The Transition from Arithmetic to Algebra: Number Patterns, or Proceptual Programming?

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Introduction

Algebra is often seen as "generalised arithmetic" and approached through number patterns. The England & Wales Mathematics National Curriculum implicitly supports this belief by making the initial algebra attainment targets a search for pattern before letters are introduced to stand for numbers. Information technology offers an alternative strategy – giving meaning to letters through programming.

In comparing the two approaches we wish to focus on the possible cognitive difficulties involved in each, and the nature of the thinking that the child may bring to algebra from arithmetic. An analysis of arithmetical thinking (Gray & Tall, 1991; Gray & Tall, in press) shows a spectrum of interpretations of arithmetic symbols. A symbol such as 3+2 can represent both a process "add three and two" and also the concept produced by that process "the sum of three and two". The manner in which the notation functions dually as process and concept leads us to formulate the notion of a *procept* – the ambiguous use of notation to stand either for a process or the object produced by that process. We shall see that those who are most successful with arithmetic treat the notation in a manner which takes advantage of this ambiguous flexibility, but the less successful are more likely to see arithmetic in terms of procedures.

Children with different attitudes to notation are likely to approach algebra in radically different ways. Algebraic notation such as $3+2x$ also has the ability to represent both process ("add 3 to 2 times x") and concept (the expression "$3+2x$", which is the result of the operations of combination). The child who tends towards procedural thinking will be faced with a dilemma. The process "add 3 to 2 times x" can only be carried out if x is known, and if it *is* known, why not just use arithmetic, why complicate things with letters? For the proceptual thinker for whom notation is more flexible, the notation can represent a *potential* process, which could be carried out when x is known, but is also more likely to be conceived as an object that can be manipulated mentally.

It may be hypothesised that those children who can confer such a flexible meaning on the notation are more likely to be successful in algebra. However, the flexible meaning, whilst being a natural vehicle

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of thought for some, can prove to be highly complex for others who are confused by the multiple meanings of the symbolism. To simplify the initial teaching we decided to have activities which focus on different aspects at different times.

To assist in conferring a flexible meaning, a curriculum programme was devised which uses the computer for programming exercises to give the symbolism meaning and a corresponding practical activity in which the children act out the internal process represented by the symbolism (Tall & Thomas 1991). The programming activity instructs the computer to carry out the process. This enables the child to focus on the results of the process rather than the internal procedure. For instance, **2*a+b** will give the same result as **b+2*a**, regardless of the values of **a**,**b**. The programming allows the expressions to be given a meaning and for equivalent expressions to be studied which have different internal processes, yet always give the same results. It eventually concentrates more on the *concept* than the process. The practical activity meanwhile focuses more on the *process*. The combination of the two activities is designed to give meaning to algebraic notation both as process and concept, and to give a more flexible foundation for algebra.

Experiences prior to algebra

To gain a better idea of why children have difficulty with algebra, we should begin by looking at their experiences prior to the stage when algebra is introduced. This experience concerns several years of arithmetic building on counting and the number concept.

When we consider the meaning of the symbol “4+5”, a child at different stages of development might respond in a number of different ways, including:

1) “count-all” where two numbers, say 4 and 5 are added by counting 4 objects, then 5 objects, putting all the objects together and counting all them to find the sum,
2) “count-on” where the first number 4 is treated as an entity and the child counts on 5 more in the number sequence,and the variant “count-on-from-largest” where the largest number, 5, in this case is taken first to reduce the work in counting on the smaller number 4,
3) “known fact” where the sum 4+5 is remembered as 9,
4) “derived fact”, where the sum 4+5 is not known, but 4+4 is known to be 8, and so 4+5 is “one more”, i.e. 9.

Of these, (1) and (2) evoke different *processes*, (3) evokes a *concept* and (4) uses a higher order process to decompose and recompose type (3) concepts (perhaps with some counting as well).

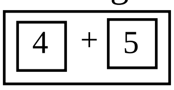
The symbol “4+5” can mean either the process “add four and five together” (which can be performed by various procedures) or the concept “the sum of four and five”.

This dualism of meaning for symbolism evoking either process or concept is a widely occurring phenomenon in mathematics. It also involves an ambiguity, in that a given symbol can either evoke a process or a concept depending on how it is interpreted. The process of addition gives a method of computation to get an answer, but the concept of sum can be manipulated at a higher level to solve even more complex problems, as 4+4 is used in (4) and replaced by 8 to give the solution of the related problem 4+5.

A good mathematician (of any age) uses this duality and ambiguity in a flexible way almost intuitively, often switching from one to the other without realising it. But the child at a given stage of development may only be able to cope with some, or even none, of this flexibility.

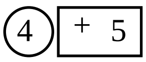
If we look at the methods (1), (2) and (3), we see that the individual symbols are treated rather differently. Using a circle to contain a symbol conceived as an object, and a square or rectangle for a symbol conceived as a process, the three methods may be written thus:

- count-all as a process of counting 4, then counting 5 then counting all:

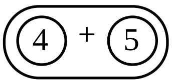
  (count four, count five, count-all)

- count-on, starting from 4, to count-on 5:

  (number 4, count-on 5)

- know that 4+5 is 9:

  (number 4, number 5, result 9).

We now begin to see more clearly what this previous experience is telling children: that notation represents a *process* to do, which can be progressively compressed to be manipulated as a mental *object*.

This duality, even ambiguity, of seeing 5 either as a counting process) or as a number concept is something which good mathematicians do almost automatically, often without being conscious of it. Once they have compressed the notation to a more sophisticated meaning, it may become very difficult for the adult without reflection to be aware of the difficulties facing the child. I would hypothesise that children with this flexibility have the mental tools which are more attuned to what is needed in algebra. As they develop them intuitively, and the teacher has them intuitively, such children are seen to be successful at algebra, although the underlying reason for success may not be explicitly understood by either teacher or child.

The cognitive difficulty of approaching algebra through number patterns

Algebra is usually conceived as "generalised arithmetic". As arithmetic is intuitively seen to be "easier" than algebra, it may seem natural to introduce algebraic ideas by using gneralised arithmetic ideas through number pattern. A typical problem might be:

continue the following sequence: 1, 4, 7, 11, ...

The human animal is a great pattern detector. A child may soon sense the rhythm of such a sequence to see it is "add 3 each time" and can readily continue to compute successive terms:

1, 4, 7, 11, 14, 17, ...
"What is the next number?" "Add 3 to get 20."

But if algebra is introduced as generalised arithmetic, is this the way to proceed? The subtle fact is that the pattern that the child has spotted is "add 3". It is a *recursive* pattern, not one given by an algebraic formula!

Recursion is something that is not part of the traditional curriculum, although the new technology may make them a more natural focus of mathematical activity. Spreadsheets are able to represent both recursion (by replicating an appropriate formula between successive entries), algebraic formulae, and other methods such as iteration (Healey & Sutherland, 1991).

Many natural pattern-spotting activities note the rhythm of the relationship between one number and the next, rather than a formula for the *n*th number. One investigation that has become a favourite in British secondary schools is to give a child a layout of consecutive numbers, say 8, 9, 10, 11, ... and ask the following question:

> Square one of the numbers, say 10 times 10 is 100. Multiply the number before by the number after, say 9 times 11, and see what happens. The answer is 99. Try this with other numbers. Can you see a pattern? Can you predict if this will always happen?

The child may *see* the pattern that the second product is one less than the square. But to realise that this pattern has an implicit reason requires an argument equivalent to splitting up the product 9x11 as

$$(10-1)\times(10+1)$$

and multiplying it out. Is this really the way to move into algebra? Is this the way to develop a version of the formula for the difference between two squares?

$$(n-1)\times(n+1) = n\times n + n\times 1 - 1\times n - 1\times 1 = n^2-1$$

This is sometimes done visually by laying out a rectangular with 9 rows each with 11 objects, and moving the last object in each row to leave 9 rows with 10 objects and 9 over which can be put in a new row, one short of a 10 by 10 array. This gives a geometric representation of (10–1)x(10+1)=10x10–1.

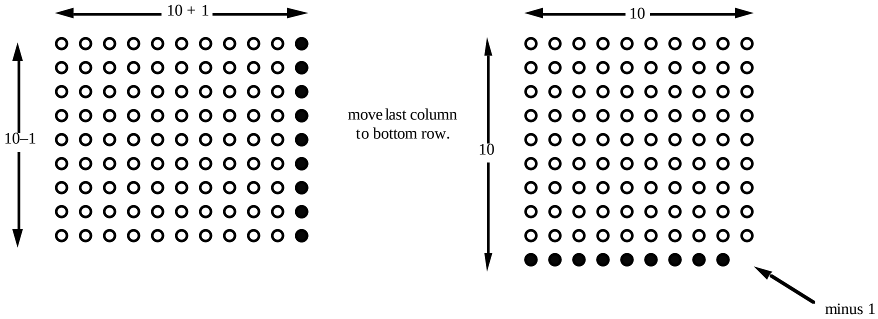


Figure 1 : A geometric interpretation of a general numerical relationship

The geometrical pattern spotting is again an interesting exercise, but is not obvious to carry out without appropriate guidance.

The final example of pattern spotting is the idea of paving round a rectangular pond, say 2 feet by 3 feet with 1 foot square paving slabs. How many slabs are needed? Clearly 2 along each short side making 4, 3 along each long side making 6, and 4 in the corners. By varying the length and width of the pool, one may hope to see the pattern that the number of slabs is twice the width plus twice the length, plus 4 in the corners, then move from this verbal description to an algebraic one.

$2 \times w + 2 \times l + 4.$

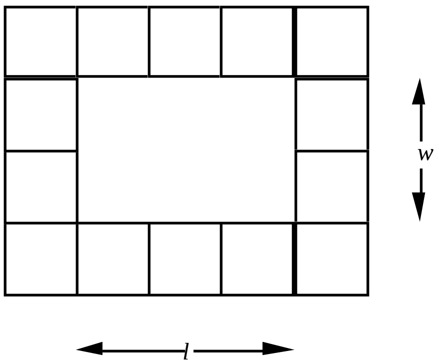


Figure 2 : An algebraic solution of a practical problem

Suddenly, with less able pupils particularly, this last stage introduces a discontinuity that seems impossible to bridge. Why is it written like this? If we know *w* and *l* (say 2 and 3), why don't we write it as 2x2+2x3+4, which we *may* be able to work out (though even this is difficult for many children to scan).

Then again, what exactly is *w*? What is *l*? Are they *lengths* or the *number of slabs* or do they stand for the slabs themselves? Children's previous experience of letters after numbers might be in terms of units, say 15*p* for 15 pence. Here the *p* does not stand for a number in the same way as it does in algebra. For instance a sentence from the National Curriculum (D.E.S., 1991) states:

> Write the total cost, *c* pence, of *n* cakes as $c = 15 \times n$ (or 15*n*) where the cost of one cake is 15*p*.

Note the different meanings here of 15*n* and 15*p*. What are children to make of them? And how are they to cope with $15\frac{1}{2}$, 151, 15*x* ?

Children who have so far seen arithmetic symbolism as representing a process that can be carried out by an arithmetic procedure suddenly find this "universal law" is violated. The expression with letters cannot be worked out unless the values are known and if the values are known, why use algebra?

There is an impasse.

It is a chasm which more able children – with an intuitive flexible use of notation as process and product – are able to span. A relatively small number of children are therefore privileged to enter the domain of algebra. But it closes the door for many more who see algebra as an unnecessary and difficult irrelevance.

In a professional course for student teachers, I posed the problem of introducing algebraic notation to children. A student responded "I always do it the way I was taught by telling the children about two apples and three bananas". Other students immediately concurred with this suggestion of giving meaning to 2a+3b in this way. It helps manipulation. 2*a*+3*b*+3*a* is 2 apples plus 3 bananas plus 3 apples which is 5 apples plus 3 bananas, or 5*a*+3*b*. But this meaning is inflexible and fails when this interpretation does not apply. What is 2*a*+3*b*–3*a*? Can it be –1*a*+3*b*? Can you have "–1 apples"? And what if the child thinks of 2*a*+3*b* as "2 apples and 3 bananas", then conceives it as "5 apples and bananas" and writes 5*a b* ?

Approaching algebra as "generalised arithmetic" through patterns has a number of difficulties, and the degeneration of algebra into "apples and bananas" symbolism to get over the initial stages is severely flawed. Using a computer offers a more promising approach.

Meaning of algebraic symbolism through programming

In BASIC (or any other suitable language), the command

```
a=3
```

followed by

```
PRINT a+1
```

gives a predictable response. If a child does not predict the answer "4", then the command can be carried out to see what happens, and consideration of similar commands such as

PRINT a–1

will begin to see what will happen.

A few examples of this sort will allow children of a wide range of ability, *even some of the very slowest learners*, to be able to predict that **PRINT a–1** for a=3 will give **2**.

By this method the child will learn to appreciate that, if the variable stands for a specific number, then an algebraic expression will give a recognisable product. In this way it is behaving like arithmetic notation – it is something that can be calculated.

Tall & Thomas (1991) report a research project which used this approach to teach algebra to mixed ability groups of children. It showed significant improvements in the experimental children's understanding of the notation.

We will return to this experiment shortly. However, before we do so, the theory that we have begun to unfold about the way that children grow in sophistication in appreciating arithmetic notation as process and concept now allows us to interpret the whole experiment in a new and illuminating way. So before proceeding to discuss algebra, let us look at a development of the theory that gives insight into the thinking processes of the child.

The dual meaning of symbolism as process and concept

The meaning of symbolism as process and concept in arithmetic is but one of a few instances of this phenomenon throughout much (but not all!) of mathematics.

Examples pervade arithmetic, algebra, trigonometry, calculus, analysis, and abstract algebra, including the following (mainly quoted from Gray & Tall, to appear),

- The symbol 4+5 represents both the process of adding through *counting all* or *counting on* and the concept of *sum* (4+5 is 9),
- The symbol 4x3 stands for the process of repeated addition "four multiplied by three" which must be carried out to produce the product of four and three which is the number 12.
- The symbol 3/4 stands for both the process of division and the concept of fraction,
- The symbol +4 stands for both the process of "add four" or shift four units along the number line, and the concept of the positive number +4,

- The symbol –7 stands for both the process of "subtract seven", or shift seven units in the opposite direction along the number line, and the concept of the negative number –7,
- The algebraic symbol $3x+2$ stands both for the process "add three times x and two" and for the product of that process, the expression "$3x+2$",
- The trigonometric ratio sine = $\frac{\text{opposite}}{\text{hypotenuse}}$ represents both the process for calculating the sine of an angle and its value,
- Speed may be calculated as a *ratio* (distance divided by time) or a *rate* (speed, say in miles per hour),
- The function notation $f(x)=x^2-3$ simultaneously tells both how to calculate the value of the function for a *particular* value of x and encapsulates the complete concept of the function for a *general* value of x,
- An "infinite" decimal representation $\pi=3.14159\ldots$ is both a process of approximating π by calculating ever more decimal places and the specific numerical limit of that process,
- The notation $\lim_{x\to a} f(x)$ represents both the process of *tending to a limit* and the concept of the *value of the limit*, as does $\lim_{n\to\infty} s_n$, $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$, and $\lim_{\delta x\to 0} \sum_{x=a}^{b} f(x)\,\delta x$.

Given such an all-pervading phenomenon, it is quite amazing to realise that this has not been blessed with a name. My colleague Eddie Gray and I realised that if we gave the phenomenon a name then we could begin to talk about it.

The idea of a procept

In Gray & Tall (1991) we formulated the following definition:

> We define a *procept* to be the amalgam of process and concept in which process and product is represented by the same symbolism. (Gray & Tall, 1991)

Later in Gray & Tall (to appear) we refined this definition to allow the notion of procept to operate flexibly in the way it is observed operating in successful mathematicians (of all ages). Here we defined:

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.

we then extend the definition as follows:

> A *procept* consists of a collection of elementary procepts which have the same object.
>
> In this sense we can talk about the *procept* 6. It includes the process of counting 6 as well as the number concept 6; it also includes a collection of other representations such as 3+3, 4+2, 2+4, 2x3, 8–2, etc. All of these symbols are considered by the child to represent the same object, though obtained through different processes. But it can be decomposed and recomposed in a flexible manner. (Gray & Tall, to appear)

The procept of number for a successful mathematical thinker allows a number like 12 to represent a whole variety of different forms: 10+2, two sixes, four threes, and so on. It is this flexibility which gives the child a generative system not only for giving mutual support for already known facts, but also to derive new facts from old with very little effort.

Children who are successful develop a flexible way of thinking in arithmetic which allows them to decompose and recompose procepts in a way which gives them great power. Those who are much less successful see arithmetic more as counting procedures. Subtraction for the more able is no more complex than addition, for if 4+2 equals 6, then 6 take away 4 must be 2. The additive triple 4,2,6 gives a proceptual structure which allows 6 to be composed and recomposed so that subtraction is just a different way of viewing addition.

The less successful child tends to be more procedural. The reverse of the *process* of addition is the *process* of subtraction. If addition is "count-on" then subtraction is "count-back", so 19–16 is performed by counting 16 numbers down starting one below 19! This is a far harder task than the use of derived facts. It shows why "slow learners" are so likely to fail. *They are performing significantly harder mathematics*.

Children are faced with the transition to algebra must build on their previous experiences, particularly in arithmetic. They bring very different views of symbolism and are likely to interpret algebraic symbolism very differently!

Procepts with built-in procedures for computation

Arithmetic procepts such as 5+3 have a built in procedure for computation: for instance, "count-on 3" as "6, 7, ***8***". Algebraic procepts only have a *potential* procedure for computation. To a successful mathematician the expression $2+3x$ means both the process "add 2 to 3 times x" and also the concept, the expression "$2+3x$" that can be treated as a procept and be decomposed and recomposed or manipulated as an object in part of a larger expression. The process within the expression can only be carried out when the value of x is known. Thus it represents a conflict with all previous experience. Some children, already more flexible with arithmetic, may be able to cope with "add two to three times *x*, *whatever x is*", and even think of the result as an object. But those children who see symbolism more as a procedure to be carried out have no hope of making any sense whatsoever of algebraic notation. It represents a process that cannot be performed, a complication that they don't understand. If we don't know *x* we can't do the sum, if we do know *x*, why complicate matters using letters, why not do ordinary arithmetic?

A cybernetic approach using a computer

The programming and evaluation of expressions can be of great value in giving meaning to algebraic symbolism. Communication between individuals involves all sorts of non-verbal meanings and shades of willingness to communicate, or misconceptions about what the other is saying. If one "talks" to a computer, by typing in commands, and the computer responds in a predictable way, then this very predictability can give meaning. If *x* has the value 3, then one soon gets to the position that the command **PRINT x+2** will give **5**. And it will happen every time.

In Tall & Thomas (1991), we extended Skemp's notion of building and testing concepts, to allow for interaction with the computer. Whilst interaction with concrete objects required the individual to interpret what was going on, interaction with a computer programmed to respond in a predictable way offered a *cybernetic* system in which the individual could build and test concepts first by observing what happens and then predicting and testing what happens. This offers a more secure method of building and testing.

Programming to focus on the concept of expression

By typing in the expressions:

```
x=3
PRINT x+2
```

the *computer* is carrying out the process. The child *sees* the result of the process. At this stage the child may think through the process to confirm what the computer is doing.

If the child considers the meaning of

```
PRINT 2*(x+1)
```

then the meaning of the brackets can be built and tested. For *x*=3, the result 8 can only be got by adding the value of *x* to 1 before multiplying by 2. "Do operations in brackets first".

If two different-looking, yet equivalent expressions are evaluated, such as

```
x=1
PRINT  2*(x+3)
PRINT  2*x+2*3
```

the child can note that the two different methods give the same result.

The program

```
10 INPUT x
20 PRINT 2*(x+3)
30 PRINT 2*x+2*3
```

will give the same results *whatever the value of x that is input*. Now each time it is run, the child will see the same numbers. Checking the computation recedes from the child's focus of attention. Instead it is possible to focus on the fact that the two expressions always give the same output. In this way, equivalence of expressions can become the focus of attention instead of the different procedures which are used to carry out the computations.

The Cardboard Maths Machine to focus on the process

In a separate activity the children took part in a game which involved playing the part of the computer in performing the calculations, so that, at a different time, they could concentrate on (a model of) the inner workings of the computer. Whilst the programming activity involved the child focusing on the expression itself and the values it gave, the game focused primarily on the *process* of evaluation.

The activity involved a *cardboard maths machine,* which consists of just two large sheets of cardboard and some smaller cards marked with letters and numbers. One piece of cardboard acted as a screen and programming instructions were placed on the screen and then carried out by members of a group of pupils. The other piece of cardboard – placed a short distance away – had a number of rectangles marked on it which represented computer stores. Each could be labelled by a letter placed above the store and each labelled store could have a number placed. Thus to carry out the instructions

```
A=1
B=A+3
PRINT  B+2
```

involves marking a store with the label A and placing the number 1 inside, then labelling another store B, looking inside the store A to find the value 1, adding 3 to get 4 and placing it in the store B, and finally looking in the store B, noting the number 4 inside, adding 2 and placing the number 6 back on the sheet of cardboard representing the screen (figure 3).

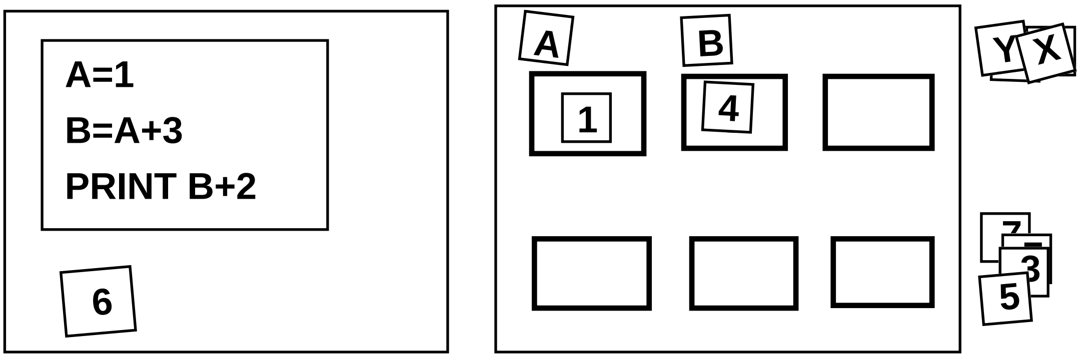


figure 3: The cardboard maths machine

The Computer Maths Machine to focus on standard notation

A further activity involved the use of a piece of software which allowed the use of standard algebraic notation (including implicit multiplication) to compute the values of one, or two, expressions for given input values of letter variables.

Here the software simply carried out the computation of the *value* of the expression, allowing the child to focus mainly on the output, and whether two expressions gave the same, or different output values for different values of the variables (figure 4).

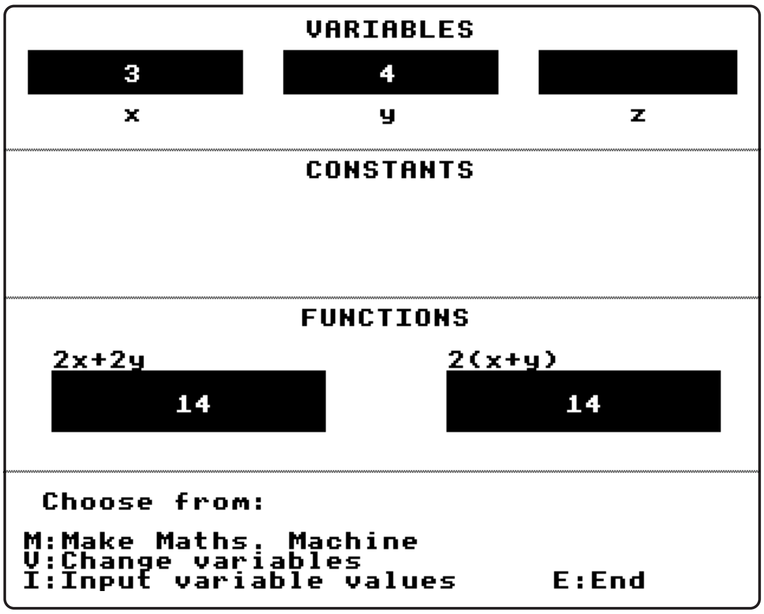


figure 4 : The Algebraic Maths Machine

Working with the algebraic maths machine clearly focuses on the *product* of computing the numerical values of algebraic expressions, just as the cardboard maths machine involves the children acting out the *process*.

Thus, at different stages, different aspects of process and object were selectively given an appropriate focus in order to produce a more versatile form of learning.

Empirical evidence for success

The results of empirical studies shows significant improvement overcoming cognitive difficulties and in acquiring a more versatile insight into the meaning of the algebraic symbolism (Tall & Thomas 1991). What was particularly clear was that those who were successful with the computer activities had a different image of the meaning of the symbolism from those with a traditional approach.

In interview, children who had used the computer showed a more meaningful use of expressions which enabled them to be composed and decomposed in a flexible way. For instance, when faced with the problem of solving the equation:

$$2p–1=5,$$

a proceptual approach might be to note that if $2p–1$ was 5, then $2p$ had got to be 6, so p must be 3, whereas a procedural approach would be to "add 1 to both sides, to get $2p=6$" then "divide both sides by 2 to get $p=3$".

One of the most striking differences occurred when children who had solved this problem were faced with

$$2s–1=5.$$

Typically in interview (Tall & Thomas 1991), control children were either unsure of the relationship between the two equations:

> Pupil C1 : *s* could be 3 as well
>
> Pupil C4 : They could both equal 4

or they would need to go through the process of solution yet again to find that *s* is also 3.

> Pupil C2 : Well what I have put is 2*p* equals 6 and 2*s* equals 6,
>
> Pupil C5 : 2s...add the 1 and 5, 6 er 2 and 2, 6, 3 times, so *s* is 3 as well.

They may be aware that the solutions are essentially the same, but they feel that solving an equation involves going through a procedure, which is essential to do to get it right, even though in this case the procedure was easy because it had been in essence done before. Typically the experimental children were more likely to say that it is "the same equation":

> Pupil E1 : I can say that *p* and *s* have the same value...its the same sum.
>
> Pupil E5 : The same. Just using a different letter.

Reviewing the data in the light of the new theory of procepts, it seems that the more successful children were more likely to have a flexible proceptual view of the symbolism that they could manipulate more successfully with meaning, whilst the control children were more likely to feel the need to use approved procedures to solve the equations.

In practice, the *procedure* for solving equations was emphasised in the teaching of all children, rather than the proceptual nature of the equation, that it was the same concept represented by different algebraic processes. As a result most children were more likely to give a procedural response, although the experimental children showed their more flexible knowledge in other ways, for instance, through recognising the fact that the actual letter used in the equation was irrelevant. Proceptual knowledge could surface in the middle of procedural activities. For instance, one child began solving

$$3x - 5 = 2x + 1$$

by rearranging to get

$$3x = 2x+6$$

and then suddenly saw that the extra x on the left must balance the extra number on the right, so x is 6. By using flexible cues again the proceptual thinker can see through a problem where a procedural thinker might need to follow through a formal procedure.

Data which we do not have, which would be interesting to seek in a future experiment is the relationship between the child's earlier proceptual/procedural view of arithmetic and their subsequent proceptual/procedural view of solving equations. It is natural to hypothesise that those with a flexible proceptual view of arithmetic are more likely to have a flexible proceptual view of algebra, and that this would be improved significantly by the use of the computer approach.

The role of procepts in arithmetic, algebra and calculus

We have seen in arithmetic the great difference between the proceptual approach of the more successful and the procedural approach of those who are less likely to be successful. We hypothesise that this will have a significant effect on the pupil's success in algebra. Indeed we hypothesise that children who have a procedural view of symbolism are likely to have limited success of a procedural kind in algebra, and are more likely to fail in the long-term in algebra. We hypothesise that those children who are already treating arithmetic symbolism in a flexible proceptual way are more likely to be able to cope with the duality of the symbolism in algebra.

The one piece of evidence in our earlier research is one that we did not fully understand when it first surfaced. Children were asked:

Is $\frac{6}{7}$ the same as $6 \div 7$?

76% of the experimental group responded “yes” but only 44% of the control group. Using our current theoretical perspective we can suggest that this essentially concerns the proceptual nature of the fraction $^6/_7$. Comments from the control group indicated that the two were not the same because

“ $\frac{6}{7}$ is a fraction, 6÷7 is a sum ”.

This reveals the perception of 6÷7 as a *process* involving value-operation-value rather than as a single entity produced by this process. This was not such a problem for the computer group. This may be interpreted as suggesting that the experimental group were more likely to have developed a proceptual view of arithmetic notation as a result of the computer treatment than those following the traditional approach.

What we now see is the fundamental role of the student’s perception of notation. In arithmetic there is a spectrum of interpretation from the successful flexibility of the proceptual thinker to the more limited success of the procedural thinker. But in arithmetic the procepts have a related internal procedure (e.g. counting) which allows the value of an arithmetic expression to be calcutlated.

In algebra the notation again functions dually as process and concept, but here the process is only *potentially* successful. It is not possible to evaluate an expression numerically without knowing the numerical values of the variables. This is likely to produce an obstacle to learning which has been called “the expected answer obstacle”, or the “lack of closure” obstacle. It has long been known that many children are uncomfortable with algebraic symbols which cannot be computed.

It may be hypothesised that this obstacle will be more serious to procedural thinkers in arithmetic than to proceptual thinkers, although even the latter may find it difficult when it is first met.

I propose that a procept which has a built-in procedure for computation is called an *operational* procept, following Sfard (1990). A procept as met in algebra, which only becomes operational when appropriate values are substituted for variables will be termed a *template* procept. The template terminology is becoming popular in computer science where a template contains place-holders for variables. Other template procepts include

$$\int_a^b f(x)\,dx$$

which can only be computed when specific values are substituted for *a*, *b* and a specific function is substituted for *f*(*x*).

Students have difficulties when they meet limit concepts in the calculus. These too are often procepts where a notation such as $\lim_{n\to\infty} s_n$

represents both the process of s_n approaching a limiting value and also the value of the limit itself. Cornu (1981) has remarked on how this causes difficulties with students because the limit can no longer be computed by simple arithmetic, but often needs to be attacked by indirect means using established theorems. I term such a phenomenon a *structural* procept, againg using an adjective popularised by Sfard (1990), although I must take the responsibility for the meaning given here. Note that a structural procept can also be of the template variety if it includes variables or place-holders for general inputs.

At this advanced level in calculus and analysis the template nature of procepts is less likely to cause difficulty. But the structural nature, which conflicts with previous experience of the student that a procept is in principle computable by a simple procedure (at least potentially, once variables are substituted in), violates the implicit belief in the implied operational nature of procepts.

Thus we see that proceptual nature of mathematical notation at each stage implicitly obeys certain rules which may be violated at later stages. These implicit beliefs constitute genuine epistemological obstacles in the cognitive growth of the subject.

The proceptual divide

Eddie Gray and I contend that the spectrum of interpretation of proceptual symbolism – from procedure to be carried out, to flexible procept dually representing either process or resultant object – leads to a spectrum of success and failure in mathematics (Gray & Tall, to appear). In analysing the nature of the procedures employed in arithmetic, we often found that the procedures were far harder than flexible proceptual methods. For instance, in computing 16–13, procedural thinkers would almost always attempt to count back thirteen numbers starting with 15, whilst proceptual thinkers would simply see the tens cancel and the result as 3. We therefore hypothesise that those who find mathematics difficult are often forced into even more difficult procedures for solving specific problems.

In the developing compression of notation in which symbols initially representing a process are compressed into objects that can be manipulated, the successful thinker reduces stress by manipulating symbols as objects instead of having to think of symbols as procedures. Procedures require more short-term memory store than compressed objects and therefore are eventually harder to handle.

This leads to an ever-widening gulf between the sucessful proceptual thinkers and the less successful procedural thinkers which we call the *proceptual divide*.

Summary

Adult mathematicians see mathematics from a mature viewpoint in which the structures have great richness and interiority. They therefore have a perception of simplicity in which this structural richness plays an implicit fundamental role. Learners do not yet have this conceptual richness. In achieving steady compression of knowledge, careful analysis shows the way in which processes of computation and objects of mental manipulation can be represented by notation that stands ambiguously and dually for both. A child's conception of arithmetic goes through such stages of compression from procedures of counting to concepts of arithmetic. Children's differing views of arithmetical processes and concepts are likely to give them different perceptions of algebra and algebraic notation. A procedural view of arithmetic may lead to the interpretation of an algebraic expression as a process which cannot be carried out and cause a considerable conceptual obstacle, which, even if overcome, may lead to a procedural view of algebra. On the other hand, a flexible, proceptual, view of arithmetic, manipulating symbols so that known relationships can be used to deduce derived facts is more appropriate to form a foundation for meaningful manipulation of algebraic symbolism.

Whilst it is natural to seek number patterns as an extension to arithmetic, this involves cognitive difficulties which may not give the best route into algebra. Computer programming and software can carry out the actual evaluation of algebraic expressions whilst the child can concentrate on the meaning of the symbolism. Practical activities can dually focus on the process of evaluation. Empirical evidence shows that this offers a way into algebra that enables children to give the symbols a more powerful meaning.

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