Current difficulties in the teaching of mathematical analysis at university : an essay review of *Yet another introduction to analysis* by Victor Bryant, (Cambridge University Press, ISBN 0-521-38835-X)

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The teaching of calculus at college level and mathematical analysis at university is currently in some turmoil. Not long ago, success in such a course was a symbol of ability to succeed at the highest level of mathematical thinking. It was a pre-requisite and a passport to further study, not only in mathematics, but in related subjects which have a mathematical basis. Now the climate is changing.

In his experience as editor of the *Mathematical Gazette*, it was often Victor Bryant's task to consider books on mathematical analysis for review. He sensed that hardly a week went past before another one arrived, and yet he felt they rarely satisfied the needs of mathematics students. In Britain there is an inexorable move at school level towards a more motivated, relevant and practical style of work. He therefore decided to write a book to reflect what he felt best served current trends. In reviewing his book I shall take the opportunity to look at the wider picture, to see the cognitive difficulties and changing social views of mathematics that give rise to the current crisis in university mathematics in general and in mathematical analysis in particular.

Cognitive considerations

We should begin by asking ourselves why it is that there seem to be so many analysis books around and yet so few that seem to satisfy the dual need of presenting the mathematics in a manner satisfactory for the professional and meaningful to the student. A major problem is apparent in the conflicting needs of these two viewpoints. Mathematics is, of its nature, a very *compressible* subject. That is, one may learn to carry out a fairly complicated procedure to solve a certain type of problem, and then that procedure becomes routinised and compressed until it can be seen, not as a process that occurs in time, but as a *mental object* that can itself be manipulated. An example might be the formal limit concept, which is defined in terms of "give me epsilon, then I will find a delta such that if something is true then something else will follow". Initially this is an extremely complex process to cope with in all its ramifications. But after considerable experience with this process the mathematician begins to be able to conceive of the limit as a *concept*, and can begin to use theorems about limits to be able to manipulate them in a manner which now seems extremely simple.

The professional mathematician therefore sees the simplicity of manipulating the compressed objects as the secret of success. However, this compression is at its most powerful when the arduous ramifications of its origins are supressed to the back of the mind, or even forgotten completely. So the professional mathematician is now likely to have a view of mathematics that is at odds with the needs of the student. In trying to explain the power of the mathematics from a mathematician's viewpoint, he (or she) extols the virtues of the compressed knowledge which proves so valuable to the expert, and is at a loss when this seems not to be understood by students.

One may surmise that this is the reason for so many relatively unsuccessful books on mathematical analysis. Each expert, in his or her wisdom, tries to formulate the theory of analysis in a way which proves most understandable to the student. But the compressed viewpoint of the expert and the cognitive needs of the student are likely to be irreconcilable. The expert is looking for logical ways to present the material that is aesthetically pleasing and well-organised. It might be a "new" way of looking at the limit concept, for instance, that continuity of functions and limits of sequences are essentially the same thing, if viewed appropriately, and so everything can be based on the definition of continuity. Or it might be that limits of sequences are considered easier than continuity, so another book bases all arguments on sequences. Another author might see the essential matter being the logic of the argument, and so everything is based on the axioms of a complete ordered field. Yet another sees these axioms as "too

Published in *Zentralblatt für Didaktik der Mathematik*, (1992), 92/2, 37–42.

complicated" for the students and so the student is assumed to be familiar with decimals, and special attention is focused only on the completeness axiom. But then, which version of the completeness axiom is the best pedagogically? And so it goes on.

I would suggest that the reason for so much failure is that the mathematician is so often looking in the wrong place for the solution to the problem. Like the drunken man who looks for his lost coin under the street-lamp because it is the only place where he can see, the mathematician looks only at the place where he has light. He looks *at the mathematics*. He shifts around the mathematics in every way possible to try to solve the problem and fails. He never looks in the darkened recesses of the student's mind.

In the last decade or so, mathematics educators have started looking at the student's difficulties. For instance, the manner in which their "spontaneous conceptions" from previous experiences might interfere with the formal definitions (Schwarzenberger & Tall, 1978; Cornu, 1981), giving them "concept images" (Tall & Vinner 1981) which might cause unforseen cognitive conflict when they attempt formal mathematics. The manner in which their "primary intuitions" (Fischbein, 1978) prove resistant to change and they may hold contradictory views of such concepts as limits and infinity (Sierpinska, 1987; Tirosh, 1985). These faltering steps at understanding the darker realms of the human mind are beginning to take a shape which can be of value in understanding the teaching of undergraduate mathematics, but there is, as yet, no single recipe for success. Indeed, Davis & Vinner (1986) suggest that there may be *unavoidable obstacles* which will face students studying the limit process.

It is known that students' spontaneous conception of the limit notion is as a *process* of getting close, and that the formal definition of a limit proves to be a very poor starting point for a cognitive development of the calculus (Tall, 1986). Therefore, in the teaching of formal analysis there is a transition to be achieved from preliminary dynamic intuitions to the formal notion of limit. But how to achieve it is something that is still the subject of research. Dubinsky (1991) is using the technique of programming functions as procedures to gain practical experience as to how they operate. For instance, a sequence s may be programmed as a function which, for any integer n, gives a value $s(n)$, and the behaviour of this value may be investigated practically for large values of n. Dubinsky uses a computer language (ISETL) which is particularly suited to matching programming to cognitive acts corresponding to mathematical theory. It includes the vital possibility of using a function as an input to another function, in such a way that the function can be conceived as an object. Even languages without this facility, such as BASIC, Pascal or Logo, have been used, with considerable success to give practical experience of the limiting process. I have found it of particular value to program a sum of a series procedurally, so that the sum function takes a value of n as an input, adds up n terms and returns the sum as output. In this way a series will be seen as a special case of a sequence, something which research shows the majority of first year students continue to confuse. Because of their much greater exposure to series at school, the notion of sequence proves to be hardly operable and becomes mentally entangled with the previous experiences of series.

A major difficulty in dealing with the usual formal definition of the limit is the need to cope with the quantifiers (Dubinsky 1991). This difficulty raises its ugly head once more in dealing with the logic of proof. Some mathematicians have even turned to non-standard analysis (Keisler 1971), partly because it seems to have a closer fit to intuitive processes and partly because it uses fewer quantifiers than the standard theory. Some mathematics educators (Artigue 1991) are very positive about the possibilities of non-standard analysis.

Others are aware of the need of the student to take an active part in the construction of the theory. At Grenoble University (Alibert, 1988) for several years students have taken part in *proof debates* where they are encouraged to make their own conjectures for communal discussion for proof or refutation. This experiment has met with a high level of student approval and a greater understanding of the processes of mathematical thinking, rather than just the learning of the product of mathematical thought.

The learning of analysis is clearly more than committing proofs to memory ready to reproduce in examinations (which seems to be the goal of so many dispirited students in current courses when they begin to lose contact with the meaning of the ideas). It requires the student to understand *why* a result might be true and *how* the proof might arise, even if the actual invention of the proof itself might be beyond the student's abilities.

The changing social context

For many years the study of Latin, or of Euclidean Geometry was seen as an intellectual exercise that honed the mind to logical thinking. Latin died as a central subject several years ago and in some countries (including Britain) Euclidean Geometry has now gone. In mathematics at the higher level, it is Mathematical Analysis that currently carries the flag as the bastion of logical deduction, and this now is severely under threat. In the United States, the traditional Calculus course at College, which is seen as the culmination of "pre-calculus" courses taken at High School, is being questioned as students fail to cope and the changing needs of technology intimate that perhaps a foundation in discrete mathematics might prove more useful. France is one of the few countries where there has been continuing research into cognitive development in advanced mathematics, and here the research has been turned into practical advice for teaching calculus and analysis (see, for example, Artigue *et al*, 1990)

In Britain the changing curriculum in schools is introducing more project work, more practical applications and more problem-solving. In consequence the students have less technical expertise in manipulating algebra, in using trigonometric formulae, and generally in the kind of activities considered as a necessary background for current calculus courses. The loss of Euclidean geometry means that students no longer have a major source of earlier ideas about proof. As attempts are made to increase participation in mathematics at university, it is becoming patently unacceptable to say that only those who are intellectually very able may succeed in mathematics. The approach to mathematical analysis in Britain is beginning to include the necessity to cope with a wider ability range than ever before, albeit a difference in ability which is still less than that of students attending higher education in America.

At last it is becoming apparent in many countries that an effort must be made to assist students in the transition from intuitive concepts of numbers, functions, and limits, to corresponding formal definitions and logical deductions of mathematical analysis. Cognitive research has shown that this transition requires considerable mental reconstruction, from familiar concepts that are *described* to similar-looking, but subtly different concepts that are *defined* and whose properties are *deduced*. It is necessary to consider very carefully what kind of theory of analysis is appropriate: should it be completely formal, should the formalities be based on intuitive experiences, and, if so, how is the transition to the appropriate level of mathematical thinking to be achieved?

Yet Another Introduction to Analysis

In this wider perspective *Yet Another Introduction to Analysis* is written from the viewpoint of the sensitive mathematician, who is trying to take account of the changing social context and the needs of the student. The author's writing style reveals him to be acutely aware of difficulties students face with definitions, proofs, and, most of all, quantifiers. As a mathematician, he is less aware of the educational research that is available but, given its developing nature, it is research which may not currently be of value to him in designing his text. However, as we shall see, it points to some possible difficulties which his students are likely to encounter.

The book covers the basic ideas of a first year British university course in analysis. It builds on the intuitive background of the student, and starts with the student's knowledge of numbers and earlier experience of the calculus. (English mathematics students study the symbolism of differentiation and integration in their last two years at school, aged 16-18, prior to their transfer the following academic year to university.) But these mechanical techniques are limited. The student knows how to calculate $f'(x)$ for $f(x)=x^2$, but not for $f(x)=|x|$. Therefore, there are new possibilities to be considered, including the need to understand functions a little better, and the nature of the number system.

In chapter 1 the author visualizes real numbers as points on a line and launches into the difficulty of representing $\sqrt{2}$ as a decimal, by looking at the real numbers L whose squares are less than 2 and the set R whose numbers exceed $\sqrt{2}$. He introduces the completeness axiom in the form which he calls “piggy-in-the-middle”, that if there exist two sets L and M where every element l of L is less than (or equal to) every element m of M, then there is a real number α such that $l \leq \alpha \leq m$. He goes on to show how this is equivalent to various other forms of completeness (least upper bounds, and so on) at the same time including concepts that will be of value later in the text. In essence, therefore, he is doing the same kind of thing as many other “introductions to analysis”. But, in what detailed ways is his approach different?

First there is a transparently honest and friendly writing style that focuses on the important details and uses unpretentious colloquial English to resonate with the students wherever possible. (When he asks “are there more rationals than irrationals”, he says “counting such huge collections is perhaps a little dodgy” and rephrases the question in the form of the probability of getting a repeating decimal if each digit is selected at random using a ten-sided die.) He is here walking a tight-rope in cognitive terms, as the colloquial language has built-in conceptual imagery which may help the student link to previous experiences, but also lead to unforseen conflicts if inappropriate imagery is evoked.

There is also a profuse use of pictures to illustrate the ideas which is a valuable way to support the conceptual development. A picture is worth a thousand words, so it is said, but pictures can also lead to limited imagery. The book here uses pictures to the fullest extent, with some excellent illustrations of basic principles liberally sprinkled with verbal comments. But the imagery is (intentionally) limited. Despite the attempt to create the idea of a “general” continuous function in later chapters, for example, there is no attempt to see in what way a function might be continuous but not differentiable, except for the limited possibility that there may be a non-differentiabilty at a single point. This is not part of the current English university scene, although it is certainly part of the new SMP A-level which uses computer software to magnify the graph so that the gradient can be seen as the magnified curve looks almost straight. A non-differentiable curve will look wrinkled, a curve with a different left and right derivative will magnify to two half lines meeting at an angle. This particular text is bent on giving students the kind of imagery that will support the formal proofs, even if the imagery is more limited than the proofs are. It should be noted, however, that the imagery is typical of most modern analysis courses which say what a differentiable function *is*, but avoid deep reflection on what it is not. It may very well be sensible for an initial course to avoid such niceties, but for full understanding, this will imply that cognitive reconstruction of the restricted concept images will be necessary at a later stage.

The author uses all sorts of devices to help students understand proofs, in some instances prefacing the proof with a discussion of the fundamental ideas, or the nature of the logical structure. He has also persuaded the publisher to use typographical layouts that more truly represent the structure of an argument. Given an “either/or” construction, for instance, he will have the two alternatives typeset side-by-side in columns for that part of the proof only, rather than sequentially one after another. He uses large brackets to associate appropriate symbols together, rather in the way that one might point to a significant relationship on the blackboard in a lecture. And, once the pattern of an argument is clear, he starts to replace some of the words by blanks, which the student must fill in to be able to complete the text. Thus passive reading must be replaced by active participation, even though this particular device only operates at a fairly rudimentary (word-recall) level. Where possible, reference is made to using a calculator or computer to carry out simple numerical procedures which illustrate important concepts, once again encouraging practical participation generating the essence of the mathematics.

The writing also exhibits a considerable sense of humour. Whilst this, in itself, may not always aid understanding of concepts, it provides relief and variety to encourage the student to persist at the task. Sometimes it underlines the informal nature of a particular exercise: an initial outline of an argument with illusions to the seaside being called a "waterproof" whilst the formal proof is referred to as being "watertight".

In addition, not only are there a large number of well-chosen exercises, there are full details of the solution to every exercise in the back which take up over 20% of the total length. This makes it possible to use the book for private study, in addition to the more obvious use in conjunction with an organised course, though it is still advantageous to have a tutor or mentor to help when things become a little more difficult.

Chapter 2 gently introduces the theory of convergence of sequences and the sum of series, thus taking the viewpoint that limits are best introduced in the context of the epsilon-N definition. However, here the author explicitly senses the difficulties that students have with quantifiers and omits them all. His definition is:

> A sequence $x_1, x_2, x_3, \ldots$ will 'tend to' or 'converge to' a 'limit' x if given any positive number ε (no matter how small) there exists an integer N such that
>
> $$x-\varepsilon < x_N < x+\varepsilon \text{ and } x-\varepsilon < x_{N+1} < x+\varepsilon \text{ and } x-\varepsilon < x_{N+2} < x+\varepsilon \text{ and } \ldots$$

Notice that this, on the face of it, simplifies the definition by replacing the explicit quantifier "for all $n \geq N$" by an open-ended collection of statements for N, $N+1$, $N+2$,...

After using this definition for a time, he introduces "some labour-saving devices" to deal in turn with the various theorems for limits, such as $x_n+y_n \to x+y$. He has a particulary nice sequence of argument to cope with the limit of $x_n y_n$ using carefully chosen preliminary lemmas. He is thus now wearing his mantle as a mathematician, leading the reader through a neat and plausible sequence of activities which the student would never have been able to negotiate without his help, though one might hope that, after the fact, the approach seems reasonable enough.

Chapter 3 introduces functions. The general terms 'range' and 'domain' are introduced, although the pictures and the types of functions used are not likely to extend the student's concept image of a function beyond that given by a formula. This restricted concept image is more firmly embedded by immediately turning to specific examples of the exponential and logarithm. Here he subtly uses his theory of sequences to define 10^x for general x by approaching x by a sequence of rationals. He introduces the idea of limit of a function f in a subtle way, using sequences, saying 'f(x) converges to a limit l as $x \to x_0$' means:

> **if** $x_1, x_2, x_3, x_4, \ldots \to x_0$
> (in f's domain and $\neq x_0$)
> **then** $f(x_1), f(x_2), f(x_3), f(x_4), \ldots \to l$

The clever part of this definition is that it introduces the limit concept without any explicit quantifiers. These are implicitly concealed within the **if : then** statement and the limit of a sequence concept (which itself has the use of explicit quantifiers simplified). The proof that f(x) does not tend to l, instead of being a difficult exercise in negating universal and existential quantifiers, simply requires one to understand some of the ideas implicitly and to find a sequence $x_1, x_2, x_3, \ldots$ which tends to x_0 whilst $f(x_1), f(x_2), f(x_3), \ldots$ does not tend to l. The quantifiers are thus not removed, they are implicit, and need to be discussed in a general way, but the method *seems* on the face of it to be easier.

The remainder of the chapter concentrates on the exponential and logarithmic functions, angles in radians, trigonometric functions, and the proof of the intermediate value theorem for continuous functions. The latter is done neatly, first using the bisection method in a practical way to find a root of an equation, and then generalising this idea to find an intermediate value for a continuous function. Finally the chapter closes with the proof of the theorem that a continuous function on a closed interval attains its maximum and minimum values. Of course, this is not constructive, in the sense that there is no computer algorithm analogous to the bisection algorithm to find the maximum value of an "arbitrary" continuous function. The author wisely does not tread in these deep waters, instead he leans on the reasonable nature of the (non-constructive) existence of a supremum of a bounded set.

Chapter 5, "calculus at last", covers the definition of the derivative, the formulae for the derivative of sum, product etc, Rolle's Theorem, the Mean Value Theorem, and Taylor's Theorem. Once more the definitions are in terms of sequences, which happens to give a neat proof of the nasty case that can occur with the derivative of a composite.

Chapter 6 is an "integrated conclusion". After an example of Riemann integration for the function $f(x)=x^2$, the definition of the general Riemann integral of a bounded function is launched immediately using upper and lower sums based on the notion of upper and lower bounds introduced at the end of chapter 1 in the discussion on alternative forms of the completeness axiom. There follows a typical sequence of theorems culminating in the Fundamental Theorem of the Calculus. A suitable climax to finish the book? The text perhaps. But the exercises contain more. The last-but-one problem asks the reader to prove that a power series can be integrated term by term, and the final question, the supreme accolade, requests the reader to follow a sequence of arguments to deduce that π is irrational.

Thus we see that *Yet Another Introduction to Analysis* is indeed another in the line of mathematically inspired introductions to the subject. But is is also an introduction with a significant difference. The author's intuitive feeling for the difficulties faced by the student help him to formulate the theory in a way that reduces the impact of quantifiers and introduces a number of techniques that give positive assistance to learning.

Inbuilt cognitive difficulties

The earlier reference to research literature will show that there are still likely to be "unavoidable conceptual difficulties" (in the sense of Davis & Vinner 1986). As some of the conceptual conflicts are almost built-in to the epistemological development of the subject, this is hardly surprising. For instance, it is likely that the students will have a limited concept image of the function concept – limited to functions given by a single formula, with little experience of the wider functions of mathematical analysis. Likewise, although *Yet Another Introduction* gives clear indications that the idea of a sequence tending to a limit means the terms may sometimes equal the limit, and even gives examples with this property, there are likely to be a substantial proportion of students who harbour misconceptions: that the terms of a sequence must be given by some algebraic formula, that they can get close to, but not equal, the limit. They may fail to understand the meaning of a decimal as representing a sequence of approximations tending to the real limit. (A strange case I found recently amongst able third year university mathematicians is that a significant proportion still do not fully understand what $\pi=3.14159\ldots$ means, interpreting the left-hand side as a *value* (π) and the right-hand side a potentially infinite *process*. They do not see the right hand-side as a notation for the *limit* of the process. They may even regard the representation as an 'infinite number', meaning an infinite process, going on forever, rather than infinite in size.)

The complexity of the theory of analysis is likely to continue always to cause problems of this nature. Walter Ledermann, an experienced writer of text-books warned me at the start of my career that analysis is a 'pop-up subject', in that if a difficulty is supressed in one place it is likely to 'pop-up' somewhere else. In an introduction to the subject it may very well be of value to supress certain difficulties (e.g. quantifiers) but one should not be too surprised if this action lays up possible traps for the future.

My own long experience of teaching analysis, particularly recent experiences with students of more modest abilities, suggests that even so, the students will still encounter difficulties, for instance with the treatment of Riemann integration through upper and lower sums. My lower ability students found it difficult to cope with the complexity that each partition gives a unique upper and lower sum, and that the latter can be represented as points on a line where every upper sum exceeds every lower sum. This particular theory is much easier when supported with programmable calculators or computers that can be used to compute the sums so that a number of upper and lower values can actually be calculated and plotted on a number line to illustrate the way the values separate into two sets, one above the other, which can be made as close as is practically possible. It is also particularly helpful when the sum from a to x with an appropriate number of strips can be represented as a function of x. Such an approach would be a natural extension of the earlier numerical work in the text on sequences.

This lack of practice with the numerical side of Riemann integration, which can now come to full fruition in the new technological age, also hides a conceptual difficulty in understanding the Leibniz notation as used in differentiation and integration. The Leibniz notation $\mathrm{d}y/\mathrm{d}x$ for $\mathrm{f}'(x_0)$ is mentioned on page 156 together with the usual British faux-pas that the symbols $\mathrm{d}y$, $\mathrm{d}x$ have no meaning in themselves. However, Leibniz's very first definition (1684) defines $\mathrm{d}x$ as any increment in x and $\mathrm{d}y$ the corresponding increment in y to the tangent. In other words, the symbols $\mathrm{d}x$, $\mathrm{d}y$ represent nothing more mysterious than the components of the tangent vector! Such an idea is expressed in the *Mathematical Gazette* of 1931 by E. G. Phillips, long before Victor Bryant became author of that venerable journal. It is also used in a book by a more recent editor of the *Gazette* (Quadling 1955).

What is less well understood is that there is a clear link between this use of $\mathrm{d}x$ and that in the integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$. If the integral is seen as the limit of a sum of strips height $\mathrm{f}(x)$, width $\mathrm{d}x$, it is useful to interpret this using the graph of $\mathrm{I}(x)$ where $\mathrm{I}'(x) = \mathrm{f}(x)$. Then the sum of the $\mathrm{f}(x)\,\mathrm{d}x$ is obtained by adding up the quantities $\mathrm{I}'(x)\,\mathrm{d}x$. These are the vertical increments to the tangent to the curve $y=\mathrm{I}(x)$, and if very many thin strips are taken, then the curve of $y=\mathrm{I}(x)$, being locally straight, is closely approximated by the tangent over a small interval. Thus the increments $\mathrm{I}'(x)\,\mathrm{d}x$ are approximately equal to the riser to the curve between points with horizontal coordinates x and $x+\mathrm{d}x$. Adding up all the risers gives the total rise from a to b, which is $\mathrm{I}(b)-\mathrm{I}(a)$, and this is the fundamental theorem. Admittedly this idea is here expressed in a loose manner. But it can easily be tightened up to show the meaningfulness of Leibniz's original theory and fulfil the aim of building theory from practical experience.

It would be churlish to base any significant criticism of the text on this particular difference of interpretation which is rooted in the difference between the continental school that followed Leibniz and the British school of Newton. It is a difference which has existed in the British psyche for three centuries and has proved difficult to eradicate.

What is clear with this particular mathematics text is that it does an extremely good job of building on the student's experiences in a meaningful cognitive way. Whilst it does not attempt to face some of the known cognitive difficulties (known to mathematics educators more than professional mathematicians), it has developed appropriate methods to minimise several of them. Others are deeply ingrained in the system and are often resistant to treatment.

I enjoyed reading *Yet Another Introduction to Analysis*. I am particularly enamoured of its continuing struggle to communicate meaningfully to the student in terms that the student is more likely to understand and the provision of complete, friendly, solutions to the exercises. As I read it, I learned mathematical ideas that were new to me and had a considerable education in techniques to make the ideas more clear to the apprentice reader. It is a book which is likely to appeal strongly to mathematicians with a desire to help their students learn, and it has all the ingredients to earn the praise of students too.

In a climate of change, where cognitive theories are beginning to grow but need intensive further study to bring them to fuller fruition, it provides an oasis of sanity and good sense. It is precisely what it sets out to be: another *introduction* to analysis. The mental objects involved – numbers, functions, sequences and limits – are cognitively developed *objects of experience* rather than *defined objects of formal theory,* with properties developed by formal proof. The proofs are based on experiences with these mental objects with a leavening of commonsense logic rather than on formal deduction. Quantifiers are avoided, but will surely need to be the subject of study in other contexts. In an introductory course to a subject with so many comlexities, such omissions are easy to justify. However, their omission will require a further difficult cognitive transition to a formal theory at a later stage. That, perhaps, should be the role of a further course in analysis, at least for those who desire a fully professional mathematical understanding. In making his own selection of topics to be emphasised in an introductory course, Victor Bryant has produced a valuable synthesis for the learner which deserves a hearty welcome.

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