THE PSYCHOLOGY OF SYMBOLS & SYMBOL MANIPULATORS: WHAT ARE WE DOING RIGHT?

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Instead of saying "wow, look at the clever math I can do with my computer", this presentation considers how mathematicians and students use symbols to think about mathematics and formulates a theory to describe how software using symbol manipulators can help and hinder the learning process.

> … the property of yielding new truths as a result of merely mechanical rearrangement of symbols […] is to be found in some form wherever a system of symbolism has been developed "to facilitate reasoning" in a particular province of thought. The late Prof Jevons actually invented a "logical machine" in which the exploration of the field of truth could be carried out by pulling levers and turning handles. It would probably not be impossible, if only it were worth while, to construct an "algebra machine" which could in a similar way be made to yield from a given formula other formulae which follow from it.
>
> (T. Percy Nunn, *The Teaching of Algebra*, 1914, p. 15.)

Symbol manipulators are reputed to "release the student from the drudgery of routine manipulation so that they can focus attention on the concepts and solving problems". Mathematicians who *know* how to do mathematics and think about symbols are using symbolic manipulators for a wide range of imaginative activities, as is witnessed by the many articles in the proceedings of this conference each year. What is equally important are the studies that consider what the students are *doing*, or even more subtly, what they are *thinking*. Mathematicians have a "symbol sense" that enables them to handle symbols in a flexible and imaginative manner, but it is clear that students do not necessarily work in anything like the same way. Formally mathematicians seek precision and unique definitions, but *cognitively* they seem to use symbols *ambiguously* to represent either processes to *do* mathematics or concepts to *think* about. This has interesting consequences for student learners, particularly those whose mental imagery of symbolism is different from that of their teachers.

Symbols as Process and Concept

The notion of symbol acting as a pivot either as a *process* to *do* mathematics or as a *concept* to *think* about mathematics is well-represented in the literature. It has a long provenance in the psychology of mathematical thinking going back certainly as far as Piaget and has entered the college level primarily through the independent pioneering work of Dubinsky (1991) and Sfard (1991). Building on Piaget's insight that "actions and operations become thematized objects of thought or assimilation", Dubinsky considered:

> … the construction which is perhaps the most important (for mathematics) and difficult (for) students … is the *encapsulation* or conversion of a (dynamic) process into a (static) object. (Dubinsky, 1991, p. 101.)

Such an idea is not without its critics, however, for instance Dörfler considered the notion of a mental object absent from his own thinking:

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> … my subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariably comes to my mind are certain patterns of dots or other units, a pentagon, the symbol 5 or V, relations like 5+5=10, 5*5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever could find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless I deem myself able to talk about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object "5".
>
> (Dörfler, 1993, pp. 146–147.)

It is in reconciling this view with the notion of encapsulation that we begin to gain further insight into the cognitive psychology of symbols.

The notion of procept

Inspired by the work of Dubinsky (1991), Sfard (1991) and others, Gray & Tall (1991, 1993, 1994) reviewed old research and initiated new studies throughout mathematical development, from young children calculating with symbols in arithmetic, older pupils manipulating symbols in algebra (Crowley *et al*, 1994) to college students developing conceptualisations of higher order concepts such as limits (Li & Tall, 1993, Monaghan *et al*, 1994). This showed the importance of the symbol acting as a pivot between process and concept. Eddie Gray and I introduced the notion of *procept* as an amalgam of three things – process, symbol, concept – to allow us to discuss the phenomenon. Thus a *symbol*, such as

$$\int_1^x \frac{\sin t}{1+\log t}\,dt$$

evokes both the *process* of integration and the *concept* of integral. The cognitive combination of the three is a *procept*.

We hypothesised that mathematicians used symbolism in an *ambiguous* way, to represent either process or concept as appropriate, flexibly changing viewpoint, say from a function as a process to a function as a concept, at will. But less successful thinkers see mathematics as inflexible procedures, seeking the security of following a tried and tested route to obtain an answer to a limited set of problems.

We found differences between those students of all ages who viewed such symbolism as cueing a procedure to be carried out and those who had a more flexible view as process or concept. In arithmetic procedural children invariably counted and placed heavy burdens on themselves as the problems became more complex. Meanwhile, more successful flexible thinkers moved easily between process and concept, deriving relationships where they proved useful or counting efficiently where necessary.

In algebra, those who saw the symbols as procedures to be carried out could not grasp the meaning of the symbolism. An expression such as $3+2x$ did not make sense *as a process* unless x was known to be able to compute the value, and then, if the value was known, there seemed no reason to complicate matters by referring to x. An equation such as

$$5x+1=11$$

might make sense as a problem where five times a number plus one is eleven, so what is the number? Here a flexible thinker might see that this meant five times the number is 10, so the number is two. The equation

$$5x+1 = 3x+5$$

would be less likely to make sense because the equals sign no longer means "makes" and there are now *two* processes to carry out, one on each side. From here, the procedural thinker may learn to respond to rules without reason, including "change sides, change sign", "move all the *x*s to one side", etc. The flexible thinker can see that the equation gives two different ways of getting the same result. Thus the two sides of the equation represent the same thing. So, adding the same thing to both of them again gives the "same thing" on both sides (but not the same "same thing" as before!). The flexible thinker has a meaningful way of manipulating equations to obtain a solution. (Tall & Thomas, 1991).

Further evidence can be gleaned from the way students interpret word problems as algebraic notation: do they write "process" equations such as "$x+4=y$" (perhaps meaning "x plus 4 makes y") or standard "assignment" equations ($y=x+4$). Crowley *et al* (1994) found that the more complex the word problem, the more students wrote the process order rather than the assignment order, with those writing the process order making significantly more errors.

At a higher level, limit concepts, such as

$$\lim_{x\to\infty}\frac{2x+5}{3x+4} \text{ or } \sum_{n=1}^{\infty} 1/n^2$$

involve symbols which represent both a process of "getting close" and also the *value* of the limit. The research literature is full of examples of students who see the process "getting closer and closer" without actually "reaching" the limit, or perhaps "only reaching the limit at infinity" (summarised in Cornu, 1991). Put simply, students often see the limit as a "process of becoming" rather than a mathematical concept. (Monaghan, 1986). At college level the student seems to face "unavoidable misconceptions" (Davis & Vinner, 1986) and copes by using localised procedures for individual problems as they occur (Williams, 1991).

It becomes manifestly clear that the flexible use of symbols is at the heart of powerful mathematical manipulations. But how can this be reconciled with the view of Dörfler that there are no mental objects which correspond to numbers and many other mathematical symbols other than the symbol itself and a host of mental relationships?

There is a resolution of this dilemma. It is clear that we do not t hink of numbers as "objects" in the same way as we think of objects in the external world. But we do use *words* to stand for objects, and we use *words and symbols* to stand for numbers, variables, expressions, and so on. It is no accident of language that we speak of the "number concept" and "the concept of variable" but not the "number object" or "the object of variable".

The number concept has a *concept image* in the individual's mind that consists of "all the mental pictures and associated properties and processes" (Tall & Vinner, 1981). The symbols in mathematics are used in our conscious mind and written more permanently on paper to allow us to link up to all these processes and relationships. Even though they may fail to have mental objects which represent them specifically does not mean that we cannot use the symbols *as if* they were mental objects. We can do this provided that we have a rich concept image structure that allows us to conjure up all the relationships required to perform the necessary computations and manipulations on the symbols which are the external manifestations of this internal activity.

Regrettably, the lack of concept imagery is so prevalent that procedural learning is seen to be the only way that the majority of students can learn. This is exemplified by the worst horrors of "college algebra" which is often accompanied by beautifully prepared, visually attractive texts, even with colour coded instructions, whose sole message it to instruct students to rote learn procedures to be "successful" in algebra.

Computer tools for manipulating symbols

In the previous discussion three essentially different kind of procepts were considered:

- Operational procepts, such as those in arithmetic, or certain procepts in higher mathematics such as the process of differentiation and the concept of derivative, which have a built-in algorithm to carry them out,

- *Template procepts*, such as expressions in algebra, which contain variables and may be evaluated by giving values to the variables, but may also be manipulated as symbols,

- *Structural procepts*, which represent a process to give a result but may not have a direct procedure to find it; instead a structure of the relationships may offer various other approaches. Examples include limits of series or solutions of certain differential equations, which may fail to have symbolic solutions but may be solved by approximate numerical methods.

Of these, *operational procepts* are the easiest to program. The four rules of arithmetic were the first on the scene, now readily available everywhere on hand-calculators.

Template procepts were the next to be attacked on symbol manipulators. Some cases, such as the solution of linear equations which cause great difficulties to many students, were suddenly were seen to be operational. A syntax such as **solve(ax+b=cx+d,x)** meaning “solve the equation $ax+b=cx+d$ in terms of x” could be used to instruct the computer to carry out the manipulations required to give the solution.

Differentiation was regarded by one of my eminent mathematical colleagues not so long ago as something requiring mathematical intelligence to perform. He said he had “a zoo of functions” in his mind which he used whilst breaking down the problem to obtain the answer. He was flabbergasted to realise that symbolic differentiation is a recursive procedure that could be programmed on a computer. The differentiation process therefore becomes *operational* for the computer even it is not seen as such by the student. Even more so, the integration process was regarded as not being given by a single process, but involved selecting from a number of different possible procedures depending on the nature of the function being integrated. The Risch algorithm killed that one too.

However, there still remain many other manipulations of symbols that do not have a clearly defined algorithm and need a different approach. For instance, the simplification of expressions requires a list of templates indicating which expressions (such as X*(–Y), say) can be replaced by other expressions (e.g. –(X*Y)) and then the list is compared with the given expression in all possible ways to successively move towards the result.

Generally, symbol manipulators are getting more sophisticated in dealing with such activities. An early version of *Derive*, simplified

```
      n   n                             n     n
(x+h) -x                           (x+h)     x
---------          to give         ----- - --- .
   h                                 h       h
```

and the limit option applied to this expression, as h tends to 0, gave not nx^{n-1}, but

```
 n LN(x)-LN(x/n)
ê                .
```

The current version of *Derive* takes this further to:

```
      n-1
   n x    .
```

Structural procepts need careful handling in symbol manipulators. Sometimes this involves an intellectual form of cheating. Whereas we might compute a limit such as

$$\lim_{x\to 0}\frac{\sin x}{x}$$

from first principles, perhaps using a visual argument, a symbol manipulator may have more success by using L'Hôpital's rule to get

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{D(\sin x)}{D(x)}=\frac{\cos 0}{1}=1.$$

Limits may involve the student in potentially infinite processes allowing something to get close, but not equal to, a specific value. Computer algebra software may attack the problem using a totally different symbolic algorithm with a finite number of steps.

Symbol manipulators therefore use a variety of sophisticated programming devices internally to carry out required processes to reveal the product to the user. They carry out the internal process without revealing the true relationship between process and concept embodied in the cognitive notion of procept.

Mental tools for thinking about symbols

The brain is a complex structure, built by the serendipity of evolution over many millennia, not a computer with a top-down design. It has around 10^{10} neurons, each with between 500 and 20,000 synaptic connections from other neurons, each firing between 5 and 500 times per second (Crick, 1994). Various groups of neurons work together, interconnecting with other groups to produce human thought processes (Edelman, 1992).

The detail of this activity is not understood. Indeed, Cohen & Stewart (1994, p. 8) tell the joke "if our brains were simple enough to understand them, we'd be so simple that we couldn't." However, some important factors about the brain are accepted. For instance, its parallel processing is so complex that the only way to cope with the bottle-neck is to filter out most activities from conscious thought at any given time and focus on just one or two. This limited "focus of attention" or "short-term working memory" causes us to work in a very special way. The information that is made part of our conscious focus must be compressed so that as much as possible can be in focus in a compressed state, and to operate at maximum efficiency, there must be powerful linkages with other mental information that can be pulled in and out of our focus of attention at will.

The compression of information into symbolism that can be manipulated is what gives mathematics its awesome power.

> I should also mention one other property of a symbolic system – its compactibility – a property that permits condensations of the order $F=MA$ or $S=\frac{1}{2}gt^2$, …in each case the grammar being quite ordinary, though the semantic squeeze is quite enormous. (Bruner, 1966, p. 12.)

By writing a string of symbols and going through what Davis (1984) calls a "visually moderated sequence" of considering the symbols, manipulating, considering the new symbols, manipulating, … , and so on until a solution is found, it is possible to use the short-term focus and long-term connections to solve problems symbolically.

To carry out manipulations meaningfully requires the building of conceptual relationships between symbols, and this in turn enhances the memory of such relationships:

> One advantage of the inclination to create connections between new and existing knowledge is that well-connected knowledge is remembered better. There are probably two explanations for this. First, an entire network of knowledge is less likely to deteriorate than an isolated piece of information. Second, retrieval of information is enhanced if it is connected to a larger network. There are simply more routes of recall. (Hiebert & Carpenter, 1992.)

But what happens if the concepts are not compacted sufficiently to fit into the focus of attention – as might be the case with long definitions in analysis – or if the connections are inadequate for flexible linkages. This leads to the inexorable attempt to use the brain's primitive facility to practice and routinize long sequences of actions. Procedural rote learning is the way the brain operates when it has failed to operate meaningfully. Instead of flexible linkages between many concepts and processes, there is *one* linear sequence of linkages – do *this*, then *this*, then …, until the problem is solved. Brighten it up with a bit of humor and a "lively approach" and it becomes a typical college algebra text.

The problem is that such procedural methods are limited in two ways. First they are *inflexible* and only operate in well-defined contexts (including carefully designed examinations asking questions in a form that students expect to answer.) Second, a procedure cannot be conceived as an entity, other than "if I have *this* problem, then I use *that* procedure". Once it is started, it must be carried through in sequence. If other procedures are required, then they must precede or follow in sequence, so the student has great difficulty in solving problems requiring two or more stages (Rashidi & Tall, 1992).

This weakness of rote-learning has been known since time immemorial[1]:

> Master. I wil propounde here ii examples to you whiche if you
> practice, you shall be rype and perfect to subtract any other summ
> Scholar. Sir, I thanke you, but I thynke I might the better doo i
> me the woorkinge of it.
> Master.Yea but you muste prove yourselfe to do som thynges that
> never taught, or els you shall not be able to doo any more then
> and were rather to learne by rote (as they cal it) than by reason.
> (Robert Recorde, *The Ground of Artes*, 1543)

The fact that so many teachers see the necessity of teaching procedurally is not a vindication of the system, but an indictment of it. Sadly, it is not a new idea to see symbol manipulation as a collection of routines to be practised, and perhaps not understood:

> If we consider the nature of Geometrical and Algebraical reasoning, it will be evident that there is a marked distinction between them. To comprehend the one,

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[1]This, and other historical quotes in this paper can be found in A. G. Howson, *A History of Mathematics Education in England*, Cambridge: CUP, 1982.

> the whole process must be kept in view from the commencement to the conclusion – while in Algebraical reasonings … the attention is altogether withdrawn from the things signified, and confined to the symbols, with the performance of certain mechanical operations, according to the rules of which the rationale may or may not be comprehended by the student.
>
> (Potts, *Euclid's Elements of Geometry*, 1845.)

The secret to understanding algebra lies in giving meaning to the symbolism, for instance, by getting the students to construct solutions to problems that use symbolism in a meaningful way (e.g. Demarois et al, 1992), or by using a computer language to find out how to "talk algebra" with a computer (Tall & Thomas, 1991, Sutherland, 1994).

Students' manifest difficulties in manipulating of symbols may be papered over by using symbol manipulators to do the work, so avoiding student errors in manipulation, but the question must be asked as to whether the symbols have any meaning, other than as a procedural interface to put in the data for the computer to process and put out an answer.

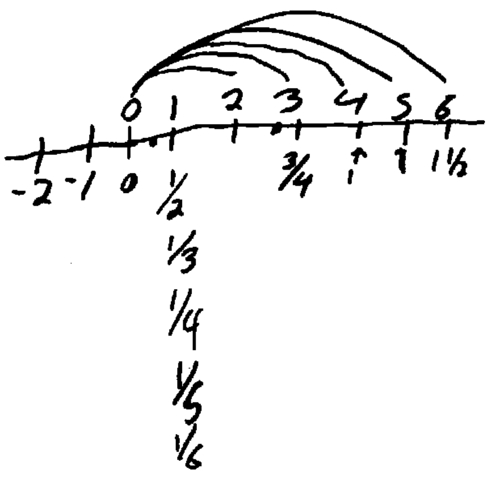
Symbols and visual representations

As the comment of Dörfler earlier warned us, the use of a symbol to represent a process and to give a result does not necessarily lead to a mental object (other than the symbol) that can be brought to conscious attention. Cognitively, Linchevska & Sfard (1991) propose that "rules without reasons" arise from the inability to conceive of processes as objects. Historically, concepts such as *negative* number, *irrational* number, *imaginary* number, *complex* number show us the way that, just because we can operate with symbols does not mean that we think of them as genuine objects.

> The imaginary expression $\sqrt{(-a)}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0-a$ is as inconceivable as $\sqrt{(-a)}$.
>
> (De Morgan, *On the Study and Difficulties of Mathematics*, 1831.)

Having a visual interpretation of these symbols. such as the number line can help enormously (Tall 1994a, 1994b). But each extension of number systems, from counting numbers to positive and negatives, to rationals, to reals, to complex, involve cognitive reconstruction. When a sixth grader familiar with the number line of integers was asked to draw some fractions, he drew an interesting diagram (Alston *et al*, 1994).

He still regarded a fraction *as a process*, so that "1/2 is "divide into two parts and take one of them". The position of fractions on the line depended on the interpretation of a fraction *of what*. He marked "½ of 2" at 1, "⅓ of 3" at 1, "¼ of 4" at 1, …, "1½ of 4" at 6.



A sixth-grader's number line

At this stage he had not encapsulated the sharing process as a new number concept. If the reader considers this to be irrelevant to college studies, it is necessary to respond that our students are a product of their development and this leaves them with a wide array of idiosyncratic concept images of mathematical concepts. Corresponding problems occur with numbers all along the line (literally) as we shall see with students' concepts of real numbers.

Failure of the mind and computer to represent "real numbers"

Symbol manipulators use representations of numbers which are familiar to students, with different types of number including integers, rationals, finite decimals, radicals such as $\sqrt{2}$, $\sqrt[10]{7}$, special mathematical numbers such as π, e. Students may have mental images of infinite decimals, repeating and non-repeating, but these can often represent "improper numbers" which "go on forever" (Monaghan, 1986). Although the "real number line" may seem "self-evident" to mathematicians, it is not to students. In fact it is not to many mathematicians either, but that is another story! Romero i Chiesa & Azcárate-Gimenez (1994) asked students a number of conceptual questions about the real number line, both in terms of decimals and the visual representation. They found absolutely no evidence to suggest that students had any intuitive idea of the mathematical "real line".

Try these three questions to see how you fare:

- Imagine a number line. What do you see?
- Imagine this is magnified, what do you see now?
- What happens at infinite magnification?

Treat them intuitively, without arguing philosophically about their meaning and write down your own response before reading on. [Yes, do it now.]

Interestingly, 47% of students questioned began by seeing the line as a whole and 28% saw elements in it – frequently reported as disks or as little spheres. At infinite magnification this changed to 20% seeing a line and 37% seeing individual elements.

What should a *mathematician* see? Perhaps a line with no thickness made up of an infinite number of densely packed points of no size? Let us not argue about how even an infinite number of points of zero size can make up a finite line segment. Bear with me and follow the implications through.

Clearly if the magnification is a linear one of the form $\mu(x)=k(x-\alpha)$, moving α to the origin and stretching by a factor k, a finite magnification gives a similar picture. But if k is *infinite* (for an infinite magnification), it requires working in a field K containing the real numbers $\boldsymbol{R}$ and an infinite number k in K but not in $\boldsymbol{R}$ to perform the magnification (Tall, 1980, 1982). We will only be interested in the image of real numbers under this microscope. Amusingly if two real numbers a, b can be seen simultaneously then $k(a-\alpha)$ and $k(b-\alpha)$ must differ by a finite number, $k(a-b)$. But for *infinite* k and *finite* $a-b$ this can only happen if $a-b$ is *infinitesimal*. As a and b are both real, this means *a must equal b*. So, under infinite magnification, only *one* real number can be seen. Did you get that?

The point to be made here is that the idea of a "common sense" version of the real line which is generally available to all, mathematician or not, is just so much nonsense. Wood (1992) found that a sizeable minority of university mathematics students believed that there was no smallest positive number (because half of it would be less) but there *is* a *first* positive number 0·00...01 corresponding to 1–0·999... .

Handling decimals, especially finite ones, seems to give a *discrete* sense to numbers, increasing a digit at a time in the last place. Thus arithmetic manipulations with such numbers conflict with the formal notion of the real numbers as a complete ordered field.

Working with symbol manipulators essentially reinforces the students' intuitive image of numbers formulated at the beginning of this section, which operate in a way that causes conflict with the formal theory.

If you don't use it you may lose it!

Focusing on certain aspects and neglecting others may cause the neglected items to atrophy. For instance, students using *Derive* on hand-held computers to draw graphs of functions did not need to substitute numerical values for the independent variable to get a table of values to draw the graph. As a result, they had little practice of numerical substitution. This had unforeseen consequences. Some students who could calculate by substitution before the course were unable to do so afterwards. The students were asked:

> "What can you say about u if $u=v+3$, and $v=1$?"

None of the seventeen students improved from pre-test to post-test and six successful on the pre-test failed on the post-test (Hunter, Monaghan & Roper, 1993).

New procedures for old

Symbol manipulators provide ways of solving problems using the software to perform the manipulations internally. But they do not remove the procedural aspects from the mathematics. Instead they introduce new procedures. For instance, using *Derive* replaces the procedure of symbolic differentiation by a sequence of keystrokes:

- select **Author** and type in the expression,
- select **Calculus**, then **Derivative**,
- specify the variable (e.g. **x**),
- **Simplify** the result.

What happened in a comparison of two schools in the UK, one following a standard course, one using *Derive* is as follows:

> Please explain the meaning of $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
>
> … All the students in school *A* gave satisfactory theoretical explanations of the expression but none gave any examples. However, none of the *Derive* group gave theoretical explanations and only two students [out of seven] mentioned the words 'gradient' or 'differentiate'. Four of the *Derive* group gave examples. They replaced $f(x)$ with a polynomial and performed or described the sequence of key strokes to calculate the limit. (Monaghan, Sun & Tall, 1994.)

Failure of some computer approaches to the limit concept

Given students' (and professors'!) idiosyncratic view of the number system, it is no wonder that they also develop idiosyncratic views of the limit concept. Programming in a computer language or numeric system with limited accuracy seems to make this worse. For over fifteen years my department has used a course in BASIC programming designed to give students intuitions to form a basis for the later formalities of mathematical analysis. Li & Tall (1993) investigated this idea and found it flawed. The calculation of the sum of a series took time so that, although 10 or 100 terms would be computed almost instantaneously, 1000 terms might take longer and 10000 terms ten times as long again. The result is that the students sensed that the process would never end, because the bigger the number, the longer it took.

There were some changes in the students' understandings during the course. In some instances there were subtle modifications of the meaning of terms such as "tends to" and "limit":

Complete the following sentences: 1, 1/2, 1/4, 1/8, … tends to ___________

The limit of 1, 1/2, 1/4, 1/8, is ___________

"tends to" / "limit"	0 / 0	$0 / \frac{1}{\infty}$	$\frac{1}{\infty} / \frac{1}{\infty}$	0 / ?	2 / 2	0 / 2	0 / 1
pre-test (*N*=25)	0	**11**	1	5	0	2	2
post-test (*N*=23)	**8**	3	3	0	4	0	2

The response "2" may indicate the sum of the *series* $1+\frac{1}{2}+\frac{1}{4}+\ldots$. An interview revealed the response "1" for the limit related to an interpretation of the "limit" of the sequence as the largest term. The most commonly occurring response changed from "tends to 0, limit 1/∞" to "tends to 0, limit 0" suggesting that the idea of 1/∞ as an indefinite number, arbitrarily small, is being replaced by the numeric limit 0.

Despite a lecture attempting to support the experiences to give a meaning to an infinite decimal as a limit, the hoary old chestnut "0·9 repeating" did not change its image:

Is $0\cdot\dot{9}=1$ ***?***	Y	**N**	?	no response
pre-test (*N*=25)	2	**21**	1	1
post-test (*N*=23)	2	**21**	0	0

Interviews revealed that students continued to conceive $0\cdot\dot{9}$ as "a sequence of numbers getting closer and closer to 1", or not a fixed value "because you haven't specified how many places there are" or "it is the nearest possible decimal below 1". The programming experiences did not change this view; the limit object cannot be constructed *exactly* in this environment, so old "process" ideas remain without becoming mental objects.

The responses to two other questions gave further insight:

(A) Can you add $0\cdot1+0\cdot01+0\cdot001+\ldots$ and go on forever to get an exact answer? *Y/?/N*
(B) $1/9=0\cdot\dot{1}$. Is 1/9 equal to $0\cdot1+0\cdot01+0\cdot001+\ldots$? (*Y/?/N*)

The favoured response on both pre-test and post test is *No* to (A) and *Yes* to (B):

Responses to (A)/(B)	Y / Y	Y / N	N / N	**N / Y**	N / ?	? / N	nr / Y
pre-test (*N*=25)	4	0	1	**18**	0	1	1
post-test (*N*=23)	**2**	2	2	**14**	1	0	0

The majority regard $0\cdot1+0\cdot01+0\cdot001+\ldots=1/9$ as *false* but $1/9=0\cdot1+0\cdot01+0\cdot001+\ldots$ to be *true*. Reading left to right the first represents a potentially infinite process which can never be completed but the second shows how 1/9 can be divided out to get as many terms as are required. Interviews suggested shades of meaning often consistent with this view, again seeing the expression $0\cdot1+0\cdot01+0\cdot001+\ldots$ as a *process*, not as a value.

Symbol Manipulation and Mathematical Proof

Symbol manipulators focus on getting results rather than taking the user through the process of manipulation to obtain the result. There is an interesting side effect. The main focus of study is on processes that *work* rather than processes that may fail. The net result is that the motivation to suggest *why* proof is necessary is often missing. For example, virtually every graph ever drawn is of a differentiable function. Efforts to use absolute values and integer parts to motivate non-differentiability and discontinuity, are regrettably not typical non-examples. For instance the idea of a "generic continuous function" which is everywhere wrinkled at any magnification is easy to imagine but rarely drawn. If such a function were part of the system (modelled in some practical way) it might be numerically integrated to give a smooth looking function whose derivative is the original wrinkled function (Tall, 1992). Or if functions were drawn that are, in some sense, modelled on the idea that they take different values on rationals and irrationals, this may help to give visual support to more interesting non-examples (e.g. Rosenthal, 1992).

Well, what *are* we doing right?

It is sometimes difficult to appreciate just how fast technology is changing. Euclidean geometry has been with us over two thousand years, the calculus over three hundred, but widespread use of the computer in education is less than a decade. Given the huge resources put into computers at college level it is natural that the first wave of activities were surrounded with hype, trumpeting the good things that are happening. Indeed, this is right to do, because without enthusiasm and belief, the doubters will hold back progress. But now the first wave is established and the mature projects are subjecting themselves to more careful scrutiny to find out what is *really* happening under the surface. As we have seen in the examples in this presentation, it is likely that any attempt to use the computer in mathematics learning will have gains and losses. Indeed, under the surface, the student's images of mathematics is very different from the mathematical formalism it is intended to embody. It is therefore right to focus our attention on the students' thinking processes and address the wider issues of what is happening in our students' learning.

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