Mathematicians thinking about Students thinking about Mathematics

Rolph Schwarzenberger (1936–1992) : In Memoriam

David Tall

Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL

Introduction

These thoughts are presented in memory of Rolph Schwarzenberger, who was a colleague, mentor and friend for the last quarter of a century, during which time we participated, initially as mathematicians, then as mathematics educators, in thinking about students thinking about mathematics.

In recent years there has been increasing attention on the difficulties that many students experience in studying mathematics at university level. It is not that such difficulties are new, but that the awareness of them has grown as the needs of the individual – both successful and unsuccessful – have increased in importance. At the same time, changes in mathematics teaching in school – with more mixed ability teaching, more investigational work, and less emphasis on routine manipulation – are perceived as affecting the nature of students arriving to read mathematics at university. The extent of these perceived changes are such that the London Mathematical Society has proposed sweeping reforms in the length and content of undergraduate degrees. Such changes, however, are perceived from the viewpoint of the mathematician without any detailed research into the actual needs of the individual student. The perceptions of one who understands mathematics are very different from those of the learner. As Freudenthal has observed:

> One finally masters an activity so perfectly that the question of how and why students don't understand them is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question. *Freudenthal ,1983, p. 469*

It therefore behoves us to consider the manner in which mathematical knowledge grows, not just from the viewpoint of the professional mathematician, but from a viewpoint which reflects on the growth of knowledge in the learner.

The observations I make and the theoretical standpoint which I shall unfold are the product of the last decade and a half, which began with

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joint work on student's conceptions of mathematical concepts with Rolph Schwarzenberger (Schwarzenberger & Tall 1978). At that time few people were performing any research into the cognitive development of thinking in mathematics at university level. In the eighties interest in this area grew and an international working group was formed whose activities culminated in the publication of the book *Advanced mathematical Thinking* (Tall, 1991) which I was privileged to edit and Rolph took part as co-author of a chapter (Robert & Schwarzenberger, 1992). This was the first book devoted entirely to theory and empirical research in mathematical thinking in both mathematicians and students.

In this presentation I intend first to look at the nature of mathematical concepts as seen by mathematicians and by students and the mismatch between what mathematicians think and what they think that their students think. Then I shall show how the thinking processes imposed on students differ radically from those of the practising mathematician. Finally I shall consider the current crisis in teaching undergraduate mathematics and review how what we know about the thinking of students can illuminate possible avenues of progress.

The nature of mathematics – as seen by mathematicians and students

The development of mathematical ideas in research may be seen as a struggle with the construction of concepts and relationships which become amazingly flexible and meaningful as they are used by mathematicians:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. *Thurston 1990, p. 847*

However, this meaning and flexibility is not necessarily shared by students. The concepts which empower mathematicians to think easily in a rich and flexible way are not yet present in this form in the learner. Some students succeed in building the ideas up in a way which enables them to become mathematicians. But others are bemused by hearing terminology for which they have not developed the same richness of meaning. It can happen all too quickly, as witnessed by the following comment from a first year mathematics major (with a B in mathematics A-level and A grades in Computing and in Business Studies), in an interview in the third week of the first term:

> Just because the tutor knows what's going on, they almost expect us to know what's going on, and so they just work through a couple of examples, showing us what happens, but without getting us to think about it. He's writing things on the board, saying "well, let's think," and you can't do both, well I can't (I expect some people can). I can't listen and think and keep up with the notes. [When looking at notes after the lecture] … then they don't make much sense, 'cause I can't remember what he said in between, so it's a non-starter from the beginning, really.
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> *First Year Male Mathematics Student in interview, 1992*

Given the quantity of mathematics considered necessary to be covered in a university degree, it seems that it is most efficient to present the material in an organised and logical form, beginning with clear definitions and going on through well-planned sequences of deductive proof. The problem is that some students may see things rather differently:

> Maths education at university level, as it stands, is based like many subjects on the system of lectures. The huge quantities of work covered by each course, in such a short space of time, make it extremely difficult to take it in and understand. The pressure of time seems to take away the essence of mathematics and does not create any true understanding of the subject. From personal experience I know that most courses do not have any lasting impression and are usually forgotten directly after the examination. This is surely not an ideal situation, where a maths student can learn and pass and do well, but not have an understanding of his or her subject.
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> *Third Year Male Mathematics Student, 1991*

All too often students perceive that what matters is to commit as much as possible to memory, ready to respond to questions in examinations:

> Since starting my University education, I have discovered that the key to advanced learning is persistence. Come the end of the year, everyone is faced with courses whose purpose they have failed to grasp, let alone its finer details. Faced with this problem, most people set about finding typical questions and memorising the typical answers. Many gain excellent marks in courses of which they have no knowledge. Most accept this as the norm, thinking that their stupidity is the problem and not considering it to have been a wasted 5 or 10 weeks of study.
>
> *Second Year Male Applied Mathematics Student, 1991*

Some appreciate the need to study mathematics in an organised fashion to get through the material in a reasonable time:

> It is important to study other people's theories and the way they write mathematics to develop one's own mathematical thinking. … The student has the chance to appreciate complex ideas which he could not have thought of for himself.
>
> *Third Year Female Mathematics Student, 1991*

But in practice the presentation of material in a logical way can lead to bewilderment because the student may fail to see where the ideas come from:

> I for one suffered a confidence crisis as answers seemed to arrive from mid-air – "Oh, we'll take $\varepsilon = \dfrac{2\delta}{\min(M,N)}$ " or similar. This evoked the 'panic' emotion … and, although frequently shown the solution, I often did not understand how to find it.
>
> *Third Year Female Mathematics Student, 1991*

This need to understand is shared by mathematicians:

> To understand the demonstration of a theorem, is that to examine successively each of the syllogisms composing it and to ascertain its correctness, its conformity to the rules of the game? ... For some, yes; when they have done this, they will say: I understand. For the majority, no. Almost all are much more exacting they wish to know not merely whether all the syllogisms of a demonstrations are correct, but why they link together in this order rather than another. In so far as to them they seem engendered by caprice and not by an intelligence always conscious of the end to be attained, they do not believe that they understand.
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> *Poincaré, 1913, page 431*

But in the crowded curriculum of a modern course, there seems so little time to build up meaning. When W. W. Sawyer tried to teach theorems in functional analysis by referring to theorems in real variables that students had been taught. he found that they had no recollection of them.

> The reason for this was that in their university lectures they had been given formal lectures that had not conveyed any intuitive meaning; they had passed their examinations by last-minute revision and by rote.
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> *W. W. Sawyer 1987*

Students confirm this degeneration into rote-learning activity:

> I say this with some personal experience, the student will look upon the subject as something to pass and then forget most of it immediately after the final exam.
>
> *2nd Year Male Mathematics Student, 1989*

Instead of being a living subject – solving problems, developing new theories – learning mathematics becomes a matter of learning theorems already established:

> I don't feel that the ability to conjecture and then check, reflect and review, is important as usually a question or theorem will be so specific that it leaves nothing to be conjectured and hence leaves nothing to check etc.
>
> *Second Year Male Mathematics Student, 1989*

There may be sense of alienation and a desire for a different approach:

> … Most maths students would say that they have enjoyed very few courses at university. ... I feel that university teaching methods are in desperate need of change.
>
> *Second Year Male Applied Mathematics Student, 1991*

Reflections on the observations

It would not be appropriate before an audience of mathematicians to draw inferences from a small number of selected examples. The quotations given so far from students are selected from essays written by students on problem-solving and interviews with students studying first year analysis. Some of the students interviewed presented the kind of qualities that mathematicians seek in their students – willingness to struggle with new ideas until they give them meaning, working as long as it takes to solve problems on examples sheets, being willing to rewrite lecture notes in a way which is more meaningful, being prepared to learn definitions in a way which enables them to be used. However, even these students, after several weeks of the course, slowly fell behind what was being demanded of them, so that the concepts they were being asked to use were still in the process of formation. Few students who take a mathematics course understand everything that is going on at the time.

We can therefore hypothesise that there is a mismatch between what is conceived by mathematicians and what is expected of students in their learning process. I conjecture that a major reason for this lies in the compression of ideas in the earlier quotation from Thurston. Having developed a way of looking at mathematical concepts which is so flexible and so clear, it is natural to wish to share this kind of knowledge with students. As Freudenthal suggested, the compression is such that it cuts off memories of how that compression occurred. When mathematicians think about students thinking about mathematics, they do so from a mathematical point of view which may no longer take into account the current development of the student.

Looking at the students comments selected earlier, a recurring theme is the pressure of having too much to learn and it may be tempting to formulate the suggestion that the way forward is to cover less in greater detail. This is a plausible conjecture. But it may also be terribly flawed.

> *If the mathematician succeeds through compressibility of knowledge, does it help the student compress knowledge better by spreading it out more?*

Mental Images of Mathematical Concepts

The methods by which individuals build up concepts depends very much on their actual experience. Even though mathematical concepts are given formal definitions, these definitions do not occur in a vacuum. They are formulated specifically to have certain properties that are useful in mathematics, so that the newly defined mathematical object is expected to have certain properties even before they are formally deduced from the definition. For example, individuals who study examples of limits of sequences will note many specific properties that are not relevant in the definition of limit. They may see that one sequence increases to a limit, another decreases, yet another oscillates on either side, and so on. Virtually all the natural examples studied seem to get closer and closer, without ever reaching the limit, and so this too is conceived by many as being a general property of the limit process – to approach, but never to reach the limit. In this way experience causes individuals to sense properties of the examples which are not relevant in the formal definition and to attach them to the formal concept in an erroneous way.

In order to move from the examples to the formal abstraction, it is necessary to construct the properties of the formal abstraction in a logical manner and to become aware what role is played, or not played, by the properties of the examples. In this way, if greater detail is given to “help” students understand, this may perversely introduce new irrelevancies which must be rationalised in dealing with the formal abstraction.

Compressibility of Concepts

The development of mathematical ideas, both in history and in teaching, often begins with problems, then procedures of solution, then formalisation of mathematical structures to lay foundations for logical deductions. As this occurs the knowledge is subtly changed. Procedures of solution might be seen as taking a variety of routes giving a choice of solution process. The symbolism involved changes subtly from being a notation such as the integral $\int_a f(x)\,dx$ being a recipe for a process to be carried out and yet also representing the product of that process. Symbols in mathematics often represent either process or product of process, the former being an action, the latter being a mental object.

> I remember as a child, in fifth grade, coming to the amazing (to me) realisation that the answer to 134 divided by 29 is $^{134}\!/_{29}$ (and so forth). What a tremendous labor-saving device! To me, ‘134 divided by 29’ meant a certain tedious chore, while $^{134}\!/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, a/b and a divided by b are just synonyms. To him it was just a small variation in notation. William P. Thurston, Fields Medallist, 1990.

Gray & Tall (1991, to appear) develop a theory of *procepts* in which the same symbolism represents both a process to be carried out and the product of that process. Examples of procepts include number (evoking both a process of counting and a concept of number), addition (2+3 representing the process of addition of 2 and 3 and the concept of sum 2+3 is 5), division (2/3 as the process of division and the concept of fraction), and many other processes/concepts of arithmetic, algebra, calculus and analysis. In group theory, for example, a group of transformations has elements which are procepts, representing both elements as objects (members of the underlying set) and also as transformations, with the group operation being the composition of successive transformations.

Often mathematical activities go through several stages of compression, and may be de-composed and re-composed in different ways. For instance the concept of continuity is initially met in schools in terms of a graph that is “in one piece”, a concept more akin to connectedness than completeness. Then it is given its full glory in university in terms of a process – “I’ll give you an epsilon and you must find a delta such that

…”. But it does not stay at the process level. Instead, specific examples are given (continuity of a constant function and the identity function) and general theorems are proved (*if* f,g are continuous, then so are f+g, f-g, $f*g$, f/g, $f \circ g$ under appropriate conditions). Now the original definition is no longer used. Instead one builds up using specific examples and general theorems. This change of tactic soon after meeting the definition means that a student who is taking time coming to terms with the definition may find the definition becomes redundant before he understands how to use it. There seems little purpose in learning a definition which is dropped before it has had time to establish itself.

Students learn to cope with isolated facts that they pick up as they go along, realising that the hard definitions and deductions will soon go away if they persevere long enough.

Various levels of compression are met in a short span and may cause bewilderment. Two weeks after being introduced to the notion of least upper bound, *none* of five students interviewed could explain it formally, although they had differing levels of understanding of the concept as “the least of the upper bounds”. Three weeks later the most successful of those interviewed could give the verbal equivalent immediately, but found greater difficulty in giving a definition in formal terms. He could say:

$\boxed{\lambda \text{ is an upper bound}}$ and $\boxed{\text{it is the least of the upper bounds}}$

but took time to express it in the form

$\boxed{\lambda \geq s, \forall s \in S}$, $\boxed{\text{if } u \text{ is another upper bound for S then } u \geq \lambda}$

and help was needed to rephrase this in the following form:

$\boxed{\lambda \geq s, \forall s \in S}$, $\boxed{\text{if } u \geq s, \forall s \in S \text{ then } u \geq \lambda}$.

To be able to develop an intuitive idea of what is going on, the verbal definition may be appropriate, but to be able to operate successfully in a formal proof requires the translation of the definition into the mathematical formalities.

The Development of Mathematical Thinking

In the desire to “cover the course” and to make certain that a given university curriculum compares in content with one at other universities, the overt need for quantity of material seems to overwhelm the need for quality of thought. Skemp commented on the need to present mathematics in a logical and orderly fashion over two decades ago:

> Some reformers try to present mathematics as a logical development. This approach is laudable in that it aims to show that mathematics is sensible and not arbitrary, but it is mistaken in two ways. First it confuses the logical and the

> psychological approaches. The main purpose of a logical approach is to convince doubters; that of a psychological one is to bring about understanding. Second, it gives only the end-product of mathematical discovery ('this is it, all you have to do is learn it'), and fails to bring about in the learner those processes by which mathematical discoveries are made. It teaches mathematical thought, not mathematical thinking.
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> Richard R. Skemp, 1971

Mathematical thinking, in terms of development of new ideas, requires creative activities which are quite different from the logical deduction of the final stage of formalisation of mathematics. Hadamard recalled the manner in which Poincaré developed his theory of Fuchsian groups through a mental struggle which went through several phases:

> .. the very observations of Poincaré show us three kinds of inventive work essentially different if considered from our standpoint, viz.,
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> a. fully conscious work
> b. illumination preceded by incubation
> c. the quite peculiar process of the sleepless night.
>
> (Hadamard, 1945, page 35.)

After a similar fashion, Polya (1945) suggested four phases as a framework for problem-solving:

- understand the problem,
- devise a plan,
- carry out the plan,
- look back at the work.

This framework forms the backbone of many subsequent attempts at formulating problem-solving strategies, though, for teaching purposes, Mason *et al* (1982) and Schoenfeld (1985) have seen the need to make the actual heuristics much more explicit and more appropriate for the learner. The idea of "devising a plan" is extremely daunting for the novice. More empathetic is the version suggested by Mason, who proposes three phases:

- entry,
- attack,
- review.

The entry phase covers the first two stages of Polya whilst attack and review correspond to Polya's third and fourth stage. In the *entry* phase the potential problem-solver gets acquainted with the problem-solving context – getting a sense of the problem by playing with the ideas, perhaps through simple specialisations, moving to a position which attempts to specify clearly what is known and what is wanted, and considering carefully what can be introduced (notation, procedures of solution, etc.) that might take the problem-solver from what is known to what is wanted. Then a qualitative change occurs with a committed *attack* on the problem using the ideas that have been introduced. This may be successful, but it can more often lead to an impasse, a seeming dead-end from which the individual should review what has been done and return to the entry phase to consider a new attack. Once some kind of solution is achieved the mood changes yet again to one of sober *review* – checking the results to make sure no error has been made, reviewing what has been done to learn of strategies that may prove useful on other occasions and then being prepared to *extend* the problem to new levels of sophistication, re-starting the entry cycle at a more sophisticated level.

Such an approach, which concentrates on the *meta-processes* of mathematical thinking – *how* to think, rather than *what* to think, have been a central feature of Open University courses for over a decade, and now are becoming an implicit part of school mathematics investigations, though they are not yet natural parts of university undergraduate courses.

Given that when a mathematical concept is introduced in a given context, each individual will create a personal concept image that may very well conflict with the formalities of mathematics, it becomes apparent that students must be aided to reconstruct their knowledge using problem-solving techniques. Students who are able to clearly delineate their current knowledge structure are more likely to be able to reconcile new, conflicting knowledge with old beliefs. Those who can only vaguely understand what is happening are faced with conflict that simply makes them more confused. They are thus forced to attempt to separate the new knowledge from the old and to learn disconnected pieces of information which will work in the restricted context at present. Long-term, with so many pieces of information to handle, each in its appropriate context, the burden is likely to be too great.

In commenting on their courses, students with no formal knowledge of psychology sense the weakness of the logical presentation in giving depth of understanding and the need for problem-solving techniques to build a coherent, supportive knowledge structure.

> In a typical lecture course the pace of work is necessarily fast, the style is anything but interactive and the lecturer has little choice but to present students with ready-made definitions, techniques and theorems rather than letting them develop their own meaningful investigations and generalisations. The courses are essentially content-based, although it is clearly hoped that somewhere along the way students will develop mathematical thinking. Nevertheless I think problem-solving techniques do hold some relevance for the lecturer who wishes his course to get the best from his students. Good lecturers often offer advice like
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> > “draw a diagram”,
> > “restate it in your own words”,
> > “try it for a numerical special case”,
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> which matches the advice in *Thinking Mathematically* but without an overall picture of problem-solving techniques students may not appreciate how valuable this advice is. I think the first thing is for lecturers to have an

> understanding of problem-solving techniques themselves, and … the sooner students are helped to become explicitly aware of what is required for improved mathematical thinking, the easier it is for them to engage in learning effectively.
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> Once lecturer and student share an understanding of the techniques involved, the lecturer can draw attention to his own use of problem-solving processes, like specialising, generalising, conjecturing and justifying, as well as pointing out the spiral nature of his enquiry process as he passes into deeper and deeper levels of complexity. For some lecturers this might involve a change in emphasis in the course structure, although the same material could be covered eventually. Rather than presenting material strictly according to content, and beginning each chapter with a catalogue of formal definitions, an approach more in tune with problem-solving techniques might be to start with simpler conjectures and definitions, coming back, refining and formalising them once students have grasped enough of the subject to see their relevance. Encouragement from lecturers for students to draw pictures, engage in concrete manipulations, draw analogies with previous problems, and to make explicit the links between structures and subjects, is often given anyway, but a knowledge of problem-solving techniques on both sides would make the encouragement more explicit and more meaningful for the students.
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> … Problem-solving techniques tell us to specialise at an *appropriate* level, which means specialising drastically so that the student is using objects (physical objects, numbers, diagrams, algebra, etc.) which are confidence inspiring and manipulable *for him*. As his mathematical sophistication grows, so will the complexity of what is considered manipulable, for example,
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> numbers,
> algebraic expressions,
> functions,
> sets of functions,
> sets of sets of functions, …
>
> *Second Year Male Mathematics Student, 1991*

In giving this advice, the student is striving to formulate an environment that encourages the student to think mathematically. In doing so, he is coming to the same conclusion as the psychologist Bruner, who ended his search "towards a theory of instruction" over a quarter of a century ago with the comment:

> A body of knowledge enshrined in a university faculty and embodied in a series of authoritative volumes is the result of much prior intellectual activity. To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as an historian does, to take part in the knowledge getting. Knowing is a process, not a product.
>
> *Jerome Bruner, 1966*

Mathematics professors designing mathematics curricula must consider very carefully the overall aim of an undergraduate mathematics course. Is it to cover a specified list of content, to teach the student to comprehend the accumulated wisdom of mathematical thought, or is it to teach active mathematical thinking? At present it appears to be the former, with the attendant corollary that many students simply learn vast quantities of material by rote only to forget most of it after the exam. As long as undergraduate mathematics courses are purely content-based they will fail the majority of our students. Let us hope it is time to provide undergraduates with an environment in which they are treated with respect like mature individuals encouraged to grow in the kind of mathematical wisdom shared by mathematicians.

This can only happen if mathematicians think about students thinking about mathematics.

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