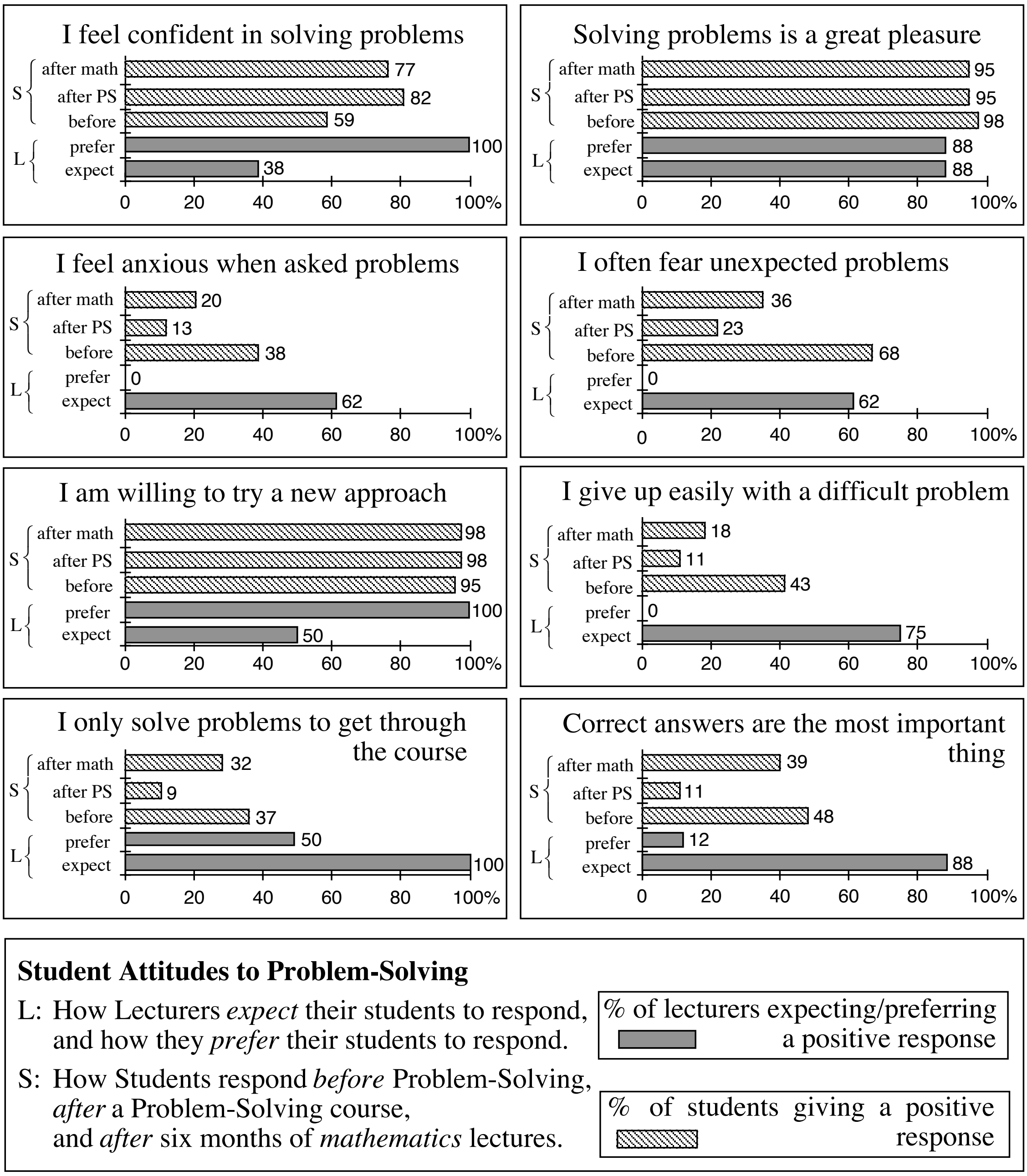

Table 2: Student attitudes to Problem-Solving (data from Yusof, to appear)

Reflections on Mathematical Thinking

Currently the university mathematics community is under some stress because it earns part of its finance from teaching undergraduates and all is not well. In the UK the London Mathematical Society produced a report which changed the British undergraduate degree structure to allow for four years instead of the traditional three. Yet when I asked the LMS to change my area of research interest to "Advanced Mathematical Thinking", the committee reluctantly refused because it was not an accepted heading in the American Mathematical Society's listing of topics. A formal request passed to the AMS through the Committee for Undergraduate Mathematics Education (CRUME) was also rejected.

Writing recently in the Bulletin of the American Mathematical Society, Thurston remarked:

> Mathematicians have developed habits of communication that are often dysfunctional.

and he went on to intimate how so many mathematicians fail to communicate in research colloquia through using highly technical language without explanation or motivation for non-experts. He continued by noting a similar problem in teaching:

> ...in classrooms .. we go through the motions of saying for the record what we think the student "ought" to learn, while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. We assume that the problem is with the students rather than with the communication: that the students either just don't have what it takes, or else just don't care. Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs. (Thurston, 1994, p. 166)

I cannot believe that mathematicians can continue to ignore the study of mathematical thinking as part of the totality of the profession, for if it is not done by mathematicians, others surely lack the mathematical knowledge to research it in depth. I suggest that the study of mathematical thinking be given a place in the canons of mathematical activity comparable with other areas of mathematics. Just as a topologist will defend a number-theorist's right to do research within the umbrella of mathematics I hope that specialists in mathematical research will similarly defend the right of mathematicians to do research into mathematical thinking. Respect will have to be earned by mathematics educators. But if opportunities to earn respect are not honoured then mathematics itself can only be the poorer.

References

Albers, D. J. & Alexanderson, G. L.: 1985. *Mathematical People – Profiles and Interviews*, Chicago: Contemporary Books.

Alibert, D.: 1988. Towards new customs in the classroom, *For the Learning of Mathematics*, **8** 2, 31–35.

Alibert, D. & Thomas, M. O. J.: 1991. Research on mathematical proof. In D. O. Tall (Ed.) *Advanced Mathematical Thinking*, (pp. 215–230). Dordrecht: Kluwer.

Davis, R. B.: 1984. *Learning mathematics: the cognitive science approach to mathematics education*. Norwood, NJ: Ablex.

Dreyfus, T.: 1991. On the status of visual reasoning in mathematics and mathematics education, *Proceedings of PME 15*, Assisi, I, 32–48.

Dubinsky, E.: 1991. Reflective Abstraction in Advanced Mathematical Thinking. In D. O. Tall (Ed.) *Advanced Mathematical Thinking*, (pp. 95–123). Dordrecht: Kluwer.

Dubinsky, E. & Leron, U.: 1994. *Learning Abstract Algebra with ISETL*, Berlin: Springer-Verlag.

Freudenthal, H.: 1973. *Mathematics as an Educational Task*, Dordrecht: Reidel.

Freudenthal, H.: 1983. *Didactic phenomenology of mathematical structures*, Dordrecht: Reidel.

Gray, E. M., & Tall, D. O.: 1994. Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal of Research in Mathematics Education,* **25** 2, 115–141.

Greeno, G. J.: 1983. Conceptual entities. In D. Genter & A. L. Stevens (Eds.). *Mental Models*, (pp. 227–252). Hillsdale, NJ: Lawrence Erlbaum.

Hadamard, J.: 1945. *The Psychology of Invention in the Mathematical Field.*, Princeton University Press, (Dover edition, New York 1954).

Halmos, P: 1985. *I want to be a mathematician – an automatography in three parts*, MAA Spectrum.

Harel, G. & Kaput, J. J.: 1991. The role of conceptual entities and their symbols in building advanced mathematical concepts. In D. O. Tall (ed.), *Advanced Mathematical Thinking*, Dordrecht: Kluwer.

Hubbard, J. H. & West, B.: 1990. *Ordinary Differential Equations* (with software for the Macintosh computer), Springer-Verlag, New York.

Koçak H.: 1986. *Differential and Difference Equations through Computer Experiments*, [for the I.B.M. Computer], Springer-Verlag, New York.

Linchevski, L. & Sfard, A.: 1991. Rules without reasons as processes without objects – the case of equations and inequalities, *Proceedings of the Fifteenth PME Conference*, Assisi, Italy, II 327–324.

Mason, J.: 1989. Mathematical abstraction seen as a delicate shift of attention. *For the Learning of Mathematics*, **9** 2, 2–8.

Mason, J., with Burton, L. & Stacey, K.: 1982. *Thinking Mathematically*, London: Addison-Wesley.

Miller, G.: 1956. The magical number seven plus or minus two: some limits on our capacity for processing information. *Psychological Review*, 63, 81–97.

Piaget, J.: 1972. *The Principles of Genetic Epistemology* (W. Mays trans.), London: Routledge & Kegan Paul.

Polya, G.: 1945. *How to Solve It*, Princeton University Press, Princeton.

Sawyer, W. W.: 1987. Intuitive understandings of mathematical proof, *Bulletin of the Institute of Mathematics and its Applications*, 23, 61–62.

Sfard, A.: 1991. On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, **22** 1, 1–36.

Sfard, A.: 1994. Reification as the birth of metaphor, *For the Learning of Mathematics*, **14** 1, 44–54.

Takagi, T.: 1903. A simple example of a continuous function without derivative, *Proc. Phys.-Math.* Japan, 1, 176-177.

Tall , D. O.: 1982.: The blancmange function, continuous everywhere but differentiable nowhere, *Mathematical Gazette*, 66, 11–22.

Tall, D. O. (Ed.): 1991a. *Advanced Mathematical Thinking*, Kluwer: Dordrecht.

Tall D. O. 1991b: *Real Functions & Graphs* (for BBC compatible computers), Cambridge University Press.

Tall, D. O.: 1992. The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof. In: Grouws D. A. (ed.) *Handbook of Research on Mathematics Teaching and Learning*, (pp. 495–511). New York: Macmillan.

Tall, D. O.: 1993. Real mathematics, rational computers and complex people, *Proceedings of the Fifth International Conference on Technology in College Mathematics Teaching*, Addison-Wesley, 243–258.

Tall, D. O. & Vinner, S.: 1981. Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, **12** 2, 151–169.

Thurston, W. P.: 1990. Mathematical Education, *Notices of the American Mathematical Society*, **37** 7, 844-850.

Thurston, W. P.: 1994. On proof and progress in mathematics, *Bulletin of the American Mathematical Society*, **30** 2, 161–177.

Yusof, Y. b. M. (to appear): Ph.D. Thesis, University of Warwick.

Yusof, Y. b. M., & Tall, D. O.: 1994. Changing Attitudes to Mathematics through Problem Solving, *Proceedings of PME 18*, Lisbon, **4**, 401–408.