The *Psychology* of Advanced Mathematical Thinking: Biological Brain and Mathematical Mind

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This paper focuses attention on psychological evidence of relevance to advanced mathematical thinking. It considers Edelman's theory of neural Darwinism and Paivio's theory of dual coding and contemplates how they reveal different perspectives in mathematics education. The former is consonant with the notion of concept image with both supportive and conflicting mental resonances, the latter is consonant with the complementary roles of symbolism and visualisation. This is related to Hadamard's evidence on mathematical thinking in research which represents "where mathematicians are". But how do students get there and what are their difficulties?

Re-thinking mathematical development from a psychological viewpoint contrasts the visual aspect of objects in geometry which are seen, described, and properties lead to deduction and proof, and the operational aspects in arithmetic and algebra where actions on objects are symbolised and themselves become manipulable mental objects. Thus we have *two* different methods of cognitive development in elementary mathematics – a Van Hiele development in geometry and a process-object development in arithmetic and algebra. In advanced mathematical thinking, the need for reliable proof brings a return of verbal side of proof from Euclidean geometry, now in the form of object-oriented concept definitions and deductions. This brings a *third* form of object construction – from verbal definitions. But these mental processes must be carried out by the biological brain and there are conflicts with the individual's established methods of cognitive construction.

Edelman's theory of Darwinian natural selection of brain function

Gerald Edelman, who received the Nobel Prize for Physiology in 1972 for study of the human immune system, later extended his theory to the workings of the brain (eg Edelman, 1987, 1989, 1992). His theory attempts to be all embracing, starting from genetics and the development of the brain in the embryo, through to its functioning at different levels from biochemistry of individual neurons to a macrostructure based on highly interconnected neuronal groups and on to broader philosophical questions. His *theory of neuronal group selection* is based on three principles.

I. The brain has a huge diversity of interconnections which allow a vast repertoire of actions both mental and physical which may or may not be useful.
II. When activities occur which prove successful, the corresponding connections are re-inforced (through chemical change in the synapses between successive neurons) which are more likely to repeat a similar action on a future occasion.

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III. These functions operate on *groups* of neurons which become hugely interconnected *within* groups and *between* groups including re-entrant linkages within a group and dual linkages between groups that allow them to work as units in co-operation.

In the lifetime of the individual, this leads to a process akin to natural selection developing facilities appropriate for successful survival.

A fundamental role of higher intelligence characteristic of the human brain is that it not only correlates current sensory input with the current memory structure of the brain (which Edelman refers to as “the remembered present”), neuronal groups also relate to others, in a manner which leads to *reflective thinking*. Being able to think about memories of the past allows a sense of time gone and opens up the possibility to plan for the future. (Edelman’s theory therefore links with meta-thinking, and with other two-level models, for instance, Skemp’s “delta-one and delta-two systems”, where delta two operates on delta one.)

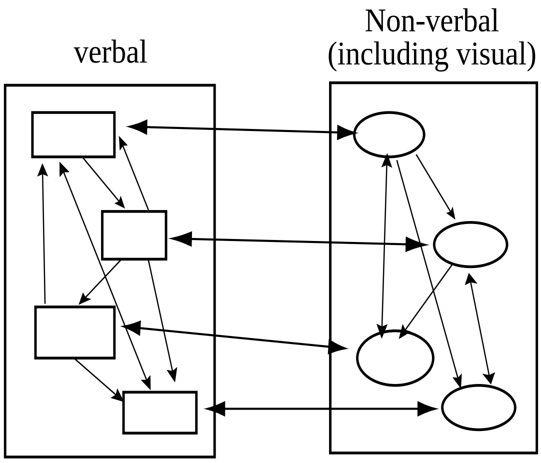
My main interest in this paper is the hypothesis that different neuronal groups function independently to produce conflicting interpretations of sensory data. Such a notion is found in the writings of many other psychologists. For instance, Gazanniga (1985) argues that “the human brain has a modular type organisation … organised into relatively independent functioning units that work in parallel” (page 4), with a special brain component acting as an interpreter, attempting to make sense out of conflicting messages. Links from this interpreter pass on the product of this sense-making to the verbal output module to “explain” what the brain thinks, (or what the interpreter thinks the brain thinks) giving the individual a sense of choice and free will.

The notion of *concept image* is particularly apposite here. In Tall & Vinner, 1981 this it is formulated in a *biological* sense, relating to the *brain*. (The original use of the term in Hershkowitz & Vinner, 1980, describes two different philosophical structures – concept image and concept definition – in the *mind*.) Clearly the notion of a concept image, dependent on the pre-history of the individual, having personal interpretations and inconsistencies, is consistent with hypothesised neuronal groups correlated in various ways, constantly being modified by new experiences. Such a biological basis of thought is clearly evident in the thinking of mathematicians and mathematics students.

Paivio’s dual coding theory for verbal and non-verbal information

Pavio (1971, 1986) proposed a *dual-coding theory* in which verbal and non-verbal systems are strongly interconnected allowing verbal and visual, or other sensory input to be interpreted, coded, stored and re-called by both verbal and non-verbal mental structures. Once again we seen different neuronal structures interlinking to process sensory input.

Paivio's research shows that, when data is coded by both verbal and non-verbal systems, it is more easily remembered and recalled than when only coded by one of them. There-fore a "natural" powerful way of coding information is to use both visual and verbal coding simultan-eously. However, the linkages between the various neuronal group-ings in the brain do not all have the same strength. The linkages between verbal codings may be quite different from those of the corresponding non-verbal codings (including visual-isations). Thus it may be that the concept images induced by the visual codings suggest powerful ideas that are not yet present in the verbal codings, or that the verbal codings may be linked by sequences of verbal relationships which may not correspond in the same way in visual codings.



Dual-coding is therefore a double-edged sword. It provides a more versatile set of mental linkages that enable creative thought to occur, thus enabling the biological brain to think about the formal mathematical structure, but it may also suggest deeply held beliefs which fail to be true in formal mathematics. It is for these very reasons that visual and other non-verbal aspects of mathematics are so valued by some, yet totally distrusted by others.

Paivio introduced his dual-coding theory because he wished to challenge

> the singular view … that performance in memory and other cognitive tasks was mediated by processes that are primarily verbal or linguistic.
> (Preface to Paivo, 1986)

Leibniz was concerned about the primacy of words:

> It troubles me greatly that I can never acknowledge, discover or prove any truth except by usng in my mind words or other things.

It is an interesting to note that in communicating this paper I must do so mainly through the medium of language. Do we play down the role of visual and non-verbal modes of thought because we have difficulty in *talking* about them? In explaining how we think, do we just *rationalise* what we *think* we think as postulated by Gazzaniga? Is it that we have to *translate* what we think we think into words to attempt to communicate our ideas?

Hadamard's evidence on the thinking of research mathematicians

It is almost exactly fifty years since Hadamard completed his book on *The Psychology of Mathematical Invention* (signing the preface on August 21st 1944). Yet his contribution, as a successful research mathematician inquiring into the processes of other research mathematicians, still remains one of the most valuable for our present discussion. He reported that:

> The mental pictures of mathematicians whose answers I have received are most frequently visual, but they may be of another kind – for instance, kinetic. There can also be auditive ones, but even these … generally keep their vague character. p. 85
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> … mathematicians born or resident in America, whom I asked, … practically all … – contrary to what occasional inquiries had suggested to Galton as to the man in the street – avoid not only the use of mental words, but also, just as I do, the mental use of algebraic or any other precise signs; also as in my case, they use vague images. pp. 83–84

In recent interviews with three research mathematicians, Sfard (1994) finds each one reporting the same kind of vague visual imagery.

Hadamard found some notable exceptions: G. D. Birkhoff, the celebrated algebraist, mentally manipulated algebraic symbols whilst doing research, Norbert Weiner reported thinking both with and without words and Jessie Douglas reported thinking with the rhythms of words rather than the words themselves. Personally, I find the latter hard to understand as it is a mode of thought that makes no sense to me. (My own images seem to be primarily verbal as if I am saying sentences to myself, together with vague pictures, diagrams, and spatial sensations.) Hadamard quotes the psychologist Ribot who found many intellectuals (other than mathematicians) who thought *predominantly* in words – for instance a physiologist who when asked to think about a dog, saw the *written* word “dog”, even though he worked regularly with these animals. Ribot reported that such people found it hard to conceive that anyone else could think differently. He identified the majority of philosophers (metaphysicians) interviewed as being of this verbal type. It may therefore be that many of those who philosophise about mathematical epistemology are as bad at understanding how mathematicians think as I am at understanding how Douglas thinks in verbal rhythms!

It is also interesting to note that the one mathematician different from all the others is the one who has done so much for mathematics education – Georg Polya. He alone of those interviewed regularly used words to focus on ideas during the creative phase of his thinking. (What does this have to say for those of us who use Polya’s methods to teach mathematical thinking?)

Hadamard gives examples of the vague mental supports used by mathematicians. For instance, in one of his own pieces of research involving the convergence of a series, he explained:

> I see not the formula itself, but the place it would take if written: a kind of ribbon, which is thicker or darker at the place corresponding to the possibly important terms; or (at other moments), I see something like a formula, but by no means a legible one, as I should see it (being strongly long-sighted) if I had

> no eye-glasses on, with letters seeming rather more apparent (though still *not legible*) at the place which is supposed to be the important one. p. 78

Notice here the need to *focus attention* on the appropriate level, to overcome the limited capacity of the brain to hold much detail in working memory. In his research Hadamard uses symbols only for elementary computations:

> I behave in this way not only about words, but even about algebraic signs. I use them when dealing with algebraic calculations; but whenever the matter looks more difficult, they become too heavy a baggage for me. I use concrete representations of quite a different nature. pp. 75,76

As an example formulated for the general reader, he explains what is in his mind when he proves Euclid's theorem that there are infinitely many primes. To show that, say there is a prime bigger than 11, he does the following:

STEPS IN THE PROOF	MY MENTAL PICTURES
I consider all primes from 2 to 11, say 2, 3, 5, 7, 11	I see a confused mass.
I form their product $2\times3\times5\times7\times11 = N$.	*N* being a rather large number, I imagine a point rather remote from the confused mass.
I increase that product by 1, say *N* plus 1.	I see a second point a little beyond the first.
That number, if not a prime, must admit of a prime divisor, which is the required number.	I see a place somewhere between the confused mass and the first point.

He remarks:

> … one can easily realize how such a mechanism or an analogous one may be necessary to me for the understanding of the above proof. I need it in order to have a simultaneous view of all elements of the argument, to hold them together, to make a whole of them … It does not inform me of any link of the argument (i.e. on any property of divisibility or primes); but it reminds me how these links are to be brought together. pp. 76, 77
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> If I should use a blackboard and write the expression $2\times3\times5\times7\times11$, the above schema would disappear from my mind as having obviously become useless, and would be automatically replaced by the formula which I should have before my eyes. p. 79

In terms of the psychological theories mentioned earlier, we see non-verbal structures in the brain cooperating in the construction of the proof and the limitations of the focus of attention of the brain which loses concious linkages with one representation when focusing on another.

Cognitive development through elementary mathematics to advanced mathematical thinking

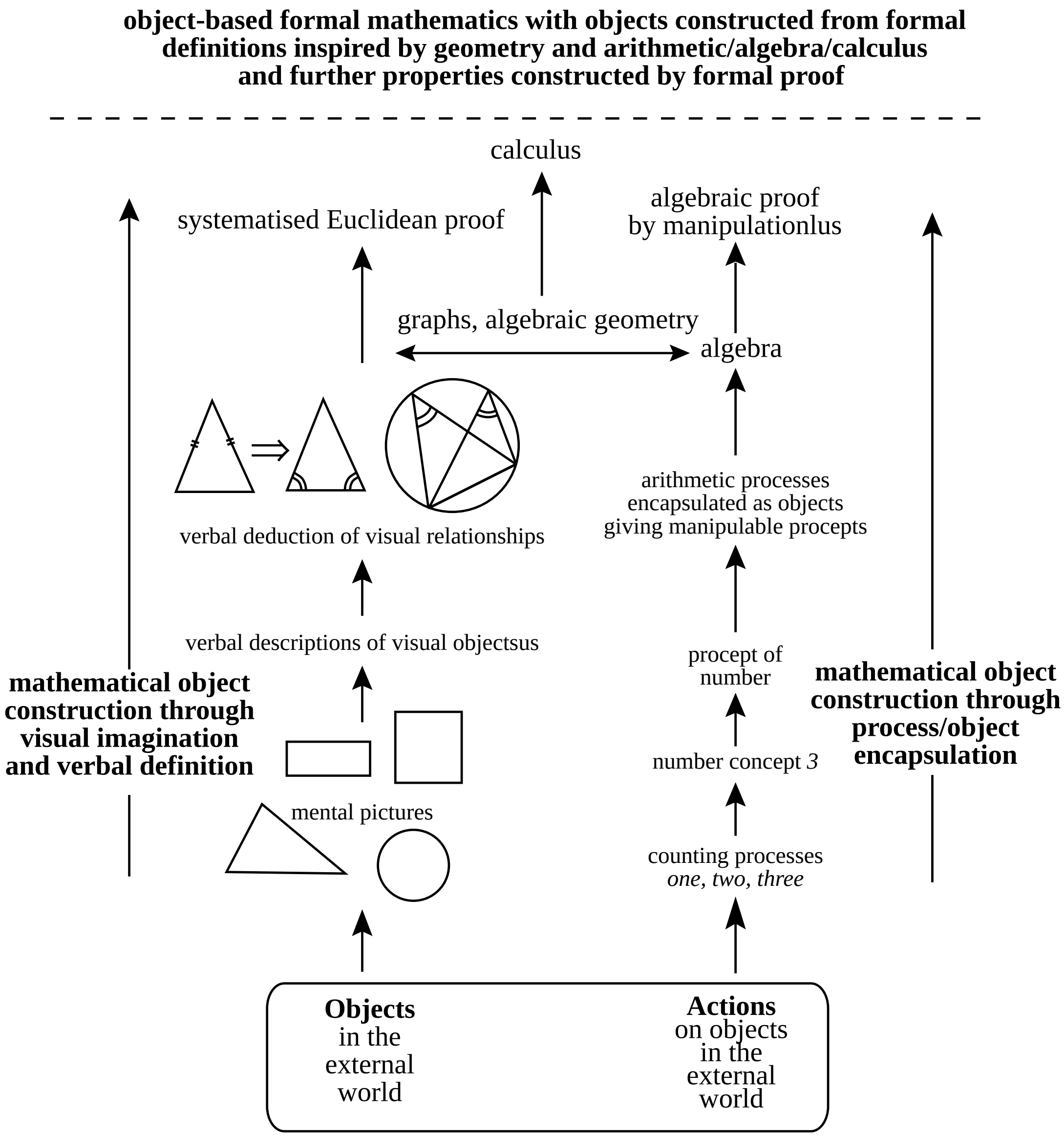
The cognitive development of mathematics occurs in a biological brain. It seems to involve the construction of mental objects which can be manipulated in the mind by analogy with actions on objects experienced in the external world. There is compression of conceptual knowledge in various ways to compensate for the limitations of the focus of attention of the brain. The way in which the development occurs seems to depend on the nature of the different forms of information presented to our senses.

In geometry, for example, objects in the external world are first seen as gestalts, then they are observed to have certain characteristics that can be verbalised, related, and later used as definitions of idealised concepts which may be used for formal deduction. This leads to a *Van Hiele style development*. On the other hand, processes in arithmetic, algebra, calculus, and so on, are either routinised so that they can be performed using little conscious attention, or, more powerfully, are symbolised so that the symbol can evoke either the process or the concept produced by that process. This gives the *process-object encapsulation theories* of Piaget-Dienes-Greeno-Dubinsky-Sfard etc and the duality-ambiguity-flexibility of symbolism as ***pro***cess and con***cept*** represented by the notion of *procept* (Gray & Tall, 1994). “Encapsulation” is a natural function of the human brain. It is the *naming* of a process that is beginning to be thought of as an object, to allow it to be used *as* an object – a natural process of selection for fitness for more sophisticated use.

Hadamard makes an intriguing comment which is consonant with my division between geometric objects and arithmetic/algebraic procepts:

> I also add that the case we have just examined especially concerns arithmetical, algebraic or analytic studies. When I undertake some geometrical research, I have generally a mental view of the diagram itself, though generally and inadequate or incomplete one, in spite of which it affords the necessary synthesis – a tendency which, it would appear, results from a training which goes back to my very earliest childhood. p. 79

Advanced mathematical thinking not only involves a greater complexity of ideas, it also *systematises* them in an organised manner, leading to the axiomatic theories of the twentieth century. Here the need for proof leads to the use of concept definitions to formulate mathematical concepts, essentially as mental *objects* having prescribed verbal-symbolic properties.



Outline of different forms of object construction

In thinking *about* formal mathematical systems, the mathematician uses his concpt imagery to guide him.

> There is no doubt that Hilbert, in working out his *Principles of Geometry*, has been constantly guided by his geometrical sense. If anybody could doubt that (which no mathematician will), he ought simply to cast one glance at Hilbert's book. Diagrams appear at practically every page. They do not hamper mathematical readers in ascertaining that, logically speaking, no concrete picture is needed.
>
> This is again a case where one is guided by images without being enslaved by them… pages 87–88.

It is this quality of using images *without being enslaved by them* which gives the professional mathematician an advantage but can cause so much difficulty for the learner.

Student difficulties in Advanced Mathematical Thinking

The different methods of object construction cause great difficulties to students in transition to advanced mathematical thinking. The introduction of axiomatic methods requires a massive mental reconstruction as they attempt to build up a new mental mathematical structure in which the mathematical objects have only those properties that follow from the definitions. The multiple linkages, particularly the dual codings of verbal and visual imagery, which have served them well in the past now must be severed until they have been re-linked in a manner which is logically deduced. But the new definitions may involve a new way of thinking which conflicts with the old.

For instance, the procepts of elementary mathematics (such as addition) have built-in methods of computation (such as counting) but procepts in advanced mathematics (eg limits) may not. Instead they are *defined* verbally using unencapsulated processes (given $\varepsilon>0$, find $\delta>0$ such that…). Yet the imagery of the concepts, such as ideas of "getting close", may lead to thought experiments which suggest false theorems (eg that a continuous function satisfies the intermediate value theorem without specifying the completeness axiom) or true theorems whose formal proof is highly intricate and does not relate to the thought experiments (such as basic theorems about limits).

Professional mathematicians have more sophisticated imagery which they are more likely to use without being enslaved by it. They use multiple-linkages including dual-coding and have the experience to think *about* mathematics in a way which is separate from the constructions performed *within* the axiomatic system. Students do not have this sophistication. They often have a crisis and suffer loss in confidence about what they know or what they think they are allowed to know. In so many cases this leads to them performing (as so many children do in elementary mathematics) by rote-learing of the symbol pushing *within* the axiomatic system without any reflective guidance from outside. The biological brain, with its richly linked concept images has yet to grow into a mathematical mind.

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