Construction of the Limit Concept with a Computer Algebra System

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Students encounter many cognitive difficulties with limit ideas: sequences never end; functions do not attain their limits; series do not produce a final answer. Limit is further both a process and an object and students usually focus on the process. Studies investigating these difficulties are noted before presenting some results from a new study that examines the limit conceptions of students learning calculus concepts with the aid of a computer algebra system. Differences with traditional approaches emerge in that process problems are suppressed and the limit as an object appears clearer but this brings its own problems. Limit is a deep notion and each approach highlights and suppresses different facets of the concept.

Conceptual difficulties with limits

The limit concept is known to cause difficulty in learning. A number of research studies reveal that students often conceive of the notion of $\lim_{x\to a} f(x)$ or $\lim_{n\to\infty} a_n$ not as a static concept but as a dynamic process of 'getting close to' a fixed value, often with the implication of 'never reaching' the limit (see summaries in Cornu (1991) and Tall (1992)).

Gray & Tall (1993) considered a wide variety of instances where a symbol can ambiguously represent either a process or a concept. They call this a *procept*. For instance 3+2 might evoke a process of addition, perhaps by counting on two, or the concept of sum. The symbols $\lim_{x\to a} f(x)$ and $\lim_{n\to\infty} a_n$ both represent either the process of getting close to a specific value, or the value of the limit itself. The limit is therefore an example of a procept. But unlike the procepts of elementary mathematics, which have a built-in algorithm to calculate the specific value of the concept, the limit value does not have a specific universal algorithm that works in all cases and, in some cases, such as $\lim_{n\to\infty} \sum_{k=1}^{n} 1/k^2$, there may be no simple algorithm. (In this instance, the theory of residues in complex integration or a sequence of key strokes on a computer algebra system shows the limit to be $\pi^2/6$.) The circuitous routes by which limits are calculated in the early stages of the theory exacerbate the difficulties students have with the concept. As Cornu (1981) observed: "mathematics no longer reduces to calculations and simple algebraic properties; infinity intervenes and it is shrouded in mystery."

Monaghan (1986) studied the growth of 16/17 year old students' conceptualisations of real number, limit and infinity over one year as the experimental group studied traditional calculus and a matched control group studied other subjects. The students' fundamental concepts of infinity and limits hardly changed in the period concerned. Their notion of real number showed them happy to manipulate whole numbers, fractions and such numbers as $\sqrt{2}$ and π, but they became less secure when attempting to deal with infinite decimals. The latter were regarded as being 'improper' and described as 'infinite numbers'. An expression like $\sqrt{2}=1.414\ldots$ does not say that "$\sqrt{2}$ can be computed exactly as (the limit of) a decimal expansion", rather that "$\sqrt{2}$ can be described to any required accuracy by approximating to a specific number of decimal places". Thus the number line is viewed as consisting of positive and negative whole numbers and fractions and combinations of other expressions including $\sqrt{2}$, π, etc., together with a more peculiar set of 'improper numbers'. In practice, computations can be carried out with finite decimal approximations but this gives a perception that such arithmetic is inherently inaccurate.

To build a precise theory of limits on such a foundation is bound to contain the seeds of cognitive conflict and sow confusion in the students' minds. A traditional approach to the limit in such circumstances is fraught with difficulty, so much so that Davis & Vinner (1986) suggested that there are *unavoidable* obstacles which the student must confront when beginning to study the topic.

Approaching the limit by functional programming

Prior to the arrival of the computer the introduction of the limit concept required the student to have considerable experience of the limit as process, so that the latter is unavoidably embedded deeply in the student's psyche. Computer software can now evaluate many limits, so the possibility arises that it may allow the curriculum to give a more balanced view of limit as concept and process by early focus on the limit as concept with the computer carrying out the process internally.

Li & Tall (1993) investigated an approach using programming in structured BASIC which allows definitions of named functions. This allowed a function to be considered either as a procedure of computation, or as an object whose name could be used in building other functions. The course was largely successful in giving a proceptual view of function as process or object and, in defining a series as a function adding up the terms of a sequence, it was able to help students discriminate between sequences and series. But it was not successful in moving from a view of limit (of a sequence) as a process to a limit as an object. This failure to encapsulate limit as an object in the majority of students was predictable in hindsight. The numerical basis on which it was built was computer arithmetic with numbers stored to approximately 8 significant figures. The experience was therefore built on a foundation of limited numerical accuracy. This was built in to the experience by computing numerical values of sequences $s(n)$ for large values of n and looking for the values stabilising to a given accuracy. The overt message was that, to this given accuracy, from some term $s(N)$ on, the terms stabilised to a fixed value, thus for $n \geq N$, the terms $s(n)$ became indistinguishable. This was used in class to 'motivate' the idea that the greater accuracy required, the greater the value of N beyond which stabilisation would occur, leading to the ε-N definition of limit. This approach should lead to the notion of (Cauchy) convergence, but after two weeks the students had forgotten most of the discussion and mainly conceived of the limit notion in a dynamic sense.

We conjecture that any computing experience intended to 'motivate' the limit notion by computing limits using approximate arithmetic may be fraught with this underlying problem. As the limit is found by a *process* of computing values of $s(n)$ for larger and larger n, it will implicitly confirm the students' belief that the limit is a *process* not a concept. Dubinsky (1991, 1993) proposes a theory of encapsulation of process as object and uses the language *ISETL* for programming functions as procedures which can then be conceived as objects and used as inputs to other procedures. This computer language is better structured for mathematical purposes than BASIC and is a good environment for conceptualising mathematical thinking in a wide variety of ways. Although it includes rational arithmetic, its fundamental numerical mode appropriate for the calculus is floating point arithmetic. It has no facility for computing the limit as a precise value, so it too is flawed as an environment for a proceptual view of the limit concept.

Using a computer algebra system

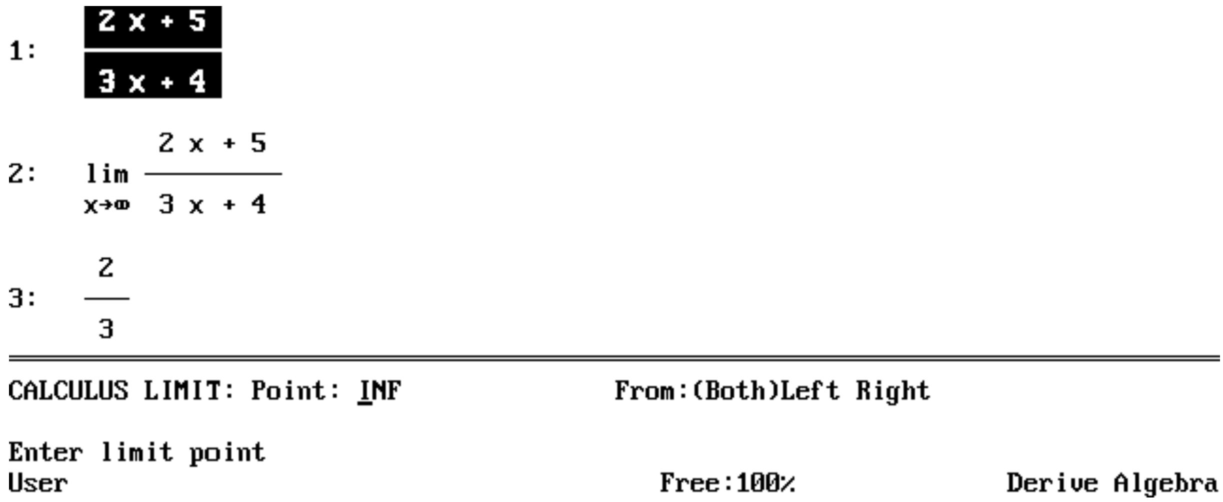
Given the student's perception of 'proper' numbers, an environment which might prove suitable for exploring the limit concept is a computer algebra system such as *Maple*, *Mathematica* or *Derive*. These allow manipulation of 'proper' numbers such as rational numbers and rational expressions in $\sqrt{2}$, π, e, etc. and do not simplify these expressions to approximate answers unless explicitly instructed to do so. All allow programming to a greater or lesser degree.

Tall (1993) suggests that the computer relieves the learner of the tyranny of having to encapsulate the process before obtaining a sense of the properties of the object. By using software which carries out the process internally, it becomes possible for the learner to explore the properties of the object produced by the process before, at the same time, or after studying the process itself. This new flexibility in curriculum development which gives new possibilities in the order in which the concepts are constructed is called *the principle of selective construction*.

Sun (1993) and Monaghan researched the effects on the limit concept of using the software *Derive* freely in the early stages. This computer algebra system was selected because it is available on a hand-held computer and the facilities are easily available through the use of simple menus. For instance, the sequence of actions to find

$$\lim_{x\to\infty}\frac{2x+5}{3x+4}$$

is shown below. By a routine sequence of key strokes the student can move from the original expression to obtain the value of the limit in the form $\frac{2}{3}$.



This is analogous to a child carrying out an arithmetic operation by a sequence of key-strokes. It is therefore behaves in a manner familiar to the student and, in one sense, is less “shrouded in mystery” than the traditional dynamic approach to limit. In another sense, however, the internal process by which the computer carries out the process remains mysterious. But, just as someone who knows what a square root *is*, but not how to calculate one, may be satisfied that the square root key gives a satisfactory approximation to $\sqrt{2}$, so the student may give some meaning to the result by other means. For instance, by computing values of the expression for large values of x, or dividing numerator and denominator by x and noting that, as x gets large, so $1/x$ gets small.

Selectively, therefore, the student may focus on the production of the limit object (using the computer) or on the limit process (by a paper and pencil or computer calculation). Therefore the student can see the two complementary facets of the limit procept as concept and process, in whichever order is desired.

The Experiment

The students in the study were able 16/17 year olds at the end of their first year of a two year Further Mathematics Advanced level course. Advanced level mathematics is open to the highest attaining quartile of 16 year olds and covers most of the differential and integral calculus of a single variable. Further Mathematics is taken by able and motivated students within this population.

The experiment was motivated by the access of the first two authors to a group of nine students in a college who had made extensive use of the computer algebra system Derive throughout their studies: 50% of lessons in rooms where Derive was ‘on call’ at desktop machines and for two months they were given palmtop computers fitted with Derive which they could use at any time. We shall call these the *Derive group*. The two authors were intrigued as to the possible effects of this exposure on students’ limit conceptions and a comparison group was found. Three schools provided 19 students with similar backgrounds who were following an identical curriculum but who had not met a computer algebra system. One of these schools, which we shall refer to as school

A and which accounted for seven students in the comparison group, had students who were very closely matched to the students in the Derive group.

A questionnaire was designed to elicit students' conceptions of the limit of a sequence, a function (graphically and algebraically) and of a numeric series. In addition discontinuity was incorporated into one question and the definition of a derivative was examined. The questions drew upon the work of Li (1992) and Monaghan (1986). The students in the Derive group were free to use Derive on their palmtop computers in the questionnaire. Within two weeks of students completing the questionnaire they were interviewed. 25 of the 28 students were interviewed including all those in the Derive group and school A. Interviews lasted for about 20 minutes and were designed to probe reasons behind specific responses.

The Results

We report on responses to three of the questions.

Q1 Please find the following limits if they exist. If there is no limit, then write 'no'.

Please explain your results.

$$\lim_{x \to \infty} \frac{2x+3}{x+2} \qquad \lim_{x \to 1} f(x) \text{ where } f(x) = \begin{cases} 3 & \text{if } x = 1 \\ \dfrac{x^2-1}{x-1} & \text{if } x \neq 1 \end{cases}$$

Eight of the nine Derive students used Derive to find the first limit and claimed they did not know any other method. The exception was the one student who 'did not like' the computer system. Of the 19 non-Derive students 12 divided and then used the concept that $1/x$ approaches 0 as x approaches ∞. Three substituted numbers and four left the question blank.

The second limit with the discontinuity is comparatively difficult and was novel to all students. There was greater diversity in responses. Nevertheless six of the nine Derive students simply used Derive to find the limit of the major part of the function and ignored the discontinuity. Of the 19 non-Derive students, 11 gave diverse answers that considered the discontinuity, six left it blank and only two ignored the discontinuity. Lack of space prevents a full analysis but the point we focus on here is using Derive as a button-pushing process that can obscure deeper consideration of the function.

Q2 a) Can you add 0.1 + 0.01 + 0.001 + ... and get an answer? Why?

b) Can you add 0.9 + 0.09 + 0.009 + ... and get an answer? Why?

Although these may appear to many as innocuous questions with easy answers, they cause great conceptual problems to students who have not pursued pure mathematics to any depth. The reason, as documented at length in Monaghan (1986), is that they never end and so you never get an answer – they are always in a state of becoming.

Only two of the 28 students distinguished between the cases, both stating ‘no’ to the first and ‘yes’ to the second because $0.\dot{9}$ rounds up to 1. We shall thus illustrate differences between the groups by using the first question. 7/9 of the Derive group indicated that there is an answer, three using the formula for the sum of a geometric progression and 4 stating it as $0.\dot{1}$. The general view in the non-Derive group was there may be an answer but there were problems. For example school A students gave similar responses to the Derive group with 2/7 using the formula for the sum of a geometric progression and 5/7 stating it was $0.\dot{1}$ but all with qualifications that this was only an approximation or that the answer tended to this. Interviews, however, revealed some similar thought behind the Derive group’s ‘object’ answers. “First I though ‘no’ because it just goes on forever and ever. Then I checked it on Derive. I did get an answer.”

Q3 Please explain the meaning of $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

It was clear from the questionnaire results that only school A and the Derive group had properly considered this notation in their mathematics lessons. We thus compare these two groups. All the students in school A gave satisfactory theoretical explanations of the expression but none gave any examples. However, none of the Derive group gave theoretical explanations and only two students mentioned the words ‘gradient’ or ‘differentiate’. Four of the Derive group gave examples. They replaced $f(x)$ with a polynomial and performed or described the sequence of key strokes to calculate the limit.

Discussion

Different proceptual ideas permeate all three questions. The ‘find the limit’ questions reveal that Derive generates a specific process for computing limits:

- select [Author] and type in the expression,
- select [Calculus], then [Limit],
- specify the variable (e.g. **x**),
- specify the limiting value of the variable (e.g. **inf**),
- [Simplify] the result.

As mentioned, this is analogous to the processes younger children use in carrying out arithmetic operations on a calculator. This has the advantage that it allows the students to focus on the limit object by suppressing problematic notions of infinity and ‘getting closer’. The other side of the coin is that we do not want students to ignore ‘closeness’ ideas and the fact that seven of the nine Derive group students ignored the discontinuity in the second part of the question suggests that this is happening.

The series questions indicate that some students are beginning to see the limit sum as an object. A possible explanation for this is that when summing the series using pencil and paper or, indeed, programming in BASIC such as Li and Tall (1993) did, the mind concentrates on the process of summing, this takes time and time becomes a factor in students' conceptions. However, when a student uses Derive to perform the summation, the mind is freed to concentrate on the outcome, the object.

The question concerning the definition of the derivative reveals an 'action schemata' in some of the Derive group: to define the derivative from first principles is to produce a sequence of key strokes as outlined above. Why are specific examples used? Is this a case of students seeing generalities only via specific examples? We believe not, but that it is only in the context of a specific example that the key strokes make sense for otherwise the key strokes would merely replicate the notation in the question. Again there are dually positive and negative aspects of this approach in that while an object is produced many of the finer points (discontinuities, stabilization, etc.) are ignored.

Comparing these focuses with those produced in a programming environment reveals some marked differences. The focus on a sequence of key strokes was not apparent in the work of Li and Tall (1993). The object of the process is stronger in the computer algebra environment. In Li and Tall's work the stabilization of a sequence was a focus of student thought but this was not the case in the computer algebra environment.

It appears that the programming approach, with its emphasis on the value of finite terms and closeness, is more like the paper and pencil approach than the computer algebra approach. This implies that the reality of approaches is much more complex than simply computer approaches vs non-computer approaches. Li and Tall (1993) posited three limit paradigms: a dynamic limit paradigm; a functional/numeric computer paradigm; the formal ε-N paradigm. To this we add a fourth: the key stroke computer algebra paradigm.

The different focuses of different paradigms has similarities to Schwarzenberger's (1980) claim that calculus cannot be made easy because the real number line is simultaneously complete, an ordered field, a metric space and a normed metric space. "A certain viewpoint may make certain calculations easy but in other directions it may make things more difficult." Limit is a related deep notion. It is not possible to make it 'easy' but, by using the process of *selective construction*, however, it should be possible to design curriculum materials that exploits the potential of all of these approaches and so gives an improved cognitive base for a flexible proceptual understanding of limit. The curriculum designer, or the student exploring the new ideas, can select which part of the new notion is to be constructed at a given time – the processes, or the resulting concepts and relationships between them.

For instance, a calculator allows the child to perform arithmetic without the process of counting or the use of the standard algorithms. It is therefore possible for the child to concentrate more on the properties of arithmetic than on the procedure of counting in the early years (Doig, 1993). Likewise it may be possible for the student to develop a more balanced view of the limit dually as process and concept by using a computer algebra system which produces a symbolic limit as a 'proper' numerical expression.

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