Conceptual and Procedural Approaches to Problem-Solving

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The introduction of problem-solving strategies has been shown to change students' attitudes to mathematics in ways that professors consider desirable. But does it change their overall strategies for doing mathematics? This paper reports data taken from students solving problems co-operatively who exhibited an overall improvement in attitudes (see Yusof & Tall, 1995). It indicates that some students who had said that "mathematics makes sense" approached problems in an open, creative way but that some lower attaining students who had stated that "mathematics does not make sense" treated problem-solving techniques as a new sequence of routine procedures.

There is a growing awareness that many students are successfully learning how to carry out routine procedures to pass examinations, but there is a concern that the system may not be providing students with the experiences to encourage them to be creative and reflective. Students at university are often given lectures that consist of theorems and proofs which do not encourage them to think mathematically. Problem-solving is seen as no more than just a skill to be acquired. Studies have shown that the traditional approach is failing the majority of the students, not only the average students but more disturbingly also successful students. Students find great difficulties in constructing their own mathematical understanding (Davis & Vinner, 1986; Martin & Wheeler, 1987; Sierpinska, 1988; Eisenberg, 1991; Williams, 1991) and have a narrow view of the mathematics that shapes their mathematical behaviour (Schoenfeld, 1989; Vinner, 1994). Nevertheless, research findings indicate that thinking mathematically or problemsolving can be taught with some success. For instance, Mason & Davis (1987) explored how people can develop their mathematical thinking, learning, and teaching by reflecting on their own experience. They argued that the technique of using meaningful vocabulary can help students to become more reflective and effective in mathematical learning. It was observed that students not only notice the use of the vocabulary and advice from tutors, but also remember it when the same language pattern (e.g. specialising, generalising, colloquial comments such as "What do I want?" etc.) was repeatedly used and their attention was explicitly drawn to it. Mohd Yusof & Tall (1995) reported that a course which provides students with experiences of sharing problem-solving activities has the effect of changing students' attitudes. Prior to the course the students generally regarded mathematics as abstract facts and procedures to be committed to memory, and had a range of negative attitudes such as fear of new problems, being unwilling to try new approaches, and giving up all too easily. After the course, students attitudes changed

Published in *Proceedings of PME 20*, Valencia (1996), vol. 4, pp. 3–10

in a positive direction. In this paper we investigate whether this change in attitude is accompanied by successful change in strategies for solving problems.

To investigate the manner in which students attack a problem, six groups were selected and given a problem which was relatively easy to state but did not have a straightforward algorithmic solution. The students taking part in the research were a mixture of third, fourth and fifth year undergraduates aged 18 to 21 in SSI (Industrial Science, majoring in Mathematics) and SPK (Computer Education), covering the full honours degree range. They had responded to a questionnaire in which they had been asked to indicate whether mathematics "makes sense" to them. Half the students agreed and half disagreed. Interestingly, the two groups had almost identical distributions of achievement in their previous year's examination. (Table 1.)

	Degree Classification					
	I	II-1	II-2	III	P	F
Group *S*	3	11	7	1	0	0
Group *N*	3	13	5	1	0	0

Table 1 : Students for whom mathematics makes sense (Group *S*) and does not (Group *N*)

During the problem-solving course they had been encouraged to work in self-selected groups of three or four students. Several groups (by chance) consisted either entirely of students who declared that mathematics made sense (*S*) or that it did not (*N*). From these, three groups were chosen with all students in group *S* and three groups with students in group *N*. (Table 2.)

	Students	**Course**	**Degree Classification**	**Gender**	**Maths "makes sense"**
group 1 (S)	Sam	5 SPK	II–1	M	S
	Abel	4 SPK	II–2	M	S
	Henry	4 SPK	II–1	M	S
group 2 (S)	Sue	4 SPK	I	F	S
	Teresa	4 SPK	II–1	F	S
	Sasha	5 SPK	II–1	F	S
group 3 (S)	Rob	3 SSI	II–1	M	S
	Kline	3 SSI	II–1	M	S
	lan	3 SSI	I	M	S
group 4 (N)	Hanna	5 SPK	II–1	F	N
	Katy	5 SPK	I	F	N
	Terry	5 SPK	I	M	N
group 5 (N)	Bob	5 SPK	II–2	M	N
	Yvonne	5 SPK	II–1	F	N
	Alma	4 SPK	II–1	F	N
	Pauline	5 SPK	II–2	F	N
group 6 (N)	Matt	5 SPK	II–1	M	N
	Al	4 SPK	II–2	M	N
	Holmes	5 SPK	III	M	N
	Ricky	5 SPK	II–2	M	N

Table 2 : The 6 groups of students selected for interview

Each group was invited at an appointed time for the session that lasted 40 minutes. The first 10 minutes served as a *relaxing* phase whereby the students were simply asked to talk about their mathematical experience at the university. For the next 30 minutes, they were given a problem to work on, as follows:

> A man lost on the Nullarbor Plain in Australia hears a train whistle due west of him. He cannot see the train but he knows that it runs on a very long, very straight track. His only chance to avoid perishing from thirst is to reach the track before the train has passed. Assuming that he and the train both travel at constant speeds, in which direction should he walk? *Mason, Burton & Stacey, 1982, p. 183.*

After being presented with the problem the students were left entirely on their own and their attempts in solving it were observed without any intervention. The interview then focused on the students' interpretation of their problem-solving experience.

During the course the students had been encouraged to view their activities in three phases – entry, attack and review, with appropriate activities for each (Mason et al, 1982). The purpose of the research was to see if the students used this structure as a framework for meaningful problem-solving.

The interview data provided some evidence of qualitatively different thinking between the various groups. For instance, the following excerpt from the beginning of the solution process when the students were in the "entry phase" indicates differences in mathematical understanding.

Students in group 1 spent a few moments establishing the meaning of "constant speed" and finally agreed it mean that both train and man were moving at different speeds.

> ABEL: Constant speed …
> HENRY: The speed of the train must be the same.
> SAM: It is not the same.
> HENRY: Constant.
> SAM: Constant means it does not increase or decrease.
> ABEL: … the train travels say at 40 mph, Ali [the man] 4 mph. Ali will always travel at 4, the train always at 40. That is constant speed.
> SAM: I agree.
> HENRY: Hmm …
> ABEL: It is not the same speed but constant speed. Ali can be faster than the train …
> SAM: Ali and the train do not move the same, not at the same speed. But at their respective speeds … the same speed all the time. *group 1 (S)*

In contrast, group 6 students started from the misconception that constant speed was relative to the man and train and thus both move at same speed. They quickly agreed with the meaning and no further reference was made to their interpretation of 'constant speed' until the end.

> MATT: … constant speed.
> AL: It means the same I think.
> HOLMES: Constant speed …, it's the same.
> MATT: Uniform ...

AL: It means the man moves with the train at the same speed. Now OK … *group 6 (N)*

During the problem-solving, it could be seen that three of the six groups (the lower attaining *N* groups 5 and 6 and the younger *S* group 3) followed the techniques taught in the problem-solving course very rigidly. Of these three, the two *N* groups seemed to be doing it more religiously than the *S* group. They were more concerned to cover each phase in a sequence and so they could be seen to be working procedurally throughout. They interpreted the problem-solving technique as a procedure that they have to follow step by step, it was as if they believed that precision in following each phase would guarantee them a solution. Most of their time was spent looking for formulas that could be used.

PAULINE: We have already <u>understood the question</u>**.** We have <u>introduced</u> what we want, what we know. We have done that. OK now we can enter the <u>attack</u> [phase].
YVONNE: What is the formula?
BOB: Speed times time.
YVONNE: The time is the same. The speed is …
ALMA: We need to define speed first.
BOB: I should remember how to do this. … Oh yes! speed is distance divided by time.
PAULINE: Now the distance, we don't know how much, right? The distance between the man and the train.
BOB: Let us assume the speed of the train is 100, the man 10.
…
ALMA: OK we did some <u>specialising</u> … *group 5 (N)*

MATT: So, first we go to the <u>entry</u> phase.
AL: OK. That is what we <u>know</u>. Now what we want is the direction in which the man should go.
MATT: Anybody feel <u>stuck</u> or anything. The question is clear isn't it?
…
RICKY: The concept of intersection. That is what we can say.
HOLMES: The intersection point is the place the man has to go.
MATT: OK, now we go to the <u>attack</u> phase. *group 6 (N)*

ROB: We are <u>stuck</u> at this point.
KLINE: Stuck. OK. write down we are stuck.
IAN: Let's <u>go back to what we want</u>. What we want is the direction in which the man should walk. Direction, the man should go … west, east …
…
ROB: We are confident our assumption is correct so far. OK now we enter the <u>attack</u> phase. *group 3 (S)*

In none of these beginnings of solutions have the students thought in a broader conceptual fashion, for instance to consider the *direction* of the train, or to draw a diagram. The other groups were more involved in considering plausible ways to solve the problem by creating their own solution method.

SAM: OK, so we conjecture that the train is moving towards west. That is according to your understanding. But I have another suggestion. To me...we go back to entry phase OK?

. . .

ABEL: OK, I got it.

HENRY: No, no, no. Hang on. The train is moving to the west. … But why should the man walk in this direction [pointing to a point on the diagram]? Why do you say that?

SAM: It is like this. Now this is just my idea. … Say the man is here [drawing another diagram] … So he cannot go this way, otherwise he will be moving parallel to the track and may never reach the train.

HENRY: So according to what you say, the direction the man should walk is this one, to the north. OK, we can conjecture that.

ABEL: We have now answered the question. Now we want to justify it [the conjecture] whether it is correct or not. *group 1 (S)*

In groups 2 (S) and 4 (N) one of the students thinks the train is moving west but others correct her and widen the issue:

TERESA: ...The train is moving to the west.

SASHA: Where does the train come from?

SUE: That is the problem. That is the one that we want to find out, it relates to the direction we want to go.

SASHA: Hmm ...We are stuck!

SUE If we know from where [the train is coming], we can find out where we want to go.

TERESA: Suppose we look at it this way. First say the man is here [pointing to a point on her paper]. Now we define where is his east, his west …

SASHA: OK. Let’s draw another diagram. *group 2 (S)*

HANNA: We are wasting our time … What I know, the question says, the train is moving towards west. So the man must go towards west as well.

TERRY: No! The question does not say the train is going west. But heard [the whistle] due west of him.

KATY: Yeah, that is my understanding too. The man heard the train whistle due west of him. But this does not mean that the train is moving towards the west. We cannot make that conclusion.

TERRY: How do we know from the whistle that the train is moving west or east... What is your reasoning?

TERRY: OK, that’s it. So we conjecture that the man should walk to the north. I think we have a solution to the problem. But we are not finished yet, we need to justify this conjecture first. *group 4 (N)*

Four of the six groups (the three *S* groups 1, 2, 3 and the higher attaining *N* group 4) gave some evidence that they are able to carry out the mathematical processes to some extent. They show that they are capable of making judgements on the content and in making mathematical decisions for themselves. They also question the meaning of the task.

The problem is very challenging. It does not require a specific formula or procedure that you have to apply to solve it. It is quite difficult. We got an idea what the answer is but to proof it is the hardest part. *group 1 (S)*

> We only managed to understand the question better towards the end of the discussion time. But I think we can solve the problem if we have more time. It is not difficult, but to generalise and to prove is very difficult … We will keep on thinking about it until we get the answer. *group 2 (S)*

> The problems in the problem-solving course are interesting. Like this one. We have to think, work out what we want, what we do know before we actually work out what we don't know. … The course is beneficial. It makes us sit down and see where to start. *group 3 (S)*

However, the other two *N* groups (5 and 6) have the notion that mathematical problems consist of direct application of facts and procedures. Their lower attainments on their examinations suggests they have less secure knowledge to bring to the solution process. Thus they are in an interesting position where they have built up their confidence to tackle problems and yet they find the problems very difficult.

> We tried to generate few possible ideas. But we felt a bit put off because we couldn't recall the formulas. … The problems are totally different from those in maths course. In maths we always know what method to use. Here we have to find it out ourselves. … I think we have more confidence now. Before the [problem-solving] course we probably would have given up very easily. *group 5 (N)*

> We found it [the problem] very difficult. We are unsure of which formulas or methods to use. Even if we got a solution, we don't know whether our solution is right. … Unlike problems in the problem-solving course, most of the problems in the maths course are simply applications of a ready rule. There is always a definite answer at the end. *group 6 (N)*

Discussion

Although none of the groups could provide a complete solution to the problem within the time limit, they were at least able to tackle the problem to make a start. *All* the student groups were very willing to tackle the problem without any overt sign of anxiety. Even though the problem remained unfinished, all three *S* groups and the higher attaining *N* group 4 considered that they could solve the problem given more time, (although based on their responses this may involve a lot more effort than they may have thought). Meanwhile, the other two *N* groups were seeking formulae appropriate for a solution and using the overall strategy of problem-solving as a procedure to attack the problem. Their response to problem-solving shows the same procedural format as their approach to traditional mathematics problems.

Byers & Erlwanger (1985) note that memory plays an important role in the understanding of mathematics. However, they suggest that it is *what* is remembered and *how* it is remembered that distinguishes those who understand from those who do not. Mathematical concepts are abstract entities which involve mental effort to construct relationships between the ideas. The students involve in this research have previously followed courses which place great demands on their success in learning procedures and applying them to solve related problems so that they perceive mathematics more as a fixed body of knowledge to be learnt.

The problem-solving course has had various positive outcomes, for instance, the students have experienced the fact that not everything they do has to be immediately correct. If they were to fear making erroneous conjectures, the may not be able to solve any real problems. Although it is essential to get the right answer by the end of the process, it is evident that after the course, the students see that it is *how* they obtain an answer which is more important; making the intellectual journey to find the right methods and correct reasoning. It is possible to conjecture that the students' success in problem-solving during the course was sufficient to give them a sense of well being.

Although the students show little emotional reactions when solving an unexpected problem, opinions expressed in an attitudinal questionnaire suggest that group 2 have positive attitude before and after the course. Majority of those in groups 1, 3 and 4 became more positively inclined after the course. Group 5 and 6's negative attitudes lessened after the course. The diminishing of fear and anxiety may be related to Skemp's idea of avoiding failure and a perceived increase in confidence during the course involves seeing the task more as a goal to be achieved. In the case of all these students, there was a general sense of satisfaction expressed at the end of the problem-solving course. However, from the evidence of these investigations, it is clear that for some, doing things procedurally is not an anti-goal for them as suggested by Skemp. To some of the students it is a goal, but it is the wrong kind of goal.

Summary

The students involved in this research have long since learned that what matters most is to be able to carry out the procedures to do the mathematics. During the problem-solving course, although the majority of students showed that they are capable of carrying out the various processes of mathematical thinking and engage actively in problem-solving, the interviews emphasise that there are differences in the quality of the students' thinking. For instance for some lower attaining students for whom mathematics does not make sense, when faced with a problem appear to be more concerned about recalling and applying learned techniques to solve the problem rather than looking for insights, methods and reasons. Perhaps their contextual understanding of mathematical concepts is limited. Thus they lack confidence in carrying out the mathematical performance. Their reaction to the given mathematical problem gives an indication that they see the problem-solving knowledge as just another procedure. While problem-solving, their emphasis is on applying learned techniques or ready rules to the task. They were using a procedural method and were not truly doing problem-solving. Their recorded discussion gave an indication of the way they do mathematics; in a procedural and a non conceptual way. After the problem-solving course, the tendency to lay emphasis on procedural aspects remains.

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