MATHEMATICIANS THINKING ABOUT STUDENTS THINKING ABOUT MATHEMATICS

A summary of a lecture presented in memory of Rolph Schwarzenberger (1936–1992) on December 12th 1992 for The London Mathematical Society at The Mathematics Institute, University of Warwick.

There is a perceived change in the nature of students entering mathematics degrees, in part from the changes in school – with mixed ability teaching, investigational work and less emphasis on routine manipulation – and in part from the widening of access to a larger proportion of the population. However, difficulties that students have in learning with mathematics are not new and have been with us for many years. It proves difficult for mathematicians to fully comprehend the causes of these difficulties for, as Freudenthal has observed, "one finally masters an activity so perfectly that how and why students don't understand is not asked any more and is not even understood as a meaningful and relevant question". A major difficulty is in the *compressibility* of mathematical concepts which Thurston (1990) described as follows. "You may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics."

Gray & Tall (1991) defined the amalgam of process and compressed concept that can be signalled by the same symbolism a "procept". This includes notions such as number, 3+2, 5/8, +2, 2+3x, the limit notation, and such things as elements of transformation groups, etc. Based on evidence from young children learning arithmetic through to university level they hypothesise that those who are successful think flexibly with symbolism that stands dually as a process to do mathematics and as a concept to manipulate mentally. They contend that mathematicians rarely pass on the flexible and ambiguous use of symbolism on to students. Instead students are often presented with the formalisms of mathematics in lectures which convey, in the words of Skemp (1971), the product of mathematical thought rather than the process of mathematical thinking.

Evidence in interviews and written comments from students show that they are well aware of the problems. One student remarked that because "the tutor knows what's going on, he expects us to know what's going on" by "working through a couple of examples", and that "I can't listen and think and keep up with the notes"; afterwards, looking at the notes "I can't remember what he said in between, so it's a non-starter from the beginning". Others report "with personal experience" that "the student looks upon the subject as something to pass and then forget most of it immediately after the exam." W. W. Sawyer (1987) found that students taking a functional analysis course remembered little of their previous analysis course on which had hoped to build because the "formal lectures had not conveyed any intuitive meaning" and the students "had passed their examinations by last minute revision and by rote."

In attempting to improve the situation, the London Mathematical Society has proposed changes in course length from three to four years. There are moves to cover the material more slowly. In my lecture I posed the question "If mathematicians succeed through compressibility of knowledge, then does it help students compress knowledge better by spreading it out more?"

The notion of "concept image" is helpful here. This is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Vinner & Tall, 1981). Through their experiences, individuals develop beliefs about concepts that are not part of the formal

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definition. For instance, that the terms of a sequence can never equal the limit, or that a function must be given by a simple formula. Students bring their own personal concept images to the formal presentations of mathematics and this can cause great conflict and prevent understanding. (Schwarzenberger & Tall, 1978).

What is needed is not just doing the same mathematics at a slower pace, but to focus on the nature of mathematical thinking and to help students reconstruct their own knowledge in a way which becomes meaningful for them.

Poincare, Hadamard and Polya are but three mathematicians to concentrate on the processes of mathematical thinking as opposed to the formal structure of final mathematical knowledge. Based on their ideas, an approach appropriate for students has been developed (e.g.. Mason et al: Thinking Mathematically, Addison Wesley, 1982). This requests students to first ENTER a problem, by thinking what is wanted, what is known, and what needs to be introduced to get to the former from the latter. An ATTACK phase follows which may be successful or unsuccessful The emotion of pleasure with success must be enjoyed for a moment before the serious business of REVIEW is undertaken to check that the arguments are formally correct and to consider how they might be generalised and extended. The negative emotions associated with being unsuccessful are seen as signals that the problem needs reconsideration, and that some gain has been made – even if only that the route taken has not proved fruitful and other routes need to be considered by returning to re-enter the problem. Such relatively simple ideas, used in a supportive and flexible context without the time pressure to solve the given problem in a fixed period, can prove dramatically successful in raising students personal esteem and develop a framework for doing mathematics meaningfully instead of just learning by rote to pass exams.

My thesis is that it is futile to focus on the undergraduate degree purely in terms of mathematical content to be learnt in a fixed time scale. Many (most?) students are now leaving university with a sense of alienation and frustration having spent three years committing mathematical content to memory sufficiently long to pass an exam. Apart from the successful minority who go on to do graduate research, few students remember the content of their courses for long afterwards. What matters is quality of learning rather than quantity of content. It is far more important for students to see mathematics at university as a time when they learnt to think mathematically and to have the confidence to tackle new problem situations in a truly mathematical way. By spending part of their time learning to use problem-solving techniques, they may be able to apply these to help them understand material given in more concentrated form in other parts of their course and to reconcile their concept images with the formalities.

This was expressed eloquently by the psychologist Jerome Bruner (1966): “A body of knowledge enshrined in a university faculty and embodied in a series of authoritative volumes is the result of much prior intellectual activity. To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as an historian does, to take part in the knowledge getting. Knowing is a process, not a product.”

True reform of undergraduate mathematics can only happen if mathematicians think seriously about students thinking about mathematics.

The full text of this talk, including references, may be obtained from Professor David Tall, Mathematics Education Research Centre, University of Warwick, COVENTRY CV4 7AL.