Student Teachers' Conceptions of the Rational Numbers

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It is well known that students and mathematicians have idiosyncratic beliefs about the real numbers. This research reveals confusions about rational numbers in undergraduates studying to be teachers. Earlier experiences of fractions and decimals in school (such as the long exposure to finite decimal expansions and the link between π and $^{22}\!/_7$) lead to subtle misconceptions. The definition of the rational numbers is rarely used to test whether specific numbers are rational other than those explicitly given as a ratio of integers, illustrating a wider problem with the use of formal definitions in mathematics.

Introduction

It has long been known that students have idiosyncratic images of real numbers (Tall & Schwarzenberger, 1978; Davis & Vinner, 1986; Monaghan, 1986; Wood, 1992; Li & Tall, 1993; Lee, 1994; Romero i Chiesa & Azcarate–Gimenez, 1994, etc). Given children's difficulties with fractions, student difficulties with rational numbers may be expected, although the problems are often unsuspected by mathematicians. This paper reports students' responses to rational numbers in the third year of a mathematics education degree preparing primary and secondary mathematics teachers. Their earlier experiences had included schoolwork using fractions and finite decimals, a first year university programming course representing numbers as finite decimals, a second year "Sets and Groups" course defining the rational numbers in the form m/n where m and n are integers with $n \neq 0$, and a third year analysis course giving formal definitions of concepts such as limits and continuity. The small number of isolated courses may lead to less coherent integration than in specialist mathematics degrees. The students proved to have idiosyncratic "evoked concept images" (Tall & Vinner, 1981) of rationals.

An analysis of solutions of twenty students on a written assessment in analysis suggested idiosyncratic concept imagery for rational numbers. For example, in response to the question

Explain why the function $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ is discontinuous for all $x \neq 0$,

the student who achieved the highest mark in the group wrote:

> "If zero is rational, then ...
> If zero is irrational, then..."

This was indicative of subtle conflicts in students' personal imagery of rational numbers.

The role of definition in mathematics and its relationship to the concept image

The use of definitions in mathematics has two very distinct purposes. On the one hand, a concept which is already familiar to the student is given a definition to identify the concept. In this case the concept determines the definition. On the other hand, in formal mathematics the definition is used to construct the properties of the mathematical concept which it defines. In this case the definition determines the concept. This reversal from *concept→definition* to *definition→concept* is an epistemological obstacle which can cause great difficulty (see, for example, Sierpinska, 1992, p. 47). It is also an essential component of the fundamental change from elementary to advanced mathematical thinking (Tall, 1995).

From a cognitive point of view, we will be concerned that the rational number concept has a concept image in individual's mind that consists of "all the mental pictures and associated properties and processes" (Tall & Vinner, 1981). The individual's experience prior to meeting a formal definition not only affects the way in which the individual forms mental representations of the concepts, but frequently becomes manifest through the efforts to resolve problems with an inappropriate "evoked concept image".

Vinner (1991, p. 69) considered the concept definition and the concept image as two different "cells" in the cognitive structure and analysed the introduction of a definition occurring in three possible scenarios:

  (i)   the concept image changes to accommodate the definition;
  (ii)   the concept image remains as it is, the definition is forgotten or distorted;
 (iii)   the concept image and definition are both present but not linked together.

Our experience is that the situation is more complex. Not only are other variants possible (eg distorted concept images produced by distorted definitions, Gray & Pinto, 1995) but, more importantly, different types of connection between concept definition and concept image typically occur in a single individual. Vinner (1995) discusses how the concept image may change as a result of *pseudo-conceptual* thought, which seems on the surface to be conceptual behaviour but lacks the reflective, analytic control procedures that characterise true conceptual thinking. It is possible for each individual to have different types of concept imagery with different links to the concept definition, including:

- *informal imagery* not deduced from a definition, which may be further subdivided into imagery *consistent* or *inconsistent* with the formal theory,
- *distorted imagery* produced from a distorted personal definition or by faulty reasoning from a correct definition,
- *pseudo-formal imagery*, which may seem consistent with formal theory but is not ultimately deduced from the definition by formal reasoning,
- *formal imagery* deduced formally from the definition.

In practice it may be difficult to distinguish between these possibilities. If no definition has been given, only informal imagery is possible. (Erroneous informalities should not be classified as distorted imagery because the student has no formal definition to distort).

When a definition is given, informal ideas continue to be available and are essential to guide the individual (including the research mathematician) to formulate new ideas which can then be deduced formally. If a student remembers a (partially) correct definition, formal constructions become possible from the (correct part of the) definition, but pseudo-formal and distorted imagery can also occur. The distinction between these requires investigation of the thought processes which led to their construction.

University teaching often introduces proof in an "informal manner", but this can cause great confusion in the students as to what it is that that can be assumed and what it is that has actually been "proved". This is more likely to occur where a "formal" definition includes "informal" elements, such as the rationals defined in terms of (informal) experiences of natural numbers.

"Proof" at university is often bedevilled with a mixture of formality and informality. Moreover, traditional teaching of number systems at this level usually overestimates students' informal understanding of numbers, assuming an apparent intuitiveness of the mathematical real line which does not exist in typical students. There seems to be an implicit belief that soon the learner will get the meaning of the symbolic representation by working on it formally, and so time is not spent on suitable discussion about numbers and relating this to the mathematical meaning of the real line.

Such an approach was perceived and described by one student during an interview:

> "I just took the numbers for granted really, the real numbers. When you first meet numbers and you learn what they are, you learn at some point about natural numbers and rational numbers and all of that. You feel they're just something you are supposed to know. Whether you do or not is another matter. The teacher just mentions that we're using real numbers and you never remember what they are, you just take them for granted."
>
> (Third year undergraduate student)

The long exposure to the approximate arithmetic which students encounter in secondary school is bound to exert its effects on their informal imagery. Monaghan (1986) found that students often perceive recurring decimals as dynamic and qualitatively different from finite decimals, so they are not "proper" numbers.

Our analysis will provide examples showing that a formal presentation of the subject matter and formal work with numbers did not encourage the students concerned to reconstruct their concept image from the definition of rational numbers.

Interviews and analysis of responses

Seven students selected for interview were each asked the questions:

"Can you define a rational number?"

"Can you define a irrational number?"

They were then given a list of real numbers to classify as rational or irrational. The list varied from one interview to another, but always included the number zero, square roots such as $\sqrt{2}$ and $\sqrt{4}$ which may or may not be rational, the special rational $^{22}\!/_{7}$ which is

used in school as an approximation of π, and various finite and recurring decimals such as 0.97853, 1.41, 0.333…, 0.343232… (the last two emphasised as "recurring").

The seven responses were subdivided for study according to the student's replies to the opening questions. Three students (A, B, C) gave (subtly incomplete) formal definitions which proved to contain implicit distortions, three (I, J, K) gave explicit distorted definitions and one (student X) was unable to recall a definition. None of the students corresponded to Vinner's category (i) whose concept image changed successfully to take account of the definition.

Students offering a formal definition

Students A, B and C stated their definition close to the formal definition although all of them failed to *explicitly* exclude zero as a possibility for the denominator of a rational:

> Student A: "An irrational number is a number that cannot be expressed as two integers, one integer divided by another integer. ... And a rational number can be expressed as two integers divided by each other, *p* over *q*."

> Student B: "Well, I would define an irrational as not a rational number, and a rational number as one which we express as *p* over *q* where *p* and *q* are both integers. And it can't be expressed *p* over *q* in an irrational such as the square root of two, and π, things like that."

> Student C: "A rational number to me is a number that can be created as a ratio of two integers, I mean, this is straight Sets and Groups. Earlier on my degree course we defined all different types of number groups, and I know that one off the top of my head. Ratio of two integers, so any two integers, which could be any positive and negative whole number, the ratio of them is the rational number. And an irrational number ... is when it's not a ratio ... So—I don't know if this is helpful at all—you've got natural numbers, which are one, two, three, whatever, your integers which are negative, plus and minus one, two, plus … including zero, your rationals are integers, ratios of two integers. You might have a negative one over seven or whatever, your irrationals are non-rationals, really. So root two, … you can't express a number in terms of a ratio of two integers. "

Student C gives additional information to amplify the definition, suggesting that the definition is not seen as a generative idea from which everything can be deduced, but as a description which sets the concept in context and may require further explanation.

Students A and B give the most succinct definitions but both experience conflicts as to whether the number zero is rational or irrational:

> Student A: "... nought, I think nought's rational, but I'm not sure. My reasoning for that is because it's part of the integers, which is why I connect it with the rationals, I think."

Student A seems to be thinking conceptually, using her knowledge that integers are rationals to come to her conclusion rather than use the formal definition she has given. But there are signs of inner conflict. When asked for further explanation, she reveals that she thinks zero might be irrational because of a distorted inference that *all* rationals should have a multiplicative inverse. Because zero has no inverse, it could be irrational:

> "Why do I think it could be irrational? Because you can't divide by nought. ... Could you do nought divide by any number, will give you nought, won't it? So if you've got nought divided by *n*, for instance, you've got nought. So that way it can be expressed as a rational, but if you've got *n* divided by nought, it doesn't work."   (Student A)

Student B built on a distorted conception identifying 0 with the illegal expression $\frac{0}{0}$:

> Student B: "Nought, I think that's nought over nought. ... I think you're allowed to use nought over nought where .... so it's rational, I think that's right."

When asked for further explanation, she replied:

> "... zero, isn't it ? I don't know. Maybe you can't divide by zero, so maybe you should leave … Maybe it's an irrational. I'm not really sure whether you can have a division by zero. ... Zero divided by zero, normally you can't have zero on the bottom of a division line because it's undefined, so if it's undefined, then it wouldn't be nought either, so therefore it can't be defined as p over q. So it must be irrational.... I'm really not sure".   (Student B)

The difficulties experienced by students A and B classifying the number zero may both be related to confusion over the notion of a rational as the quotient of two integers where the numerator may be zero but the denominator may not. Partially remembered this may lead to confusion as to whether the numerator or denominator may be zero. The case of zero may be further compromised when rationals are described as "positive or negative fractions or zero", treating zero as a special case distinct from familiar fractions.

Positive rationals often have informal every day meanings as fractions:

> Student A: "To represent parts of things. You use naturals for the whole things and rationals for parts of that thing. You split up into whichever number is on the bottom part, say 3 over n, you have to split your object into n parts and then 3 of those n parts will make your 3 over n."

But the following lengthy response reveals an uneasy separation between informal fractions and the technical use of rationals (as in Vinner's category (iii)):

> Student C: "I look at it as just a fraction, five eighths. ... just using it as five pieces of eight, that's just a fraction really, that's all it means to me. ... The natural number is to me the basis of maths and you probably use that every day rather, as opposed to five eighths, the fraction five eighths. To me when I would use natural numbers it would just come about every day, you are just using so many examples and uses than fractions. I would say so, just off the top of my head. ... Natural numbers to me are the simplest form of groups of numbers, there aren't many complications, you don't have to deal with negatives. ... Definitely, if you say rational number, if you went out into the street and said can you tell me a rational number, I don't know anyone, well not many people, would be able to tell you what a rational number was, in that sense. If you said give me a fraction, I mean it's like when you said what does that mean to me, that meant a fraction to me, it didn't mean a rational number, that's what it is. I'd say definitely natural numbers, more than rational and again rational is just a ... to me it's just another way of saying fractions, a fraction's more of an everyday ... more people would be able to tell you a fraction rather than a rational number. ... Again, another difference between fraction and rational number would be, say, realising that, I mean not many people would give you a negative fraction, which obviously would be a rational number, say minus five over eight. It's still a rational but they wouldn't necessarily say… I don't think that minus five over eight is a proper fraction, they would think of minus, you see what I'm saying at all? *When you said what does that mean to me, I didn't think rational number. I just thought fraction, which is a subgroup, a section of rational numbers, but that's what it means to me immediately* and now I'm putting myself in maths degree mode I'll think—Yes, that's a rational number. ... *When would I use 0.59? ... Just when you are dealing with decimal work, a part of again I suppose a fraction of unity, if you like. Or point fifty nine*. I'm trying to think now. You mean as opposed to using this, say, or you know like using, I mean fifty nine over a hundred or whether I'd use point fifty nine ..."

Instead of expanding the familiar image of fractions, a separate technical meaning has been given to rational numbers. The phrase "point fifty nine" instead of "point five nine"

also indicates informal conceptions that may contain classical conflicts (such as "point fifty" being bigger than "point five", or "point eleven" being bigger than "point nine").

Although the definition given was used by all three students to *describe* rational numbers, it was not used in an operative sense to *test* whether a given number is rational, so correct responses could be pseudo-formal rather than formal. In spite of declaring the finite decimal 0.97853 correctly to be rational, Student A said:

> "I'm not sure about ... I can never remember that one."

Student B, meanwhile, classified the finite decimal 0.97853 as *irrational*, showing that she does not operate mathematically with the definition, in spite of being aware of it.

Students stating distorted definitions

Students I, J and K were not able to give a formal definition of a rational when asked to do so. Instead they attempted to describe what they thought a rational number would be, referring to their informal concept imagery. The dominance of concept image over concept definition is more transparent in these cases. For instance, students might relate the notion of rational to their experiences of finite decimals:

> Student I: "A rational number is a number that can be defined by ... it's easier to say it's not an irrational number.... An irrational number is a number with an infinite ... it cannot be defined to a finite number of decimal places. ... Yes, so a rational number would be a number that can be defined to a specific number of decimal places."

In the absence of an operative definition, Student I made decisions about the rationality of numbers in a variety of ways. For instance, she classified recurring decimals as irrationals, and the fraction $22/7$ was also considered as an irrational:

> "Because that's the way that π is represented and we've been told all through school life that π is an irrational number and can keep going for ever and ever." (Student I)

Idiosyncratic definitions often start by being stated in descriptive form, but the use, or rather lack of use, of the definition suggests an unawareness of the role of a definition in a formal context for deduction of the properties of the concept so defined. It is possible for a student to have a coherent personal definition and yet evoke other parts of the informal concept image to give a distorted image. For example, student J's definition seemed to be appropriate for some use in discriminating rationals and irrationals:

> "A rational number is a number that can be written as a fraction. For example, one is a rational number, because it can be written one over one, a half is a rational number because that's obviously one over two. An irrational number is the opposite to a rational number, you can't write it as a fraction." (Student J)

Student J correctly classified "point three recurring" as the fraction "one third", and then classified "point nine recurring" as being rational, because it was "nine over nine". The finite decimal 0.97853 and the recurring decimal 0.343232… were then classified as irrationals because in each case *she could not imagine a fraction that could represent it*.

Student K's descriptive definition of a fraction gave little opportunity for deduction:

> Student K: "A rational number? I know this one, I think. A rational number is a number that is divisible, say a whole number or a number like a half or seventeen over sixteen or

something like that. It can be positive or negative, as opposed to irrational numbers which are square roots of some numbers."

Instead she ignored the definition and classified numbers by evoking idiosyncratic concept images. For instance decimal expansions of numbers, some finite (1.41 and 0.97853) and some recurring (0.343232…), were classified as irrationals:

> "Because when it's in decimals, I can't visualise in my head what it is, so I don't really know". (Student K)

In classifying point nine recurring correctly as a rational, she presented the argument:

> "If you rounded that up it would be a rational number." (Student K)

Further questioning revealed the pseudo-conceptual nature of the thinking based on informal imagery:

> "I don't know, it's like point nine, nine, nine ... is so close to one, but I don't know whether that makes any difference to a rational or an irrational number — being so tiny. I'm just guessing." (Student K)

Students unable to give a definition

Only one student interviewed could not formulate a definition in words, instead giving an indication of how she copes with tasks involving such knowledge:

> Student X: "I always look these up when I need to know what they are."

But she had an informal concept image of a rational number as follows:

> "I think it's rational if it can be written in terms of numbers like that" (pointing to $\frac{6}{8}$, a number in the given list). (Student X)

In her classification, all the decimal expressions were classified as irrationals, so she was again asked if she had a definition in mind. She replied in a manner which revealed how she avoided working with such definition:

> "Yes, but I don't know if it's right. That's what I remember. When I have to use a rational or irrational number, I've got a list of all different symbols and what the things mean; and I usually refer to that when I need to know, but it hasn't stuck yet." (Student X)

She appeared to be working with a mental list of specific examples of irrationals (such as π and $\sqrt{2}$) and rationals included numbers such as $\frac{2}{5}$ explicitly written in fractional form, but if the expression could not be readily converted into rational form *by her*, it was considered irrational.

Conclusion

The evidence presented in this paper reveals the rich diversity of imagery students can have relating to the concept of rational number. Although three of the seven students interviewed were able to give an (almost) satisfactory definition of the concept, none consistently used the definition as the source of meaning of the concept of rational number. Instead they used their rich concept imagery developed over the years to produce conclusions which were sometimes in agreement with deductions from the formal definition, but often were not. Whole numbers and fractions were often seen as

"real world" concepts, with rationals as more technical concepts (except when positive rationals were identified with fractions). Long exposure to approximate arithmetic with finite decimals gives the latter a primacy of meaning that is sometimes erroneously identified with the fraction concept. The meaning of "rational" may also be interpreted as *the individual's ability to express the number as a quotient of two integers*, so that familiar decimals such as 0.1 or 0.333… are interpreted as rational, but less familiar decimals (particularly recurring decimals) may not be.

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