Procedural and Conceptual Aspects of Standard Algorithms in Calculus

Maselan bin Ali
Department of Mathematics
Universiti Teknologi Malaysia
Locked Bag No 791, 80990 Johor Bahru
Malaysia

David Tall
Mathematics Education Research Centre
University of Warwick
Coventry CV4 7AL
United Kingdom

This research studies the different methods students use to carry out algorithms for differentiation and integration. Following Krutetskii, it might be conjectured that the higher attainers produce curtailed solutions giving the answer in a smaller number of steps. However, in the population studied (Malaysian students in the 50th to 90th percentile), some higher attaining students wrote out solutions in great detail, so little correlation was found between the attainment of students and the number of steps taken. On the other hand, the higher attainers had less fragile knowledge structures and were significantly more likely to succeed. But with problems that can be simplified by a non-algorithmic manipulation before using a standard algorithm, the higher attainers were more likely to use some form of conceptual preparation.

Introduction

In his renowned study of the different problem-solving styles of children, Krutetskii (1976) showed that, of four groups (gifted, capable, average, incapable), the gifted were likely to *curtail* solutions to solve them in a small number of powerful steps, whilst the capable and average were more likely to learn to curtail solutions only after considerable practice, and the incapable were likely to fail. This may be related to the strength of the conceptual links formed by the more successful students in their cognitive structure (Hiebert and Lefevre, 1986) which helps the individual utilise knowledge in an efficient and powerful way.

The brain is a huge simultaneous processing system that must filter out most of its activity to be able to focus attention on a small amount of data for decision making (Crick, 1994, p. 61). Therefore the ability to code information efficiently—to make appropriate links between concepts and to develop methods that economise on processes—is likely to increase the brain's capacity to perform mathematics.

Davis (1983) suggested that at least two kinds of procedures exist: a visually moderated sequence (VMS) and an integrated sequence. In a VMS, the whole sequence is not yet apparent and the student carries out a manipulation to produce new written information which is then operated on in turn until the problem is solved. In an integrated sequence, the student is aware of the whole algorithm built up from smaller component sequences.

Hiebert and Lefevre *et al* (1986) contrasted *procedural* and *conceptual* methods of processing mathematical information. Following Dubinsky (1991) and Sfard (1991), who focused on the way in which process becomes encapsulated (or reified) as mental object, Gray & Tall (1991, 1994) introduced the notion of *procept*: the amalgam of

Published in *Proceedings of PME 20*, Valencia, (1996), vol. 2, pp. 19–26

*pro*cess and con*cept* represented by the same symbol. They hypothesised that less flexible thinkers see the symbol more as a process to be carried out using fairly inflexible procedures. The more flexible thinkers are hypothesised to view a symbol both as a process to *do* mathematics and a concept to *think* about. Evidence with young children doing arithmetic showed that whilst the less successful clung to (often idiosyncratic and inefficient) counting procedures, the more successful not only showed flexible ways of thinking conceptually, but also often chose more efficient procedures to carry out required processes.

In this study we consider a population of students solving problems involving standard algorithms in differentiation and integration. Three groups, each of twelve students, were selected attaining grades A, B, C respectively in recent examinations. Following Krutetskii, one might hypothesise that the more successful make sophisticated links to reduce the manipulation involved and curtail their algorithms to make them more effective, whilst the less successful are likely to use more rigid procedural methods that have longer and more fragile connections which may break down. However, the population studied does not fully reflect these hypotheses. It consists of Malaysian students following degrees involving mathematics taken from the 50th to the 90th percentile of the total population (because the highest attaining 10% travel abroad to study). It was found that in this population there was little correlation between attainment and curtailment of solutions (because the higher attainers included those who wrote out painstakingly detailed solutions). The major difference between higher and lower attainers in standard questions was that the low attainers had more fragile connections in their knowledge structure and were more likely to break down.

However, the higher attaining grade A students were more likely to show the capacity to use subtle initial simplifications to simplify the overall manipulation required. Specially designed problems, such as finding the derivative of $\frac{1+x^2}{x^2}$ benefit from an initial *conceptual preparation* to make the differentiation algorithm simpler to apply. Those who fail to carry out a conceptual preparation and tackle the problem using the standard algorithm may not only be applying a more complex algorithm, but have to follow it up with a more complex *post-algorithmic simplification*.

It was found that in certain questions, higher attainers were more likely to use conceptual preparation than lower attainers. On other occasions where the preparation required was more subtle or the gain was not so obvious, their confidence in symbolic manipulation led some high attainers to use a standard method even when they were aware of a possible alternative. Just as with the more successful children in arithmetic, who would confidently use efficient procedures when they did not immediately recall the relevant number facts, the more successful calculus students developed a powerful combination of conceptual and procedural methods whilst the less successful were often faced with a more difficult manipulation and therefore were more likely to fail.

Curtailment of solutions

A crude method of determining the degree of curtailment of a solution process is to count the number of steps carried out by the students. Some students may begin with the given formula, others may write a simplification as their first line. The latter case needs to include the implicit simplification in the first line in the line count. In addition, the final form of the solution is often written in a conventional manner, and when a student writes a solution which is not yet in this form, a note should be made that to attain the canonical form to be comparable with other students may require one (or more) further steps.

The following tables show typical solutions of the integration problem

$$\int \sqrt{3x^3}\, dx.$$

for the number of steps given in each column. (Each column may represent slight variants, but the most common solution is written out.) Some solutions do not end in the conventional form $\frac{2\sqrt{3}}{5}x^{5/2}+c$, so these could be considered as requiring one more step to attain standard form for the sake of comparability.

Grade A students all responded correctly and their solutions vary in length from two to six steps (the latter possibly being equivalent to seven steps if the last line were further simplified to its conventional form). (Table 1.)

Typical solutions of grade A students				
All responses correct (12)				
1 student	2 students	5 students	2 students	2 students
$\int \sqrt{3x^3}\, dx$ $= \sqrt{3}\frac{x^{5/2}}{5}(2)+c$ $= \frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int \sqrt{3}x^{3/2}dx$ $= \sqrt{3}\frac{x^{5/2}}{5}(2)+c$ $= \frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int (3x^3)^{\frac{1}{2}}\, dx$ $= \sqrt{3}\int x^{\frac{3}{2}}\, dx$ $= \sqrt{3}\left[x^{\frac{5}{2}}.\frac{2}{5}\right]+c$ $= \frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int \sqrt{3x^3}dx$ $= \int \sqrt{3}(x^3)dx$ $= \sqrt{3}\int x^{3/2}dx$ $= \sqrt{3}\left[\frac{x^{3/2+1}}{3/2+1}\right]$ $= \sqrt{3}\left(\frac{2}{5}\right)x^{5/2}$ $= \frac{2\sqrt{3}}{5}x^{5/2}+c$	$\int \sqrt{3x^3}dx$ $= \int (3x^3)^{\frac{1}{2}}dx$ $= \int 3^{\frac{1}{2}}(x^3)^{\frac{1}{2}}dx$ $= 3^{\frac{1}{2}}\int x^{\frac{3}{2}}dx$ $= \sqrt{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+c$ $= \sqrt{3}\times\frac{2}{5}x^{\frac{5}{2}}+c$ $= \sqrt{3}\frac{2}{5}x^{\frac{5}{2}}+c$
2 steps	3 steps (including unwritten first line)	4 steps (one solution non -conventional)	5 steps	6 steps Both in non - conventional form

Table 1: Grade A student responses to an integration problem

Grade B students produced many errors, with five correct and seven incorrect solutions. Amongst the correct responses, three used four steps and two used six steps. (Table 2.)

Typical solutions of grade B students				
Correct responses (5)		Errors (7)		
3 students	2 students	2 students	3 students	2 students
$\int (3x^3)^{\frac{1}{2}} dx$ $= \sqrt{3}\int x^{\frac{3}{2}} dx$ $= \sqrt{3}\left[x^{\frac{5}{2}}.\frac{2}{5}\right] + c$ $= \frac{2\sqrt{3}}{5} x^{\frac{5}{2}} + c$	$\int \sqrt{3x^3}\, dx$ $= \int (3x^3)^{\frac{1}{2}} dx$ $= \int 3^{\frac{1}{2}} x^{\frac{3}{2}} dx$ $= 3^{\frac{1}{2}} \int x^{\frac{3}{2}} dx$ $= 3^{\frac{1}{2}}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + c$ $= 3^{\frac{1}{2}}.\frac{2}{5}.x^{5/2} + c$ $= \frac{2}{5}\sqrt{3}x^{5/2} + c$	$\int \sqrt{3x^3}\, dx$ $= \int (3x^3)^{1/2} dx$ $= \frac{(3x^3)^{3/2}}{3/2}$ $= \frac{2(3x^3)^{3/2}}{3}$	$\int \sqrt{3x^3}\, dx$ $= \int (3x^3)^{\frac{1}{2}} dx$ Let $u = 3x^3$ $\frac{du}{dx} = 9x^2$ $dx = \frac{du}{9x^2}$ $\therefore \int (u)^{\frac{1}{2}} dx = \int u^{1/2} \frac{du}{9x^2}$ $= \frac{1}{9x^2}\int u^{1/2} du$ $= \frac{1}{9x^2}\left[\frac{2}{3}u^{3/2}\right] + c$ $= \frac{2u^{3/2}}{27x^2} + c = \frac{2(3x^3)}{27x^2}$ $= \frac{6x^3}{27x^2} = \frac{2}{9}x$	$\int \sqrt{3x^3}\, dx$ $= \int 3x^{3/2} dx$ $= 3\int x^{3/2} dx$ $= 3x^{5/2}.2/5$ $= 6/5\, x^{5/2} + c$
4 steps	6 steps	Overgeneralisation of integration	Mixture of substitution and direct integration	Algebraic Misconception

Table 2: Grade B student responses to an integration problem

Grade C students have only four correct solutions but one has only 2 steps, one has 3 steps and two have 4 steps. (Table 3.)

From these solutions of students in grades A, B, C we note that the higher attainers in grade A are all successful but vary considerable in the number of steps taken. Grade B students are less successful (5 out of 12) and the correct solutions vary from 4 to 6 steps. The grade C students are even less successful (4 out of 12) and the four successful students have solutions varying in length from 2 to 4 steps. It cannot be asserted that there is any clear pattern between curtailment and attainment. However, there is a clear diminution in lower attaining students successfully completing the problem. The difference between the performance of Grade A and Grade B is statistically significant using the χ^2-test with Yates correction ($p<0.01$), and between Grade A and Grade C even more so ($p<0.0025$). The zero entry in the Grade A failures greatly biases these results, nevertheless the differences are clearly striking.

Typical solutions of grade C students					
Correct responses (4)			Errors (8)		
1 student	1 student	2 students	3 students	2 students	3 students
$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\sqrt{3}\frac{2}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx$ $=\int\sqrt{3}x^{3/2}dx$ $=\frac{\sqrt{3}x^{5/2}}{5/2}+c$ $=\frac{2}{5}\sqrt{3}x^{5/2}+c$	$\int(3x^3)^{\frac{1}{2}}dx$ $=\sqrt{3}\int x^{\frac{3}{2}}dx$ $=\sqrt{3}\left[x^{\frac{5}{2}}\cdot\frac{2}{5}\right]+c$ $=\frac{2\sqrt{3}}{5}x^{\frac{5}{2}}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x)^{1/2}dx$ $=2/3(3x^3)^{3/2}+c$	$\int\sqrt{3x^3}dx$ $=(3x^3)^{\frac{1}{2}}$ $u=3x^3$ $=\int(u)^{\frac{1}{2}}$ $=\frac{u^{3/2}}{3/2}+c$ $=\frac{2}{3}u^{3/2}+c$ $=\frac{2}{3}(3x^3)^{3/2}+c$ $=2x^{9/2}+c$	$\int\sqrt{3x^3}dx$ $=\int(3x^3)^{1/2}dx$ $=9\int(x^3)^{1/2}dx$ $=9\left[\frac{(x^3)^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$ $=9\left[\frac{(x^3)^{3/2}}{\frac{9}{2}}\right]+c$ $=2[(x^3)^{3/2}]+c$
2 steps Non-conventional solution	3 steps	4 steps	Over-generalisation of direct integration	Mixture of substitution and direct integration	Algebraic Misconception

Table 3: Grade C student responses to an integration problem

Conceptual Preparation

When the manipulation involved in using an algorithm becomes more complex, it may be possible to devise alternate methods to simplify the solution For example, the problem to determine the derivative of $\frac{1+x^2}{x^2}$ using the standard algorithm for the derivative of a quotient involves the student needing to use the formula in a cumbersome way and then simplifying the result:

$$y=\frac{1+x^2}{x^2},$$

$$\frac{dy}{dx}=\frac{(2x)(x^2)-(2x)(1+x^2)}{(x^2)^2}=\frac{2x^3-2x-2x^3}{x^4}=-\frac{2x}{x^4}=-\frac{2}{x^3}$$

However, if the expression is first simplified as $x^{-2}+1$ then its derivative is straight away seen to be $-2x^{-3}$, affording a considerable reduction in processing. Students may shorten their solutions in various ways, for instance, the initial simplification might be conceived as a succession of formal manipulations:

$$\frac{1+x^2}{x^2} = \frac{1}{x^2} + \frac{x^2}{x^2} = x^{-2} + 1.$$

However, often students compress this further to a single written step:

$$\frac{1+x^2}{x^2} = x^{-2} + 1.$$

Some do this by reading the symbol $\frac{1+x^2}{x^2}$ as two fractions in this way:

$$\frac{1}{x^2} \quad + \quad \frac{x^2}{x^2}.$$

translating $\frac{1}{x^2}$ immediately as x^{-2}, then writing $\frac{x^2}{x^2}$ as +1, to perform the simplification in a single composite step.

Out of thirty six students, twenty of them simplified the expression $\frac{1+x^2}{x^2}$ before carrying out the differentiation, for example by writing:

$$y = x^{-2} + 1,$$

$$\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}.$$

Fifteen students failed to conceptually prepare and so led to a more complex version of the algorithm and the need to perform more simplification afterwards. All but one student were successful in this task, the remaining student making a single slip by writing a '+' sign in the numerator of the quotient algorithm instead of a '–' sign:

$$\frac{dy}{dx} = \frac{2x(x^2) + 2x(1+x^2)}{(x^2)^2} = \frac{2x^3 + 2x + 2x^3}{x^4} = \frac{4x^3 + 2x}{x^4} = \frac{4}{x} + \frac{2}{x^3}.$$

The students in the various grades performed as follows:

Students' grade	Conceptually prepared	Post-algorithmic simplification	No further simplification
A	10	2	0
B	6	6	0
C	4	7	1
Total	20	15	1

Table 4: Student responses to a differentiation problem

Here the number carrying out conceptual preparation reduces from 10 out of 12 in grade A to only 4 out of 12 in grade C. Using a χ^2 test with Yates correction, this is significant at the 5% level (with p=0.038). The numbers involved are small and the differences between groups A and B and between B and C are not statistically significant.

The fragility of conceptual preparation

The conceptual preparation for a solution depends very much on the nature of the problem. There is no obvious algorithm to cover all possible cases. For instance the derivative of $y = \frac{1+x^2}{x^2}$ is simplified by separating the expression into two parts, but the derivative of

$$y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2}$$

is found more easily by adding the two expressions together and factorising the numerator.

$$y = \frac{1}{1+x^2} - \frac{x^4}{1+x^2} = \frac{1-x^4}{1+x^2} = \frac{(1-x^2)(1+x^2)}{(1+x^2)} = 1-x^2,$$

$$\frac{dy}{dx} = -2x.$$

In this example, only six of the twelve Grade A students added the terms together and factorised the numerator. Conceptual preparation therefore varies considerably from case to case and is not given by a single algorithm, so students may use some form of conceptual preparation in some problems, but not in others.

Sometimes it may not even be clear whether some form of conceptual preparation may be advantageous. For instance, the problem

Find $\frac{dy}{dx}$, when $y = \left(x + \frac{1}{x}\right)^n$

is best solved by using the chain rule with $u = x + \frac{1}{x}$ to obtain the derivative in the form $nu^{n-1}\frac{du}{dx}$. However the problem

Find $\frac{dy}{dx}$, when $y = \left(x + \frac{1}{x}\right)^2$

happens to be easier by expanding the bracket to differentiate $x^2 + 2 + x^{-2}$. In this case there is a tension between using the generalisable chain rule method and the particular method expanding the bracket, which happens to be marginally shorter. This is reflected in the performance of the grade A students where six used the chain rule and six expanded the bracket. In interview, four of the six using the chain rule could see a possible advantage in the alternative method but preferred to use the more general strategy and trust their facility in manipulation.

Conclusion

In the group of students studied (between the 50th and 90th percentile in the whole population) there is no obvious correlation between the number of steps taken in carrying out a routine symbolic algorithm and the level of attainment of the student. Thus the curtailment spoken of by Krutetskii in higher attaining children solving problems does not occur here. The more successful Grade A students include those who write out algorithms in greater detail as well as those who curtail the solution. The most obvious difference between the Grade A and Grade C students is the ability of the former to complete the procedure correctly.

However, when problems are designed which can be simplified by an initial conceptual preparation, the more successful students are more likely to conceptually prepare than the less successful students. With problems where the preparation involves using a more specific method that is shorter or a generalisable method which happens to be longer, the more successful students are likely to be aware of the alternatives, some using the shorter method, some preferring the more general method and having confidence in their ability to carry out the manipulation. This is in accord with the notion of proceptual thinking in arithmetic (Gray & Tall, 1994) where the more successful select appropriate conceptual methods or have the power to carry out the procedures correctly. It is also in accord with the value of having both conceptual and procedural knowledge (Hiebert & Lefevre, 1986).

References

Crick, F. (1994). *The Astonishing Hypothesis*. London: Simon & Schuster.

Davis, R. B. (1983). Complex mathematical cognition. In Ginsburg (Ed.), *The development of mathematical thinking*, (pp. 253-290). New York: Academic Press.

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D.Tall (Ed.) *Advanced Mathematical Thinking*, (pp. 95–123) Kluwer:Dordrecht.

Gray, E. M. & Tall, D. O. (1991). Duality , ambiguity and flexibility in successful mathematical thinking. *Proceedings of PME XV*. Assisi, Italy, Vol. II, pp. 72-79.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal of Research in Mathematics Education*, **25** 2, 115–141.

Hiebert, J & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

Krutetskii, V. A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*. Chicago: The University of Chicago Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, **22** 1, 1–36.