Facets and Layers of the Function Concept

Phil DeMarois
William Rainey Harper College
1200 W. Algonquin Rd
Palatine IL 60067
USA
e-mail: pdemario@harper.cc.il.us

David Tall
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL
UK
e-mail: David.Tall@warwick.ac.uk

This paper considers different aspects that make up the function concept, taking critical account of several current theories of multiple representations and encapsulation of process as object to build a view of function in terms of different ***facets*** *(representations) and different* ***layers*** *(of development via process and object). An interview technique is used to determine the profile of students according to this view.*

Facets and Layers of the Function Concept

The function concept has been a major focus of attention for the mathematics education research community over the past decade. (See Dubinsky & Harel, 1992, for example.) Schwingendorf *et al* (1992) contrast the *vertical* development of the concept in which the process aspect is encapsulated as a function concept and the *horizontal* development relating different representations. We refer to these as *depth* and *breadth* respectively (noting that increasing depth here means higher levels of cognitive abstraction) and investigate the way in which the student's concept image of function can be described in terms of these two dimensions.

The breadth dimension is often conceived as consisting of various *representations*, including *geometric*, *numeric* and *symbolic*. However, there is increasing criticism of the theory of the mental representations involved—what they actually represent, and how they are linked cognitively:

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation... The core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.
> (Thompson, 1994, p. 39)

Crick (1994, p.10) notes the way in which the brain is built as "a messy accumulation of interacting gadgets" promoted by evolution, so that "if a new device works, no matter in however odd a manner, evolution will try to promote it". However clean and neat we attempt to formulate the mathematical theory in terms of external representations, the internal workings of the brain operate in a far more complex manner.

To acknowledge this debate, we use the word *facet* to build up a description of the breadth dimension. Webster's *New World Dictionary* (Guralnik, 1980 p. 300) defines a facet as "any of a number of sides or aspects." The facets of a mathematical entity refer to various ways of thinking about it and communicating to others, including verbal

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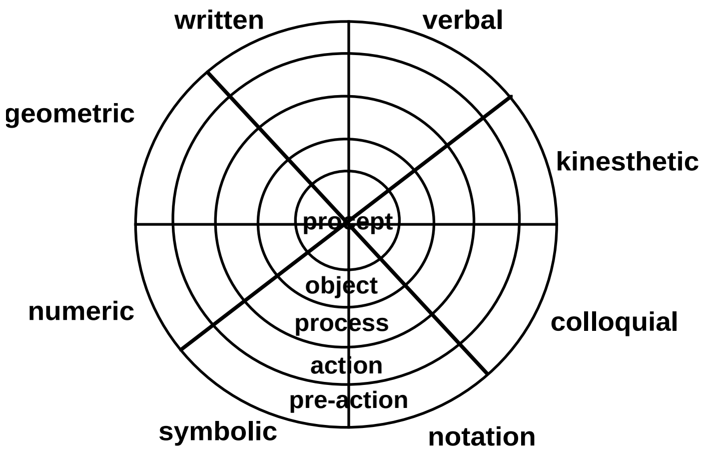
(spoken), written, kinesthetic (enactive), colloquial (informal or idiomatic), notational conventions, numeric, symbolic, and geometric (visual) aspects. These are not intended to be independent or exhaustive, but provide a suitably broad framework to begin an analysis of the function concept in this paper.

We use the term *layer* to refer to various levels of the depth dimension in the development via cognitive process to mental object. This has been discussed extensively in the literature, including Dubinsky's Action-Process-Object construction in which mental actions (on objects) become repeatable processes which are encapsulated as objects (see Breidenbach, *et al*, 1992; Dubinsky & Harel, 1992). In a similar way, Sfard (1992) begins with a *process* acting on familiar objects which is first *interiorized*, then *condensed* "in terms of input/output without necessarily considering its component steps" (corresponding to Dubinsky's notion of formation of a process) and then *reified* as an "object-like entity".

Gray & Tall (1994, p. 121) describe a *procept* essentially as the amalgam of three things, a *pro*cess (such as addition of three and four), a con*cept* produced by that process (the sum) and a symbol that evokes either concept or process (e.g. 3+4). Following Davis (1983), they distinguish between a process which may be carried out by a variety of different algorithms and a procedure which is a "specific algorithm for implementing a process" (p. 117). A procedure is therefore cognitively more primitive than a process.

Webster's Dictionary describes a "layer" as "a single thickness, coat, or stratum." In this paper, *action*, *process*, and *object* are considered as three layers of increasing depth. One new layer is added before action, called *pre-action*, for students at the ground floor, so to speak, with respect to a concept. After the object layer we place a *proceptual* layer, to indicate the flexibility to move easily between process and object layers as required.

The two aspects can be combined diagrammatically with the layers as concentric circles representing increasing depth, sliced into sectors representing various facets.



Facets and layers of a concept

Facets and Layers of the Function Concept

Three facets of the function concept—*numeric* using tables, *geometric* using graphs, and *symbolic* using equations—have been widely discussed in the literature (eg Cuoco, 1994; Schwingendorf *et al*, 1992; Sierpinska, 1988; Thompson, 1994). Written and verbal descriptions of function represent two other facets and the function notation is the notational facet. We will explore the colloquial facet using the notion of function machine. Finally, the kinesthetic aspect might be represented by asking students to act out their understanding about function.

Note that several of these facets have sub-facets. For example there are several ways to represent a function symbolically using symbolism such as $f(x) = x+1$ and $f: x \to x+1$. Visually, a two-dimensional coordinate graph provides a visualization for functions of one variable from the real numbers to the real numbers. Other visualizations, such as drawing correspondences from domain to range, can also be used for the geometric facet.

An area that has received much attention is students' ability to move comfortably between facets. This implies that they can choose the most appropriate facet to use for a given problem. Cuoco (1994, p. 125) suggests that the connections between "representations" are properties of a "higher-order function." While these are not the subject of this paper, it is important to appreciate the subtleties involved in linking the facets of a concept.

The layers of the function concept, especially the action-process-object layers, have received extensive treatment in the literature. According to Cuoco, "Students who view functions as actions think of a function as a sequence of isolated calculations or manipulations" (Cuoco, 1994, p. 122). Specific procedures are regarded as being at the action level. Students at this level are dependent on the procedure performed to obtain output from input. Cuoco suggests that "students who view functions as processes think of functions as dynamic (single-valued) transformations that can be composed with other transformations" (ibid, p. 122) and goes on to suggest that when students can view functions as "atomic structures that can be inputs and outputs to higher-order processes," such students have an object conception of function (ibid, p. 123). Students reach the most depth (the procept layer) when they can demonstrate flexibility in viewing a function as either a process or an object, as required by the problem situation.

Student Conceptions of Function

A number of community college students were interviewed to begin to analyze their concept image of function in terms of facets and layers. In this paper we report interviews with one student who had just completed a "reform" developmental algebra course. The text (DeMarois, McGowen & Whitkanack, 1996) focuses on student investigation of problems. This is based on a pedagogical approach that uses a constructivist theoretical perspective of how mathematics is learned (Davis *et al*, 1990). The authors subscribe to the theoretical perspective that the main concern in mathematics should be "with the students' construction of schemas for understanding concepts. Instruction should be dedicated to inducing students to make these constructions and helping them along in the process." (Dubinsky,1991, p. 119). Each unit begins with an investigation of a problem situation. Following the gathering of data, students work collaboratively on tasks based on the investigation activities. A discussion in the text summarizes essential mathematical ideas. The instructor orchestrates inter-group and class discussions of the investigations. Explorations are assigned to reinforce the knowledge students are expected to have constructed during successive steps of the cycle.

The materials focus on development of mathematical ideas using a core concept of function. Each function is based in a problem situation. Functions are often investigated numerically, graphically, and with function machines before the symbolic form is created. As Sierpinska writes: "The most fundamental conception of function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another." (Sierpinska, p. 572) As each new function arises, investigations support multiple facets and wise choices in terms of what facet might be best for analyzing a specific problem. Tables, equations, graphs, function machines, verbal and written descriptions are all used to analyze relationships. Graphing calculators provide excellent support for the tables, equations, and graphs.

We summarise the interviews with DK who had completed two semesters of developmental algebra, receiving an "A" in both semesters.

Layers of the Verbal Definition Facet

The first question explores the student's verbal definition of function.

Int Explain in a sentence or so what you think a function is. If you can give a definition for a function then do so.

DK A function comprises, I think, the whole general idea of what we have been doing. And some functions you go into relationships, from there you go into equations, models, quadratic or linear. I mean, everything comes off of the function. I think that's a basic idea in mathematics, the function.

Int Let's narrow it down a bit more.

DK If I saw an equation, I would call that a function.

Int Anything else?

DK A relationship. To me a function is the whole general idea.

DK tends to be very non-specific about function. It seems that she has spent so much time studying problems that relate to functions that she has overgeneralized. When asked to be more specific, she says she would call an equation a function, but she places no restrictions on equation. Finally, she uses a key description that the materials emphasize: relationship. However, she places no conditions on the relationship. Ultimately her verbal definition of function shows no depth suggesting that she is at the “pre-action” layer with respect to the verbal definition facet.

Layers of the Notation Facet

Another crucial part of working with functions is understanding function notation. DK expressed some confusion about when to read a string such as $y(x)$ as multiplication and when to read it as function notation:

Int What do you think when you see the notation $y(x)$?

DK That’s a function notation. x means–x is the input. y is the output. When you substitute a number in with the x, then you would, on the other side of the equals sign, apply that number to all the xs in that equation.

While DK has interpreted the notation correctly, she seems very procedural in her use of it. She finds it difficult to accept $y(x)$ alone without setting it equal to an algebraic expression that describes the process. Thompson (1994, p. 24) suggests that “the predominant image evoked in a student by the word “function” is of two written expressions separated by an equal sign.” DK seems to have this image. We might interpret that she is at an action layer with respect to the notation facet.

Using Function Composition to Probe the Depth of Numeric, Geometric and Symbolic Facets

None of the students had been exposed to composition of functions before. The interviewer provided some brief comments on the meaning of function composition prior to the following questions. In addition to gaining information about students’ ability to answer questions about the numeric, geometric, and symbolic facets, these questions permitted more analysis of the students’ understanding of the notation facet.

The students were given two input-output tables, one for function f and the other for function g.

x	$f(x)$
1	3
2	–1
3	1
4	0
5	–2

x	$g(x)$
–2	3
–1	1
0	5
1	2
2	4

Int What is the value of $f(g(2))$? Why?

DK exhibits much confusion.

Int Let’s break it down a bit. Can you tell me what g of 2 is?

DK g of 2 is 1. Input of 1. g of 2 is the x of 1.

Int Is 2 a value for input or output in this case?

DK 2 of x is the input. I better put a 2 over here (she points to the input column).

Int So g of 2 is equal to what?

DK 4.

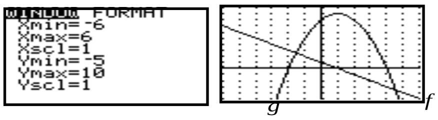
Int So why don’t you substitute 4 for $g(2)$ is the expression $f(g(2))$?

DK f of 4 is zero. It equals zero. f of g of 2.

This was the correct answer. DK tried to analyze what she had done, but became hopelessly confused between input and output.

The interviewer continued to question DK on composition, this time focussing on the geometric facet. She initially demonstrated that she could interpret expressions of the form *f*(a) and *g*(b) where *a* and *b* are given from the graph.

Int Consider the following graphs for functions *f* and *g*. The graph of *f* is the line. The graph of *g* is the parabola.



Approximate the value of *g*(*f*(2)). Describe what you did.

DK was unable to even begin to answer this question, revealing a weakness to deal with the geometric facet.

Composition was then investigated in the symbolic facet:

Int Consider functions *f* and *g* defined as $f(x) = 3x - 5$ and $g(x) = x^2 + 1$.What is *g*(*f*(3))? Describe what you did.

DK Well, *f* of 3 equals *x* squared plus...

Int What did you do?

DK Maybe I am confused.

Int Suppose I covered this (the *g* of) up.

DK Then I would use *f* of 3.

Int And what would you get?

DK 4.

Int So this could be interpreted as *g* of what?

DK *g* of 4.

Int And what would that mean?

DK Oh, okay. *g* of 4.

DK writes 17, revealing that, though she has difficulty dealing with the numeric aspect of composition of functions using tables, she is perfectly capable of interpreting the symbolism numerically. This was confirmed by another example:

Int Suppose I want *g* of *f* of 1.

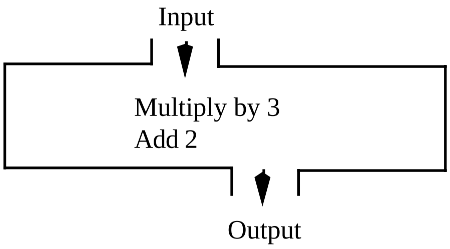
DK Well, I'd do this first. I'd go *f* of 1 equals 3 *x* minus 5. f of 1 equals 3 minus 5. *f* of 1 equals –2. You want *g* of *f* of 1? And then you take that and *g* of negative 2 equals *x* squared plus 1. Positive 4 plus 1 equals 5. Is that right?

DK was more able to deal with function composition symbolically than in the other two representations. This may partly be due to the fact that she had dealt with the concept in two previous problems. She still, however, initially needed help interpreting the notation. She appears very adept (procedural?) at finding an output given an input symbolically. She appears very comfortable with pushing the symbols and performs satisfactorily with the action level of the symbolic facet yet struggles with the action facet of the numeric and graphic.

Layers of the Colloquial (Function Machine) Facet

The materials use function machines extensively to analyze functions. This is the first facet of function, after a written, informal definition, that the students interact with.

Int Consider the following function machine.



What is the output if the input is $y(x) = x^2 - 5x$? What did you do?

DK What is the output if the input is–okay. If the input is *x* squared minus 5 *x*. You'd multiply it by 3 and then you'd add 2. Do you want me to work it out?

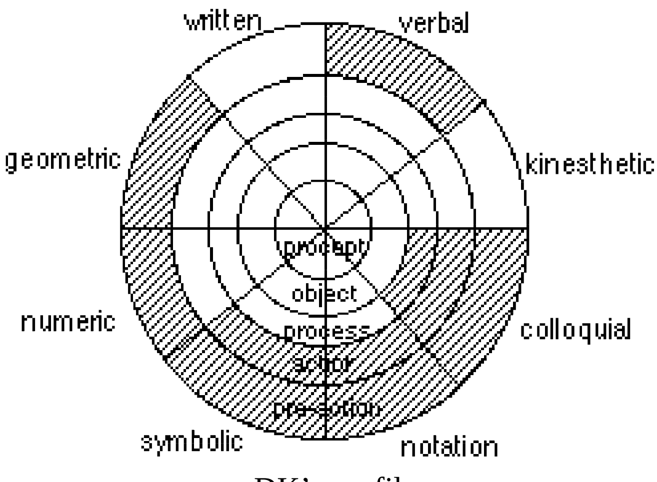
Int Sure

DK 3 *x* squared minus 15 *x* plus 2.

The symbolic input caused little trouble. On the colloquial facet, she continues to be able to handle a symbolic as well as a numerical input, suggesting she is moving towards the process layer.

Analysis

Based on the responses to these few questions, we begin to develop a profile of the student's understanding of function. The shading indicates the number of layers the student has demonstrated in their understanding of a specific facet. The student's knowledge of a specific facet has not been assessed if the outermost layer (pre-action) is unshaded, in this case the written and kinesthetic facets.



DK's profile

Reflections

These interviews underline the complexity of the function concept, for instance that the student concerned can operate more successfully in the symbolic facet than in the numeric facet, even though symbolism seems to be more sophisticated than numeric representations. Further work is necessary to complete the student's profile and critical issues have arisen in the classification of facets, including the links between them and deeper analysis of sub-facets. Nevertheless, the profile provides useful insight into a highly complex issue, re-focusing our attention on the nature of the cognitive structure of the function concept.

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