Can all children climb the same curriculum ladder?

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This article presents evidence that the way the human brain thinks about mathematics requires an ability to use symbols to represent both process and concept. The more successful use symbols in a conceptual way to be able to manipulate them mentally. The less successful attempt to learn how to do the processes but fail to develop techniques for **thinking** about mathematics through conceiving of the symbols as flexible mathematical objects. Hence the more successful have a system which helps them increase the power of their mathematical thought, but the less successful increasingly learn isolated techniques which do not fit together in a meaningful way and may cause the learner to reach a plateau beyond which learning in a particular context becomes difficult.

Introduction

At the age of 16 the number of children passing their General Certificate of Secondary Education at grades A, B, C is currently increasing year by year (although there is a worrying trend that the number with the lowest grades is remaining stubbornly stable). At age 18 there is an improving spectrum of passes at A-level, although mathematics is becoming a less popular subject. Despite an apparent trend for top students to get better marks in school examinations, university lecturers claim that the students arriving at university lack basic skills. In particular:

(i) They lack fluency in arithmetic and algebraic skills,

(ii) They are less able to solve problems involving several steps,

(iii) They do not perceive the need for absolute precision and proof in mathematics.

The contrast between the apparent success as seen from a school perspective yet failure from a university perspective has led to unseemly accusations flying in all directions. It appears that the two viewpoints are focusing on different things

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and arguing at cross-purposes. To be able to unravel the conundrum, we need get an insight into what is happening when individuals learn mathematics and begin to “think mathematically”. By doing so it is hoped that some light can be shed on the situation.

Initially some fundamental questions need to be asked to see how it is that individuals learn to use mathematics in a powerful and productive way.

Cognitive considerations

The human brain is a huge simultaneously processing system. To be able to make conscious decisions using such a mechanism requires the individual to filter out inessential detail and to focus attention on the important essentials.

> The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) “object(s)” can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once).
>
> (Crick, 1994, p. 61)

The process leads to the sensation in which the conscious mind focuses attention on things of current interest, manipulating them in the mind, then passing on to related ideas which occur in the conscious train of thought:

> There seems to be a presence-chamber in my mind where full consciousness holds court, and where two or three ideas are at the same time in audience, and an ante-chamber full of more or less allied ideas, which is situated just beyond the full ken of consciousness. Out of this ante-chamber the ideas most nearly allied to those in the presence chamber appear to be summoned in a mechanically logical way, and to have their turn of audience.
>
> (Galton, *Inquiries into human faculty and its development*, 1883)

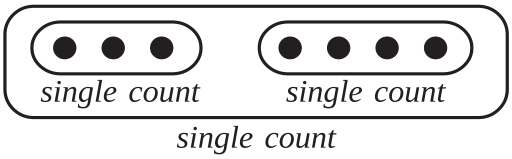
This limited focus of attention means that thinking is enhanced by two complementary processes:

- *compressing* data to fit in the focus of attention,
- *linking* data in the brain to be able to bring other ideas in the huge long-term memory into the focus of attention for processing.

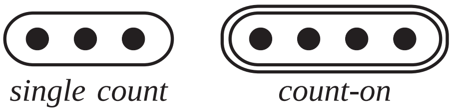
In mathematics, compression of data occurs in various pragmatic and elegant ways. For instance, we may draw diagrams to represent information succinctly, use words to stand for complex ideas, or mathematical symbols to represent problem statements that can be manipulated to produce solutions. The last of these proves to be particularly powerful and subtle.

The problems that occur later in school and university begin much earlier, starting with the way that the processes of simple arithmetic develop (or even before). In level 1 of the National Curriculum, children are expected to be able to handle addition of numbers up to ten by the process of "count-all", counting a set of objects, then another set, and putting the two together to count them all to obtain the sum. Subsequently various other techniques are developed showing greater efficiency:

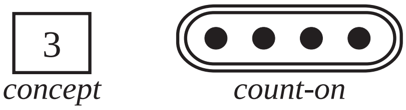
- **Count-all**: a succession of three simple counting processes, 3+4 is "1, 2, 3", "4, 5, 6, 7", then count "1, 2, 3, 4, 5, 6, 7", usually pointing at specific objects,



- **Count-both** is two counting processes, a simple count, e.g. "1, 2, 3", then a count-on "4, 5, 6, 7", usually using some technique (e.g. four fingers) to keep a tally of the count-on,



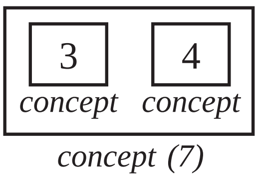
- **Count-on** is a concept "3" and a process to count-on 4 after 3 as "4, 5, 6, 7,"



- **Count-on from larger** is a variant of count-on which shortens the counting process,

  (This is more spectacular where the numbers differ greatly in size, for instance, computing 2+9 by counting on 2 after 9 rather than 9 after 2).

- **Known fact** regards the symbols as number concepts and recalls the result as another number concept, "3+4 is 7."



- **Derived fact** uses the number facts themselves as manipulable mental objects, operated on them to give new facts, e.g. "3+4 is one less than 4+4=8, so it is 7."

(Fuson, 1992; Gray & Tall, 1994)

As children develop they initially interpret symbolism in process terms so that, for them, the symbol 3+2=5 means “3 plus 2 *makes* 5”, and the equation 5=3+2 lacks meaning because 5 does not *make* 3+2. The initial introduction of what adults may see as “algebra” in solving equations such as

$$4+\square = 7,$$

$$\square + 3 = 7.$$

represents completely different problems for children. At a certain stage the first may be read “how many do I count-on after 4 to get to 7?” and the second is much harder if it is read as “Which number do I start at to count on 3 and reach 7?”.

Figure 1 shows the results of a test given to children in years 1, 2, and 3, involving such problems (Foster, 1994):

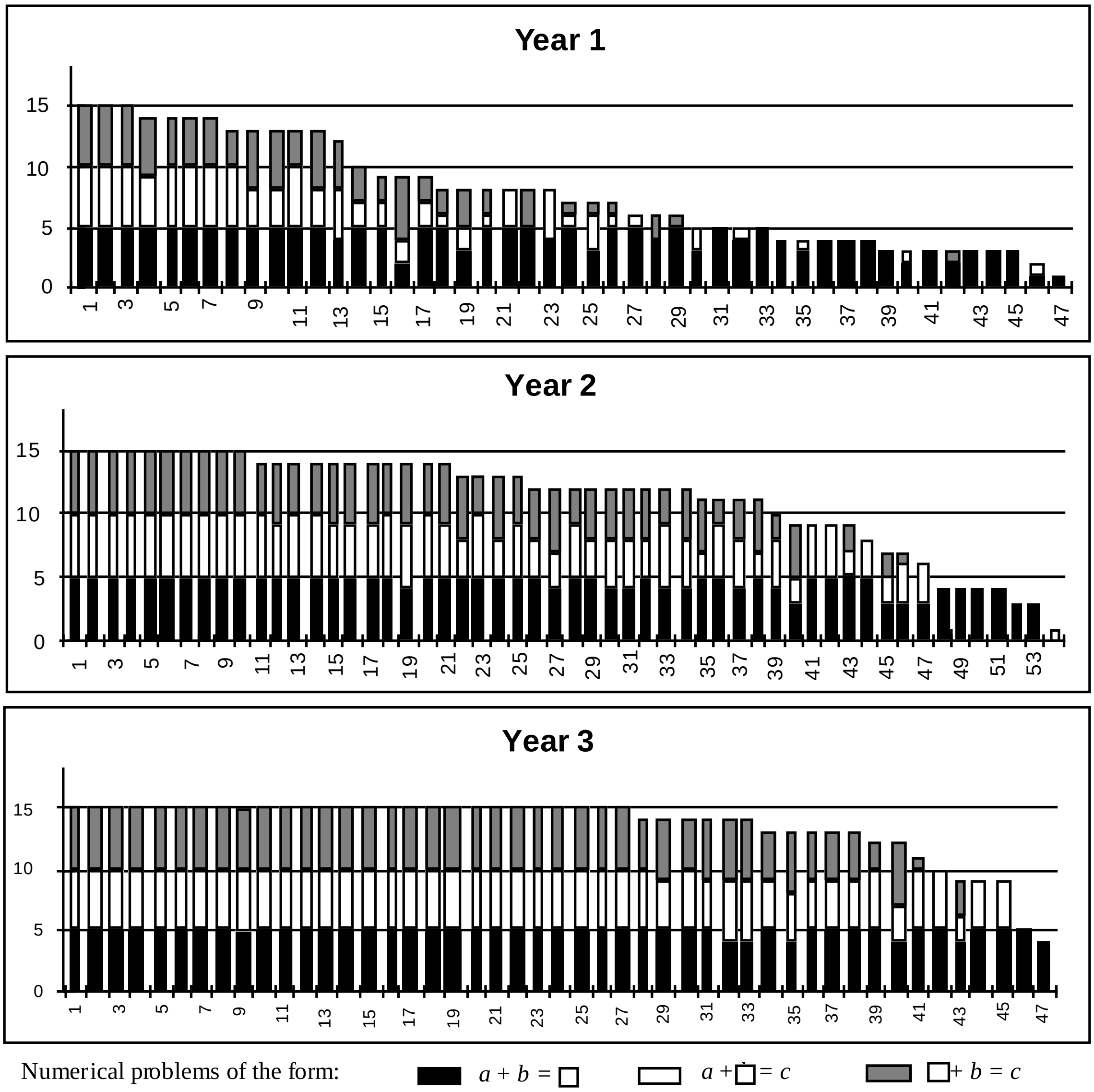


Figure 1: Children’s performance on arithmetic problems

Each bar represents an individual child. The three parts of each bar represent the number of correct responses to five written arithmetic problems (fifteen in all) in the form:

■ : $a+b=\square$ (1)

□ : $a+\square=c$, (2)

▒ : $\square+b=c$. (3)

This reveals a familiar statistic—as the children get older, the performances get better, which is the basis of the idea of successive levels of attainment which children reach. But this is a grossly oversimplified interpretation. If the classes are divided into thirds as “higher”, “middle” and “lower”, then qualitative differences are revealed in the spectrum of performances.

Year 1	(1)	(2)	(3)
Higher	99%	*81%*	*83%*
Middle	86%	29%	29%
Lower	59%	5%	1%

Year 2	(1)	(2)	(3)
Higher	100%	96%	96%
Middle	93%	*76%*	*80%*
Lower	74%	38%	20%

Year 3	(1)	(2)	(3)
Higher	100%	100%	100%
Middle	100%	98%	98%
Lower	93%	73%	53%

Percentage of correct responses in each category by year at different levels of attainment

Apart from tiny reversals shown in italics, the performances are all ordered (1)≥(2)≥(3). In year 2 the higher attainers reach almost 100% whilst the middle attainers do not do so until year 3. The higher and middle attainers all show (2) approximately equal to (3), but the first year lower attainers cannot do items (2) or (3) and those in years two and three continue to show a difference between each of the three categories in the order (1)>(2)>(3).

It is not simply a question that the lower attainers do the same thing but slower. The *marks* improve, but the *methods* at extremes of the spectrum may be very different. Whilst the five-year olds who succeed are already beginning to use their number facts in a flexible manner as both processes and concepts. Those who have their first success at a later age are more likely to rely on procedural counting. Those nearer the top end of the spectrum are developing a way of thinking which compresses knowledge into flexible symbolic form suitable for more abstract developments. Those nearer the bottom end are focusing more on procedures on physical objects which lock them in a mode from which abstract thought is much more difficult, even, perhaps, impossible.

The notion of procept

It often happens that a mathematical process (such as counting) is symbolised, then the symbol is treated as a mathematical concept (such as number) which is then manipulated as if it were a mental object. Here are just a few examples:

symbol	***process***	***concept***
1, 2, 3, ...	counting	number
3+2	addition	sum
–3	subtract 3, 3 steps left	negative 3
3/4	division	fraction
3+2*x*	evaluation	expression

These mental constructions go beyond elementary mathematics to more advanced mathematics. For example, the processes of differentiation and integration in the calculus have their corresponding concepts of derivative and integral. The student who operates procedurally learns how to carry out procedures and hence can solve routine problems in examinations, but may not be able to think of the symbols as mental concepts or be able to use them in more complex problem solving situations.

Given the wide distribution of this phenomenon of symbols evoking both process and concept, it is useful to provide terminology to enable it to be considered further.

The term *procept* is used to express this duality of ***pro***cess and con***cept*** . For example 3+4 can be seen as both the process of addition or the resulting concept of sum. The symbol evokes a process to arrive at an answer or the concept itself. (Gray & Tall, 1994). The person carrying out the process can *do* the arithmetic, but the person with the concept is able to *think* mathematically.

These definitions are of a cognitive concept to model what seems to happen in the cognitive structure of the individual. The mathematically oriented child does not only think that 2+3 “makes” 5, but that 2+3 *is* 5, as are 4+1, 7–2, $^{10}/_{2}$, and so on. The procept “5” grows to include all these different ways of making five, so that it becomes cognitively richer as the child grows. It is this cognitive richness that gives the individual power as a mathematician. It represents the way that individuals seem to use symbols to give great *flexibility* in thinking – using the *process* to *do* mathematics and get answers, or using the *concept* as a compressed mental object to *think about* mathematics – in the sense of Thurston:

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, ‘134 divided by 29’ meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a*/*b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation. (Thurston, 1990, p. 847)

Perhaps the reason why mathematicians have not formulated the definition of a procept (as they have done other profound simplicities such as “set” and “function”) is that mathematicians seem to *think* in such a flexible ambiguous way often without consciously realising it, but their desire for final precision is such that they write in a manner which attempts to use unambiguous definitions. This leads to the modern set-theoretic basis of mathematics in which concepts are defined as *objects*. It is a superb way to systematise mathematics but is cognitively in conflict with developmental growth in elementary mathematics where arithmetic and algebraic processes *become* mathematical objects through the form of compression called encapsulation. In this way the viewpoint of the mathematician is not synonymous with the needs of the child in developing mathematical knowledge.

The more mathematically oriented children seem to develop this flexibility early on, the less mathematical do not and this imposes a great strain on their focus of attention, causing them to fall back on rote-learning of procedures which will *do* mathematics at the time, but are far less likely to be suitable for holding in the small focus of attention to reflect upon and build more sophisticated ideas.

We suggest that greater mathematical success comes not from remaining linked to the perceptions of the world through our senses, but through using the symbolism that is especially designed for *doing* mathematics and for *thinking* about it. We have only so much conscious focus of attention to use at any one time and rely on the richness of internal connections to build up the flexible characteristics of the mathematical mind. Those who use internal conceptual connections to fit together mathematical processes and concepts in a flexible manner have a greater chance of success than those who burden their already strained memory with long procedures needing concrete support.

Fractions

If the difficulty of compressing counting procedures into number concepts is unresolved, the problem grows still larger in later developments. For instance, in the study of fractions, a child who does not have flexible knowledge of arithmetic of whole numbers will find it more difficult to co-ordinate the notion of equivalent fractions and a child who sees a fraction $\frac{2}{3}$ as a process “take three equal parts and select two of them” is going to have great difficulty in operating on fractions. For instance if $\frac{1}{2}+\frac{2}{3}$ is interpreted as “divide something into two equal parts and take one of them and add this to the result of dividing something into three equal parts and take two of them”, then it is hardly likely to make sense to someone already at the limit of their cognitive capacity.

Thus it is that fractions prove difficult for some and impossible for many and so have no place in a democratic curriculum on a ladder which all will climb. But the more successful students, who would find such things cognitively simpler, may be denied adequate exposure to the topic to give it meaning.

The beginnings of algebra

Algebra suffers in the beginning from the difficulty many children have with meaning. For instance, if an arithmetic expression such as 2+3 is considered as performing a sum to get an answer, then an algebraic expression such as $2+3x$ causes confusion. If it is read as a process, then it asks the child to add two and three then do something with x and, if x is unknown, the operation cannot be done. There is the further difficulty of parsing the expression (breaking it down to do it in the appropriate sequence). If it is read in the usual order from left to right, it is 2+3 (which is 5) and then x, which gives $5x$.

There is research (Macgregor & Stacey, 1993) which shows that when children are asked to write a simple sentence such as "y equals the sum of 4 and x" into algebraic notation, many get it wrong, often writing $y=4x$, perhaps because the word "and" between 4 and x is interpreted just by writing the symbols next to each other (because the child knows no mathematical symbol equivalent to "and").

If the symbols are interpreted correctly then an equation such as

$3x+1=7$

is inherently easier than

$3x+1=2x+1$.

But this is not simply because the second is "more complicated". In essence, the first can be read as a *process* "3 times x plus 1 is 7", and this can be unravelled by seeing that if "something plus 1" is 7, then that "something", in this case "3 times x", equals 6, and this in turn shows x must be 2. The second equation is more difficult because it seems to have a process on either side and therefore may not make sense to children who regard "=" as "makes". One way of giving meaning is to consider each side as an expression, whose values are equal. The solution method taught in school involves "doing the same thing to both sides", so that the two sides remain equal but are now different from what they were before—a most sophisticated and confusing idea for the less successful.

One "solution" to these difficulties suitable for a wider range of children has involved linking them to the real world by interpreting an equation as a physical balance to explain "adding the same thing to both sides". We suspect this is a pseudo-scientific approach. Those who are insightful enough to give algebra an appropriate meaning do not need it and those who need it develop an approach which sometimes seems to the casual observer to be mathematics but in reality is likely to be something subtly different.

Thus traditional algebra proves to be cognitively difficult for the majority of students. Many educators attempt to improve the situation by giving it "real meaning". But mathematicians think powerfully precisely because they use the links *within* mathematics and do not relate constantly to the real world. It may therefore be that curriculum designers who desire to make algebra widely understood are using methods that are broadly applicable but may be less suitable for the mathematically oriented who eventually need a more powerful abstract form of algebraic thinking.

Limiting processes

The idea of a limit also has notation that has connotations both as a limiting process and a limit concept. This has proved a significant obstacle to understanding the mathematical idea of a limit. For example infinite decimals are considered as "going on forever" and "never reaching" the "final value". Although students may regard 0.333... as a repeating decimal whose value is the fraction ⅓ because of its familiarity, a decimal such as 0.4576121212... with the repeating pair 12 may not be seen as a fraction. For many years students are becoming familiar with approximate decimals and so find the precision of mathematical analysis and the formal limit process are foreign to their experience.

Thus the limit is a procept which behaves quite differently from the procepts of arithmetic which have built-in operations to give an answer in a finite number of steps. The limiting process is potentially infinite and causes difficulties because many students seem to believe that it never reaches its conclusion.

Changes in the nature of proof in elementary mathematics

Euclidean geometry requires more than the proof structure of Euclid to make it meaningful:

> The deductive geometry of Euclid from which a few things have been omitted cannot produce an elementary geometry. In order to be elementary, one will have to start from a world as perceived and already partially globally known by the children. The objective should be to analyze these phenomena and to establish a logical relationship. Only through an approach modified in this way can a geometry evolve that may be called elementary according to psychological principles. (van Hiele Geldof, 1984, p. 16)

The development of geometrical knowledge is very different from that of arithmetic and algebraic knowledge, involving teasing out the properties of geometric shapes, describing and refining their meaning to give verbal definitions of imagined "perfect" platonic figures. The difficulty of the development of geometrical knowledge and proof has long been acknowledged.

The Mathematical Association was formed as the "Association for the Improvement of Geometry Teaching" in 1871, yet the teaching of geometry has never proved satisfactory for a wide range of pupils. In the sixties there was an attempt to replace synthetic Euclidean geometry by transformation geometry. In terms of the theory presented here this is a significant move because the transformations are both processes of transformation and objects of a transformation group, and so they are procepts. The attempt was therefore to make geometry computation-oriented rather than proof-oriented. This however proved to be intellectually too demanding for the wide mass of pupils and it failed. So geometry as both a deductive and a computational science has largely been lost to elementary school to be replaced by broader concepts of “space and shape”.

Again there is a loss for mathematically oriented students who might find delight in seeing how proof is constructed in a verbal way based on visual representations of geometric figures.

Reflections

The result of this analysis is that the mathematics of arithmetic, algebra and calculus uses symbols both as processes and concepts and the mathematically oriented student develops flexible ways of using them, both as compact symbols that can be considered as mental objects in the limited focus of attention and as ongoing mathematical processes to be able to obtain answers to problems. There is evidence to support the hypothesis that the less successful student tends to cling more to the security of known procedures to “get answers” but that these are less suitable for thinking about than flexible symbols which can also be considered as mathematical objects to be compared, related and operated upon.

The idea that we all go through the same developmental stages but perhaps at a different pace is comforting to curriculum developers who may therefore decide to design such a curriculum in successive levels for children to study. It is a straw for politicians to grasp, for example in the legally enforced National Curriculum in England, for it suggests that progress may be measured by the level attained. According to this theory, children’s progress can be monitored by assessing progress through specified stages and the changes will give a measure of the quality of learning and teaching.

But it is a naive and damaging assumption. It is not only the things that children can *do* that measures progress, but *how* they do them, and whether their methods are of a kind that can be built on in subsequent development. This discussion has shown that there is a broad spectrum of performance in which those who are successful develop a flexible way of handling symbols so that they may be flexibly manipulated to derive new facts from known ones. Those at the other end of the spectrum learn fewer facts, are often unable to derive new facts from old and in arithmetic fall into inflexible counting procedures related to physical representations that tend not to generalise to problems involving larger numbers. Short-term success might be bought for a time by their learning routine procedures and attempting to rote-learn facts but this may only store up problems for a later stage.

A curriculum built up on evaluating what children can do, in which it is possible to succeed at a given level by radically different thinking processes, can lay foundations for eventual failure for those who do not develop methods that will lead on to later developments and may limit those whose cognitive structure develops in a way which is suitable for more powerful thinking.

Democracy in education does not therefore mean giving every child the same sequences of learning, but at different paces. It means giving each child the education that best suits the child's individual needs appropriate for his or her growing cognitive structure. And this in turn may very well mean that many children need help with the physical meanings and relationships with real world referents, but those who are succeeding in a flexible manner may need a more reflective learning environment, less dependent on physical referents, that encourages growth of more powerful conceptual relationships.

In the debate between university mathematicians and schoolteachers there may be a situation in which each is focusing on different forms of mathematical need, the schoolteacher for the need of a wide spectrum of children and the mathematician on that part of the spectrum which may move on to study university mathematics.

The perceived difficulties formulated at the beginning of this paper are all consonant with the possibility that the more mathematically oriented children in school are not getting the kind of curriculum for which they are capable, in the name of producing a curriculum ladder for all. It is a curriculum ladder which demands too much of the less successful so that they reach various plateaux where their cognitive structure is no longer able to cope with the increasing complexity, yet fails to support the mathematically able who need a more powerful approach to build the long-term development needed in professional mathematics.

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