Students' Difficulties in Calculus

Plenary presentation in Working Group 3, ICME, Québec, August 1992

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1. The Calculus

It should be emphasised that the Calculus means a variety of different things in different countries in a spectrum from:

1. *informal calculus* – informal ideas of rate of change and the rules of differentiation with integration as the inverse process, with calculating areas, volumes etc. as applications of integration

to

2. *formal analysis* – formal ideas of *completeness*, ε–δ definitions of limits, continuity, differentiation, Riemann integration, and formal deductions of theorems such as mean-value theorem, the fundamental theorem of calculus etc.,

with a variety of more recent approaches including

3. infinitesimal ideas based on non-standard analysis,

4. computer approaches using one or more of the graphical, numerical, symbolic manipulation facilities with, or without, programming.

In some countries the first of these is taught in secondary school and the second to mathematics majors in college. In others a subject somewhere along the spectrum between the two is taught as the first major college mathematics course. In a few countries (e.g. Greece), the formal ideas are taught from the beginning in secondary school.

The details of these approaches, the level of rigour, the representations (geometric, numeric, symbolic, using functions or independent and dependent variables), the individual topics covered, vary greatly from course to course.

2. Difficulties in the Calculus

The calculus represents the first time in which the student is confronted with the *limit* concept, involving calculations that are no longer performed by simple arithmetic and algebra, and infinite processes that can only be carried out by indirect arguments. Teachers often attempt to circumvent the problems by using an "informal" approach playing down the technicalities. However, whatever method is used, a general dissatisfaction with the calculus course has emerged in various countries round the world in the last decade.

Published in *Proceedings of Working Group 3 on Students' Difficulties in Calculus, ICME-7 1992, Québec, Canada*, (1993), 13–28. ISBN 2 920916 23 8.

In France, the birthplace of the logical structures of Bourbaki, mathematics educators realised that formal approaches to learning had fundamental flaws and the IREMs (Instituts de Recherche sur l'Enseignement des Mathématiques) have relentlessly pursued the need to make the development of the subject matter more meaningful to students (Artigue *et al*, 1990). In the UK a recent report of the London Mathematical Society acknowledges the difficulty of university mathematics and the need to reduce the content and reorganise the course. (London Mathematical Society, 1992). In the USA it is acknowledged that of the 600 000 students taking college calculus in 1987, only 46% obtained a pass at grade D or above (Anderson & Loftsgaarden, 1987). This atmosphere of general dissatisfaction led to the "Calculus Reform Movement" in the USA, with a heavy investment in development and technology but with little initial investment in cognitive research. The latter omission is being remedied, with a considerable increase in publications on cognitive difficulties in the calculus but, with a few notable exceptions, the reform movement itself still awaits independent analysis.

2.1 Fundamental difficulties with limits and infinite processes

Whichever way the calculus is approached, there seem to be inherently difficult concepts which seem to cause problems no matter how they are taught. The *limit* concept creates a number of cognitive difficulties, including:

- difficulties embodied in the language; terms like "limit", "tends to", "approaches", "as small as we please" have powerful colloquial meanings that conflict with the formal concepts,
- the limit process is not be performed by simple arithmetic or algebra, infinite concepts arise and the whole thing becomes "surrounded in mystery",
- the process of " a variable getting arbitrarily small" is often interpreted as an "arbitrarily small variable quantity", implicitly suggesting infinitesimal concepts even when these are not explicitly taught,
- likewise, the idea of "*N* getting arbitrarily large", implicitly suggests conceptions of infinite numbers,
- students often have difficulties over whether the limit can actually be reached,
- there is confusion over the passage from finite to infinite, in understanding "what happens *at infinity*".

paraphrased from Cornu, 1981
Schwarzenberger & Tall, 1978
Orton, 1980ab, 1983ab
Robert, 1982
Sierpiń ska 1985, 1987
etc., etc.

How does the student handle such conflicts? Two methods are possible:

- reconcile the old and the new by re-constructing a new coherent knowledge structure,
- keep the conflicting elements in separate compartments and never let them be brought simultaneously to the conscious mind.

As the first of these is very difficult, many students (and most teachers!) prefer the latter, separating troublesome theory from the practical methods to solve problems:

> [In] the official French programme … books generally devoted a chapter to the general limit concept including a formal definition, a statement of its uniqueness, and theorems about arithmetic operations applied to limits. The exercises, however, did not concentrate on the limit *concept*, but on inequalities, the notion of absolute value, the idea of a sufficient condition and, above all, on *operations*: the limit of a sum, of a product, and so on. These exercises are far more related to algebra and the routines of formal differentiation and integration than to analysis. … Given such a bias in emphasis it is therefore little wonder that students pick up implicit beliefs about the way in which they are expected to operate. (Cornu 1992, p. 153)

> … [American] students often considered the ease and practicality of a model of limit more important than mathematical formality. This is particularly true in the sense that models of limit that allow them to deal with the realities of limits in the classroom, the kind they see on tests, tend to be seen as sufficient for the purposes of most students. It was noted by several students that neither formal nor dynamic models of limit figure heavily in the procedures students use to work problems from their calculus class; their procedural knowledge (e.g., substituting values into continuous functions, factoring and cancelling, using conjugates, employing L'Hôpital's rule) is largely separate from their conceptual knowledge.
> (Williams, 1991, p. 233)

Various studies show that what the students believe is related to the dominant work that they do, and paying lip service to formalities may satisfy the teacher but it has very little impact on the learner. Ervynck (1981) concluded that most students have a prerigorous understanding of limit but few ever achieve full understanding of the rigorous definition.

How can such difficulties be avoided? Davis & Vinner (1986) attempted to do so by avoiding reference to the language of limits in the initial stages, but came to the conclusion that "avoiding appeals to such pre-mathematical mental representation fragments may very well be futile." They show that specific examples dominate the learning, so that if, for example, monotonic sequences dominate the students' early experiences of sequences then they will also dominate the students' concept images of sequences and their limits. Thus it becomes almost impossible to give students simple experiences without giving them correspondingly simple long-term conceptions of the concepts being introduced.

There is evidence that students apply arguments not globally, but use different arguments suitable for each case, allowing them to keep disconcerting conflicts in separate compartments. For instance, a student might use different conceptions of limit selected according to the particular context being considered, without being concerned about possible overall consistencies:

> And I thought about all the definitions that we deal with, and I think they're all right – they're all correct in a way and they're all incorrect in a way because they can only apply to a certain number of functions, while others apply to other functions, but it's like talking about infinity or God, you know. Our mind is only so limited that you don't know the real answer, but part of it. (Williams, 1991, p. 232)

Students learn the things that will get them through the exams:

> Much of what our students have actually learned ... – more precisely, what they have invented for themselves – is a set of "coping skills" for getting past the next assignment, the next quiz, the next exam. When their coping skills fail them, they invent new ones. The new ones don't have to be consistent with the old ones; the challenge is to guess right among the available options and not to get faked out by the teacher's tricky questions. … We see some of the "best" students in the country; what makes them "best" is that their coping skills have worked better than most for getting them past the various testing barriers by which we sort students. We can assure you that that does not necessarily mean our students have any real advantage in terms of understanding mathematics. (Smith & Moore, 1991)

When students meet difficulties, a dominant strategy for coping is to concentrate on the procedural aspects that are usually set in examinations. Because the teacher knows that conceptual questions are rarely answered correctly, the vicious circle of procedural questions is set in motion. Indeed, for those students who take an initial calculus course based on elementary procedures, there is evidence that this may have an unforeseen limiting effect on their attitudes when they take a more rigorous course at a later stage. Commenting on the results of a large study comparing the results of students taking advanced placement calculus courses in school, Ferrini-Mundy & Gaudard (1992) found that

> it is possible that procedural, technique-oriented secondary school courses in calculus may predispose students to attend more to the procedural aspects of the college course. (Ferrini-Mundy & Gaudard 1992, p.68)

Perhaps this can be solved by confronting the student with discrepancies between personal imagery and new conflicting data in an attempt to encourage a re-construction of knowledge on a more sophisticated level. Williams (1991) selected 10 students with concept images of the limit (such as 'gets close to, but does not reach') at variance with the formal definition and attempted over a series of five interviews to confront the student with new examples that conflicted with the old. There was little change:

> The data of this study confirm students' procedural, dynamic view of limit, that is, as an idealization of evaluating the function at points successively closer to a point of interest. The data also suggest that there are numerous idiosyncratic variations on this theme, some of them extremely difficult to dislodge. Given the complex nature of cognitive change, it is not surprising that the students in this study failed to adopt a more formal view of limit after only five sessions. (Williams, 1991, p.235)

It becomes apparent that firmly held concept images can prove notoriously difficult to dislodge, even when they conflict with the formal definition.

On the other hand, if formal ε–δ methods are taught from the start (as in the Greek curriculum) it can reduce the incidence of infinitesimal methods whilst having its own peculiar difficulties:

> ... the English have no formal instruction about limits on the real line, contrary to the Greek case. We find the English use 'infinitesimals" which often confounds the completion of a limit process, whereas the Greeks sometimes display difficulties in using formal symbolism and reasoning, suggesting that little insight is given by the strict definition. (Mamona-Downs, 1990, p. 69)

> The Greeks, although they did not use for example the "ε–δ" definition and preferred to use standard procedures, did seem to be able to accept a limit as a mathematical object rather than a "dynamic" process. ... However a few Greeks did show some conflicts between the dynamic and static approaches, suggesting that the first is more natural to their original intuition. (ibid., p.75)

Difficulties are revealed whichever approach is taken. It seems like a case of "Heads you lose, tails you don't win".

Perhaps the solution might be to lead on not to formal standard analysis, but to non-standard analysis. Sullivan (1976) gave evidence of the apparent success of such an approach. But although it continues to have its adherents, it has failed to take root on a wide scale. There are cognitive reasons why such an approach may seem on the surface to have success (just as the informal procedural approach often has procedural success). Even though the informal use of infinitesimals may seem to be closer to non-standard analysis, students' spontaneous beliefs are often inconsistent with non-standard theory too, for instance it is often believed that "nought point nine recurring" is the "last number less than 1" whereas if $x<1$ then $\frac{1}{2}(x+1)$ is also less than 1, so there can be no "last number less than 1" in either standard or non-standard theory.

An informal approach is therefore likely to involve factors which potentially conflict with any formal theory whilst a formal approach may prove too difficult a starting point, lacking insight.

2.2 Other difficulties in the calculus

Having considered the limit concept in some detail, we list some of the other difficulties students encounter in the calculus, each worthy of extended investigation, including:

- restricted mental images of functions,
- the Leibniz notation – a 'useful fiction' or a genuine meaning,
- difficulties in translating real-world problems into calculus formulation,
- difficulties in selecting and using appropriate representations,
- algebraic manipulation – or lack of it,
- difficulties in absorbing complex new ideas in a limited time,
- difficulties in handling quantifiers in multiply-quantified definitions,
- consequent student preference for procedural methods rather than conceptual understanding.

Restricted mental images of functions are not always seen as provoking a difficulty in elementary calculus particularly when the subject is seen as focusing on the differentiation and integration of standard functions given as formulae. Nevertheless it causes difficulties as soon as the student is faced by examples slightly beyond their experience, such as calculating $\int_{-3}^{3} |x+2|\, dx$ (Mundy 1984) or finding *a*, *b* such that

$$f(x) = \begin{cases} ax, & x \le 1 \\ bx^2 + x + 1, & x > 1 \end{cases}$$

is differentiable at 1 (Selden, Mason & Selden, 1989), then the students fare extremely badly. Unless students meet the concept of function in a broader context, such difficulties should be expected.

Difficulties in translating real-world problems into calculus formulation are part of the folk-lore of the subject (though there seems to be little cognitive research). Many examinations for calculus examinations focus on the symbolic manipulation rather than problem-solving (see, for example, the selection of examination papers quoted in *Calculus for a New Century* (Steen 1988, p. 179 et seq.).)

The Leibniz notation $\frac{dy}{dx}$ proves to be almost indispensable in the calculus. Yet it causes serious conceptual problems. Is it a fraction, or a single indivisible symbol? What is the relationship between the *dx* in $\frac{dy}{dx}$ and the *dx* in $\int f(x)\, dx$? Can the *du* be cancelled in the equation $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$? Giving a modern meaning to these terms that allows a consistent meaningful interpretation for all contexts in the calculus is possible but not universally recognised. On the other hand, failing to give a satisfactory coherent meaning leads to cognitive conflict which is usually resolved by keeping the various meanings of the differential in separate compartments ($\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ in differentiation and *dx* means “with respect to *x* in integration). This can only exacerbate conceptual chasm between the notation and any possible coherent meaning.

Difficulties in selecting and using appropriate representations are known to be widespread. Robert & Boschet (1984) reported that the students who were the most successful were invariably those who could flexibly use a variety of approaches: symbolic, numeric, visual. Dreyfus & Eisenberg (1986, 1991) report students’ reluctance to visual concepts in calculus. They give examples where visual representations would solve certain problems almost trivially, yet students refrain from using them because the preference developed over the years is for a numerical, symbolic mode of approach. Yet research shows that visual images *can* provide vital insights. However, it may sometimes prove difficult for students to link the global gestalt to a sequential deductive form of thinking.

Algebraic manipulation is the preferred mode of operation for many students. Wider access to calculus courses may have the effect of allowing students with less manipulative facility to take the course. In the UK, for instance, the latest school curricula require less algebraic manipulation. Students are beginning to take calculus with less facility in manipulating polynomials and with little knowledge of trigonometric formulae. As a result, routine manipulation in algebra can no longer be taken for granted.

Difficulties in absorbing complex new ideas in a limited time occur throughout college mathematics. Yet the difficulties in calculus are often acute. The concepts change their nature as time passes. A limit might be initially an intuitive process of "getting close", then an "epsilon-delta" definition, but then after a few theorems have been proved, the definition is suppressed and the theorems are quoted instead. Thus students at different stages of coping with this transition are faced with different meanings for the concept. Azcarate (1991) studied the changing concept of instantaneous speed, from a primitive ratio concept (distance/time), via an approximation idea, calculating the speed over an interval of decreasing size, to the notion of "instantaneous speed" represented as the gradient of the tangent to a time-distance graph. Interviewing students before, during, and after the teaching showed students at various stages of development during the teaching.

Limit			1 15 34 35 39 44 62 64 81 87 105
Approximation		1 15 35 44	21 83 93
Mixed Approximation/ Primitive ratio	39 87	21 34 39 64 81 93	
Primitive ratio	1 15 21 30 34 35 38 44 53 62 64 79 81 83 93 105	30 38 53 62 79 83 105	30 38 53 79
	First interview	Second interview	Third interview

Evidently, if students are attending lecture courses in topics which quickly change in sophistication of treatment, those left behind will soon experience great difficulties.

Multiply quantified definitions (For all positive epsilon there exists a delta such that...) are not part of informal calculus, but they become important once the theory is formalized. They put an intolerable strain on students. (On taking a sample of 12 students attending an analysis course in which the epsilon-N definition of convergence of a sequence was given and had been used in lectures for two weeks, *none* of the students could reproduce the definition from memory, although they could use various tests of convergence effectively, such as the comparison test, or the ratio test. When asked to complete the definition:

$$\lim_{n\to\infty} a_n = a \text{ means "Given } \varepsilon>0, \ldots\text{"}$$

responses included such things as "Given ε>0, as $n\to\infty$ so $a_n\to a$" and "Given ε>0, a_n gets within ε of a". None could quote the required definition "because it isn't used in the examples we have to do …".)

Yet *some* students win and succeed in calculus. But how? The evidence (e.g. Robert & Boschet (1984)) suggests that those who tend to succeed in an analysis course are those who are more versatile in using different representations – using visual, numeric, or verbal cues, whichever proves the more appropriate at a given stage. To do this they need flexible knowledge. They have, or they develop, an ability to cope with the complexity of the subject by turning, almost intuitively, to the representation that will prove to be useful in the particular cause. It may be that calculus works for those more able students who can think flexibly and fails for those who look for more procedural guidance to get them through their problems.

3. How can students' learning be improved?

Various hypotheses have been put forward suggesting ways in which students' understanding might be improved, including:

- active learning
- build up intuitions suitable for later formalizations
- computer graphics
- computer programming
- symbol manipulators

Active learning by the students, instead of passive reception of lecture material, was advocated by Cummins (1960) over 30 years ago, in an "experience-discovery approach" using "materials ... to develop understanding in the use of some of the fundamental ideas before these concepts were subjected to critical discussion", with "a series of study-guide sheets ... by which students, either independently or with the help of class discussion, could arrive ... at some methods and facts of the calculus." Experimental students scored at exactly the same level as control students in traditional skills, but significantly higher on a questions requiring conceptual understanding. For example, 25 out of 38 experimental students were successful at giving an explanation for the quotient, product or chain rule for differentiation compared with only one out of 24 control students. Eighteen out of 38 experimental students could explain the logical connection between differentiation and integration, compared to none from the control group.

More recently, Alibert (1988), Case (1991), Farmer (1991), Dubinsky (1992) and others advocate student discovery over reception learning from lectures. Dubinsky's approach is a particularly sophisticated one, in which students cooperate in groups to reflect on programming constructions in a computer language (ISETL) designed to support the growth of mathematical thinking. For example, functions may be specified as procedures and then conceived as objects that can be used as inputs to other functions. This "encapsulation" of the function process as an object he sees as a fundamental step in the learning process. While procedural approaches focus on processes that can be carried out in specific circumstances to solve problems, encapsulation allows a process to be conceived both as a process that can be carried out and as an object that can be mentally manipulated on a higher level. This is designed to enable the individual to think in a flexible and powerful manner. His approach is one of the few in the Calculus Reform Movement that is cognisant of cognitive difficulties and attempts to resolve them. Given the nature of student difficulties and student attitudes mentioned with respect to the limit concept, it is clear that the deep cognitive obstacles will not be solved by the students alone without the action of an external mentor providing them with activities appropriate for reflective (re-)construction of concepts.

Many computer experiences are designed to **build up intuitions for later formalizations**. For instance **interactive computer graphics** may be used to help students *see* concepts, like local straightness through zooming in on a differentiable curve to *see* it locally as a straight line with a visible gradient. My own approach – in the knowledge that the limit concept causes many cognitive difficulties – was designed to study the idea of limit *implicitly* through zooming to lay foundations for later formal ideas (Tall & Sheath, 1983; Tall, Blokland & Kok, 1990). I saw versatile thinking as an important focus of learning, linking the visual ideas of local straightness, looking along a curve to *see* the changing gradient, and drawing the graph of the gradient function linked with numerical and symbolic computations of the gradient.

I soon realised that graphics alone were unsatisfactory (Tall 1986), and saw the need for versatile movement between representations. Graphics give *qualitative global insight* where numerics give *quantitative results* and symbolics give *powerful manipulative ability*.

The use of the three representations (graphic, numeric, symbolic) throughout calculus is a focus of the Harvard approach (Gleason et al 1990, Hughes Hallett, 1991).

> One of the guiding principles is the 'Rule of Three,' which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and where students see each major idea from several angles. (Hughes Hallett 1991, p. 121)

This approach is based as much on mathematical beliefs as on cognitive growth. My own sense is that mathematicians selectively focus on the *most useful* representation, so that *versatile movement between representations* is more important, and cognitively more natural, than focusing on all three representations at once.

Computer programming is seen by some as introducing extra problems that increase the difficulty of the mathematics, and by others as a constructive activity that allows the student to learn by telling the computer how to carry out the required algorithms. Sometimes this is an additional activity, as in the Harvard programming in True BASIC, where students were given programming activities to complement their symbolic paper and pencil manipulations. Sometimes this is an integrated activity aimed at enabling the students to program the mathematics itself, as in the work of Dubinsky and Schwingendorf (1992) in the language ISETL.

Initial evaluations of these and other projects are beginning to be published. For instance, Cowell & Prosser (1991) report a mixture of "good and bad news" in the use of True BASIC.

> The students largely agreed that the computer assignments were well integrated with the rest of the course, and that learning the necessary programming was easy, but they disagreed that the computer enhanced their interest in the course material, they disagreed that the computer should be dropped and they were divided on whether the computer assignments were a valuable part of the course:

	Disagree			Agree	
	SD	D	N	A	SA
The computer assignments were well integrated with the rest of the course	3.0	17.8	13.3	50.4	15.6
Learning how to program in True BASIC for this class was relatively simple	3.0	23.7	25.9	36.3	11.1
The computer assignments enhanced my interest in the course material	19.1	36.0	23.5	16.9	4.4
I would have preferred Math 3 if the computer were not used at all	12.6	28.1	23.7	17.0	18.5
Overall the computer assignments were a valuable part of this course.	5.1	22.8	28.7	33.1	10.3

(Cowell & Prosser, 1991, pp. 152, 153)

Comparing the scores on examinations with scores on the previous non-computer course showed virtually identical median and quartile scores. However, note that the computer tasks were added to the curriculum basically for mathematical rather than cognitive reasons.

Dubinsky's approach on the other hand is designed specifically to encourage students to make the necessary mathematical constructions and to reflect on them to gain meaningful understanding. In a comparison with students following a standard course, he was able to show that students could perform as well or better on traditional tasks, but considers such comparisons of little value because they are coloured by implicit beliefs as to what constitutes success, in the case of this comparison, traditional manipulation.

Symbolic Manipulation Software is now being used more extensively in teaching calculus, from courses based on software notebooks that include symbol manipulation and graph-drawing in *Mathematica* (Brown, Porta & Uhl, 1990, 1991a), to laboratory workshops added to standard courses in *Maple* (e.g. Muller, 1991), and research projects (e.g. Palmiter, 1991). Brown, Porta & Uhl (1990, 1991b) report sophisticated student usage of the symbol manipulators to solve problems, although they also admit an alarming 30% dropout in the first course which fortunately has not been repeated to the same extent elsewhere. Muller (1991) is one of the projects that has included and published evaluations of the course as it has proceeded. After a first course (1988) of enthusiastic volunteers, two successive compulsory courses (1989, 1990) still show some gains, though at a more realistic level:

		1988	1989	1990
Would you recommend laboratories to a friend taking the course next year?	yes	86	25	41
	no	2	46	32
Student ranking of laboratories versus other modes of learning	high	29	7	11
	low	41	73	68
Student assessment of laboratories as a learning aid	high	47	12	15
	low	27	67	56
Confidence of being able to succeed in the course	high	52	57	61
	low	15	12	12
Enjoyment in doing mathematics	high	41	44	45
	low	27	15	15

An important factor in continuing with the Maple experiment is the significant reduction in student withdrawal rates and failure rates.

Palmiter (1991) used the symbolic software MACSYMA to teach one cohort of students a first course in integration for five weeks whilst a parallel cohort studied a traditional course for a full ten weeks. The MACSYMA students used the software to carry out routine computations whilst the traditional students were taught the techniques. Both groups took a conceptual examination and a computational examination at the end. The conceptual examination was taken by both groups under identical conditions, the experimental students were allowed to use MACSYMA in the computational examination but had only one hour whilst the control students were given two hours. The results showed a significant improvement in the students using the computer over those without:

	Class			
Examination	MACSYMA	Traditional	T^2	$p<?$
Conceptual	89.8 (15.9)	72.0 (20.4)	1.20	$p<0.001$
Computational	90.0 (13.3)	69.6 (24.2)	0.92	p<0.001

This gives clear indications that a “student plus manipulation tool” can be more successful in conceptual and computational tasks that a student working in a traditional manner.

The Calculus Reform Movement is in the first stage of enthusiasm, vigorous attack and mutual competition. What is remarkable is the small extent to which the work of the movement interrelates to the research on student difficulties which has been discussed earlier. At this stage it would be helpful to have a period of impartial reflection and evaluation. A significant difficulty in this process is the wide variety of goals set by the different participants – by what criteria is the success of the operation to be considered?

4. Future Developments

Where do we go from here? In the short term we have the opportunity to discuss what evidence we have for student difficulties. However, the idea of looking for difficulties, then teaching to reduce or avoid them, is a somewhat negative metaphor for education. It is a physician metaphor – look for the illness and try to cure it. Far better is a positive attitude developing a theory of cognitive development aimed at an improved form of learning.

The Calculus Reform Movement began from a general atmosphere of dissatisfaction rather than any clear empirical base. Given that it is based on a number of very large projects, it is natural that the initial activities are based on enthusiasm for the positive gains that are envisaged. Serious money is at stake in terms of selling course materials widely. It is therefore not surprising that independent critical evaluation has not featured widely in the initial stages. Now the euphoria of the beginnings of the movement have had its day, a more sober mood of empirical investigation and objective evaluation will become more appropriate.

Progress in the next four years will profit from:

- More empirical evidence
- more reflection on the evidence
- better theories of learning appropriate for practical teaching

It is the purpose of this working group to reflect on the difficulties encountered by students of differing abilities and experience, to obtain unbiased empirical evidence to build and test theories of learning to enable more fruitful learning experiences for students in the calculus.

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