

Students' Mental Prototypes for Functions and Graphs

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This research study investigates the concept of function developed by students studying English A-level mathematics. It shows that, whilst students may be able to use functions in their practical mathematics, their grasp of the theoretical nature of the function concept may be tenuous and inconsistent. The hypothesis is that students develop prototypes for the function concept in much the same way as they develop prototypes for concepts in everyday life. The definition of the function concept, though given in the curriculum, is not stressed and proves to be inoperative, with their understanding of the concept reliant on properties of familiar prototype examples: those having regular shaped graphs, such as x^2 or $\sin x$, those often encountered (possibly erroneously), such as a circle, those in which y is defined as an explicit formula in x , and so on. Investigations reveal significant misconceptions. For example, threequarters of a sample of students starting a university mathematics course considered that a constant function was not a function in either its graphical or algebraic forms, and threequarters thought that a circle is a function. This reveals a wide gulf between the concepts as perceived to be taught and as actually learned by the students.

The concept of a function permeates every branch of mathematics and occupies a central position in its development, yet it proves subtle and elusive whenever we try to teach it in school. Internationally its difficulty is acknowledged (see Tall 1990 for a survey of current research). Yet the idea of a function machine – with an input number giving a corresponding output – is now part of the U.K. National Curriculum in the primary school (algebra attainment target 6, level 3 for children around the age of eight, DES 1989).

The set theory in the “new mathematics” of the sixties and seventies introduced the concept of function in the secondary school in terms of domain, range and rule relating each element in the first with a unique element in the second. The notion proved difficult for most pupils. Somehow the general concept seems to be *too general* to make much sense. Although we may teach pupils about general concepts such as the *domain* on which the function is defined and the *range* of possible values, these terms do not seem to stick in their memories. Instead, they gain their impression of what a function is from its *use* in the curriculum, implanting deep-seated ideas which may be at variance with the formal definition.

In essence the idea of *defining* a concept is at variance with the child's everyday experience. Here a concept such as “bird” would be developed

through encounters initially with examples and then focussing on salient features. “That is a bird. ... A bird *flies*, ... it has *wings* ... and *feathers* ... and a *beak* ... and *lays eggs*”. Then there comes the testing of new creatures against these various criteria. Is a chicken a bird ? ... It has wings, feathers, a beak and lays eggs, but it doesn’t fly. OK, some birds don’t fly. We will say a chicken is a bird. Is a bat a bird? It flies and has wings, but it is really a flying mouse, so it is not a bird.

In this way the individual builds a complex of interconnected prototypes which help to test whether newly encountered examples can be classified as instances of the general concept. (Smith 1988). Is a penguin a bird? It has wings (of a kind), a beak and lays eggs, but doesn’t fly. OK – it has similar attributes to a chicken, so we will accept it. Is a flying squirrel a bird? Highly unlikely – in the same way as a bat isn’t a bird.

In everyday life our development of concepts depends on perpetual negotiations of this kind, which are a deep-seated part of the human psyche. It therefore comes as no surprise that students are likely to apply similar criteria when faced with concepts in the mathematics class.

We hypothesize that the students develop “prototype examples” of the function concept in their mind, such as: a function is like $y=x^2$, or a polynomial, or $1/x$, or a sine function. When asked if a graph is a function, in the absence of an operative definition of a function, the mind attempts to respond by resonating with these mental prototypes. If there is a resonance, the individual experiences the sensation and responds positively. If there is no resonance, the individual experiences confusion, searching in the mind for a meaning to the question, attempting to formulate the reason for failure to obtain a mental match.

We shall see that positive resonances may be erroneous because they evoke properties of prototypes which are not part of the formal definition. For instance, that a function should be described by a formula, or that the familiar graph of a circle is a function. Negative resonances may be equally incorrect: for instance that a strange looking graph cannot be a function because it does not match any of the prototypes, or that a function cannot be constant, because a function depends on a variable and it is considered essential that this variable actually appears in the expression.

Students’ conceptions of a function

Following ideas of gathering evidence about student conceptions of functions in Vinner (1983) and Barnes (1988), we asked a group of twenty eight students (aged 16/17) at the end of their first year of study in a British sixth-form to:

Explain in a sentence or so what you think a function is.
If you can give a definition of a function then do so.

They had studied the notion of a function as part of their course preparing for 16+ exams over a year previously and since then had *used* functions in the calculus but without any emphasis on the technical aspects of domain, range and so on. None gave satisfactory definitions, but all gave explanations, including the following:

- a function is like an equation which has variable inputs, processes the inputted number and gives an output.
- a “machine” that will put out a number from another number that is put in.
- an expression that gives a range of answers with different values of x .
- a form of equation describing a curve/path on a graph.
- a way of describing a curve on a cartesian graph in terms of x and y coordinates.
- an order which plots a curve or straight line on a graph.
- a mathematical command which can change a variable into a different value.
- a set of instructions that you can put numbers through.
- a process that numbers go through, treating them all the same to get an answer.
- a process which can be performed on any number and is represented in algebraic form using x as a variable.
- a series of calculations to determine a final answer, to which you have submitted a digit.
- a term which will produce a sequence of numbers, when a random set of numbers is fed into the term.

It is pleasing to note the number of students who have some idea of the *process* aspect of function – taking some kind of input and carrying out some procedure to produce an output. But not one reply mentions that the process can only be applied to a certain *domain* of inputs, or that it takes a *range* of values, despite the fact that these definitions had been given to them earlier in their studies. Note also the number of technical mathematical words, such as *term*, *sequence*, *series*, *set*, and so on which are used with colloquial, rather than mathematical, meanings. Here lies an inextricably difficult part of the human communication process for both students and teachers. With each of the responses above a teacher may empathise with what the students say and realize that it contains within it the grains of truth. But can we be sure that what another human being says is what we think has been said, or even that the speaker has said what (s)he intended to say? It substantiates the difficulty enunciated by Malik (1980) that *teachers engaged in teaching the function*

concept face enormous difficulties in communicating this abstract concept in the classroom.

Graphs as functions

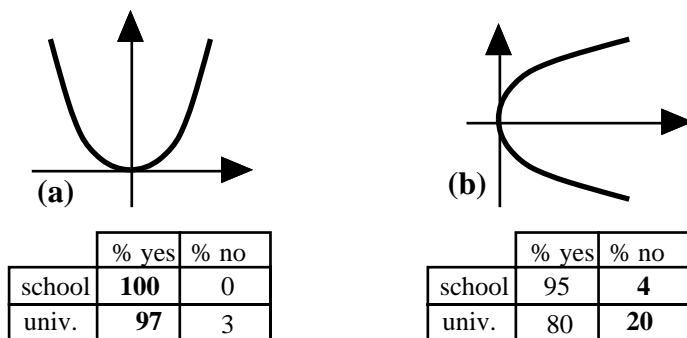
School mathematics is intended to give students experiences of mathematical activities, rather than plumb the formal depths of logical meaning. The formalities may be mentioned, but they are not stressed because they do not appear to be appropriate until the student has a suitable richness of experience. But the collection of activities inadvertently colours the meaning of the function concept with impressions that are different from the mathematical meaning which, in turn, can store up problems for later stages of development.

To investigate this, we asked the twenty eight sixth-formers mentioned earlier to state in a written questionnaire which of a given number of sketches could represent a function. The same questionnaire was given to one hundred and nine students in their first year of university prior to any university study of the function concept. The latter therefore represent the state of development of more able mathematics students at the end of their two years of sixth form study. It would be expected that these students would have a better idea of the function concept, and this was confirmed, but they still had aspects in their concept of function at variance with the formal definition.

Students were given nine graphs, as shown below and asked:

Which of the following sketches could represent functions? Tick one box in each case. Wherever you have said no, write a little explanation why by the diagram.

Here we show each graph followed by a table giving the percentage responses “yes” or “no” for each group. They do not always add up to 100% partly through rounding errors but also due to a small number of non-responses. The response which is more likely to be adjudged correct is given in bold face type. As we shall see, sometimes it is possible for the alternative response to be correct also...



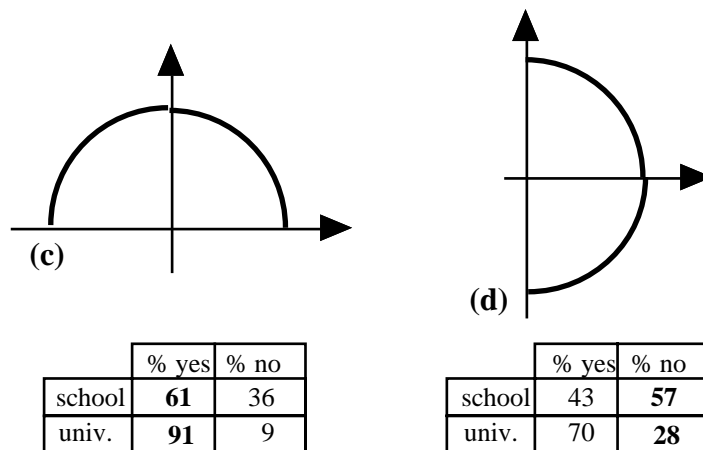
We see that virtually all students agreed that (a) is a function, with the vast majority asserting (b) is also. It was only after we asked this question that we realized that it was formulated in an ambiguous manner. It assumes the usual

mathematical conventions – that the horizontal axis represents the independent variable and the vertical axis the dependent variable. But we did not *say* what we meant – although we think we meant what we said! There was no written evidence that any school student considered (b) to represent x as a function of y . But two university students interpreted the graph in this light - one asserting “look at it a different way”, the other saying “ $f(y)=x$ ”. The increased percentage of university students suggesting (b) was not a function often did so with a comment equivalent to the fact that this “sometimes has two y ’s for each x ”.

A more simple explanation for so many students responding positively to both (a) and (b) is that the term “function” is usually associated with familiar graphs in the sixth form. Both graphs resonate with students’ mental prototypes for functions, so the students respond positively to them.

The single school student who apparently responded correctly to (b) gave no reason and failed to give consistent answers on the rest of the questions. Only one school pupil made any comment at all. He initially thought that (b) was not a function, saying “you have got two y -values for one x value”, then changed his mind and crossed out his comment. It was as if he did remember the function definition, but then his thoughts were overwhelmed by more recent experiences of the function concept loosely linked to familiar graphical prototypes.

When the same question was asked in an analogous case using semicircles instead of parabolas, the responses were radically different:



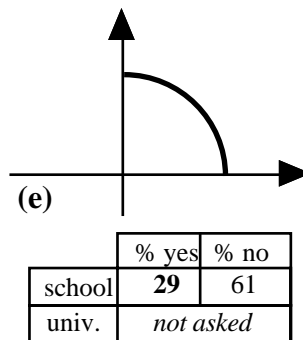
There is a drop to 61% of school pupils thinking figure (c) is a function and 57% now correctly respond that figure (d) is not. The drop in belief in figure (c) compared with (a) was accompanied with comments such as:

- “if a function the graph would continue, not just stop”,
- “stops dead, values are not limitless”,
- “the lines would have to continue”,
- “functions are usually continuous, \therefore needs a condition”,
- “this could not apply to any value”.

Here the word “continuous” does not seem to have its usual mathematical meaning, but the colloquial meaning of “continuing without a break”. Several of the explanations allude to ideas such as “continue, not just stop”, “stops dead”, “could not apply to any value”, which suggest that there is a feeling that functions should not be unnaturally curtailed. One student dotted in an extension of the graph to “continue” it for more values of x . This time there was no written evidence that any students were regarding x as a function of y , but this remains a possibility, certainly amongst the large number of positive university students.

The functions the students have handled in their course are polynomials, trigonometric functions, and their like, which are naturally defined by a formula almost everywhere (except a few odd points where the expression may be undefined). Thus we may conjecture that their prototypes are “naturally defined everywhere the function is defined”, leading to apparent unease with “artificial” functions such as the top half of a circle.

The idea that a function should not be unnaturally curtailed is given more credence by the fact that only 29% of school pupils regarded (e) to be a function (this graph was not given in the university questionnaire):



Reasons for this included:

“couldn’t apply to any value”

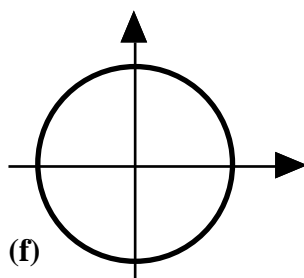
“if a function the graph would continue, not just stop”,

again suggesting a sense of unease when the graph seemed arbitrarily restricted to a smaller domain. The school pupil’s belief in a graph being a function through pictures (a), (c), (e) drops from 100% to 61% to 29% as the graph passes from parabola to semicircle to quadrant, becoming less familiar and restricted to a smaller and smaller domain. As one pupil wrote about the quadrant:

the graph is “not complete”.

Discussion afterwards revealed that he thought of it as *part* of a circle, so it was not a function because it was not all drawn. To this student a function is a natural totality given by a formula, and it is essential to have it *all*, not an unnaturally selected part.

Although a quadrant of a circle (which is the graph of a function) is considered not to be a function by most pupils, the situation is reversed with a complete circle. Approximately two thirds of the students in school and university incorrectly considered the circle in figure (f) to be a function:



	% yes	% no
school	64	29
univ.	65	35

Those thinking it was not a function included two from school saying:

“You can’t work a function that goes back on itself”.

and

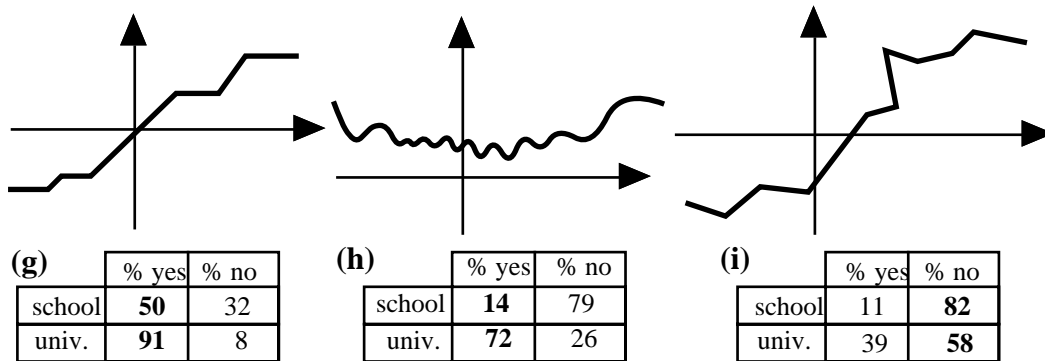
“equation is $x^2+y^2=25$ ”,

which implicitly – but not explicitly – suggests that y is not determined uniquely by x . Amongst the minority of university students who (correctly) thought it was not a function, most alluded to the idea that each value of x might be related to more than one value of y .

The persistence of two thirds of the students thinking a circle is a function once more suggests that familiarity with the graph evokes the function concept. This belief bears little relationship to most of the descriptions of a function given by the pupils in terms of processes.

Another highly probable reason for so many pupils thinking that a circle is a function arises from the use of language in the mathematical classroom. Many of us still use the term “implicit function” (or “many-valued function”) to describe such a relationship, and the circle is a prototype example of this phenomenon. Indeed one of us has published an “Implicit Function Plotter” (Tall 1985) which draws just such a graph... *et tu Brute!*

The final three pictures presented to students – (g), (h) and (i) – presented even more conflict. They look strange, so none of them fit the students’ mental collection of prototypes.



Both (g) and (h) could satisfy the function definition, but not (i) because there is a part of the graph where one value of x corresponds to more than one value of y .

In general the university students cope better with these more general curves. The fact that more school pupils seem successful with (i) is an illusion, due to the fact that they deny that (i) can be a graph because it looks unfamiliar, rather than because of any formal property of a function. Time and time again they respond that a graph cannot represent a function because it looks too irregular or because they cannot think of a formula to represent it:

- (g) (h) (i) are not functions because: “graphs are usually smooth, either a straight line or curve, not a combination of the two, nor staggered, when dealing with a function”,
- no - “because the lines above are totally random”,
- “non-uniform”,
- “these are absurd”,
- (h) is “too complicated to be defined as a function”,
- (h) is “totally irregular and couldn’t be represented by a function”,
- (h) has “no regular pattern too difficult to be defined by a function”,
- (h) is not a function because “curves and straight lines don’t mix”,
- (h) is “too irregular”.

Even when (i) is correctly stated to be not a function, the reasons are often related to the irregularity of the pattern or the lack of a formula. Again we ask if the concept of “regularity” of a function is actually taught. We think not. None of these graphs match their mental collection of prototypes for the function concept. Because their experience is usually in terms of graphs given by a formula which tends to have a recognizable shape, their prototypes tend to be “given by a formula”, have a “smooth” graph, seem “regular” and so on. They therefore verbalize some of their perceived mismatches in their own words.

Three school pupils do focus on the part of the graph where there are three y -values for each x -value:

“here the curve goes back on itself”,

“this goes back on itself”,

“there is an irregular peak which could not be created from a function”.

They are beginning to evoke the restriction that each x should have only one y . But they have not applied this test consistently in the earlier examples, and the definition of a function given by each of them does not mention this fact. For these three a function is:

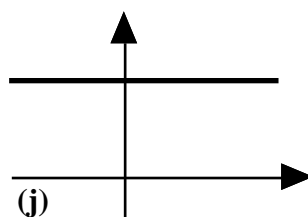
“a mathematical command or identity”,

“an equation with a variable factor – tells us what happens to a variable factor, e.g. $f(x)=x+2$ ”,

“the product of a series of numbers which the numbers must undergo”.

Thus *not one* of the school pupils consistently evokes a coherent function concept. Only eight of the university students (7% of the total) gave a consistent set of replies to all the graphs, with one further student giving consistent replies in which he allowed x to be a function of y as well as y to be a function of x .

One graph was given to the university students, but not to those at school (in lieu of graph (e) above):



	% yes	% no
school	<i>not given</i>	
univ.	55	44

Here almost half the students at university think that a constant is not a function. It appears that they are concerned that y is not a function *of* x , because y is independent of the value of x . Where do students pick up such ideas? Which of us teachers actually teach them this interpretation? Very few of us would admit such a heinous crime. Yet one of us writing a module for the new 16-19 SMP A-level found himself writing that the differential equation $dy/dx=1/x$ describes dy/dx as a function of x but *not of* y . In different contexts we use the same words with different meanings. Clearly implicit in school mathematics is that the notion of a function is to do with variables, and if a variable is missing, then the expression is not a function of that variable.

Algebraic expressions as functions

To look at the meaning of a function in terms of formulae (as in Barnes, 1988), we asked the university students to say which of a number of symbolic expressions or procedures could represent y as a function of x . Some of these

were algebraic equivalents of the pictorial representations mentioned earlier. The responses are given in table 1. Thirty eight of the 109 students explicitly mentioned at least once in their response that, for each x there must be one y , or that the function must be “many-one” or equivalent. In addition to the total percentage of students responding yes or no, we include two extra columns (“% yes*” and “% no*”) representing the percentages of these 38 “more knowledgeable” students. The latter have, at some stage in their earlier career, encountered and now remember more technical aspects of the function concept and we would expect them to perform better. The rest, of course, may have discussed such technical aspects but do not evoke them explicitly in their response.

	University students ($N=109$)		Subset showing some technical knowledge ($N=38$)	
	% yes	% no	% yes*	% no*
(1) $y = x^2$	96	4	95	3
(2) $y = 4$	30	69	47	53
(3) $x^2+y^2 = 1$	62	37	40	60
(4) $y = \frac{3}{x}$	91	9	84	16
(5) $xy = 5$	82	17	82	18
(6) $y = \pm \sqrt{4x-1}$	67	33	34	66
(7) $y = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } x > 1 \end{cases}$	92	7	95	5
(8) $y = 0$ if x is a rational number	50	48	42	58
(9) $y = 0$ (if x is a rational number), $y = 1$ (if x is an irrational number).	75	22	79	21

Table 1

Once again the expression $y=x^2$ is almost universally regarded as a function, but the constant $y=4$ is not. As in Barnes (1988), a majority of all students consider the circle $x^2+y^2=1$ to be a function. In each of the latter two cases those exhibiting a more technical knowledge perform better, but still only 47% think that $y=4$ is a function whilst 60% think that $x^2+y^2=1$ is not.

Expressions (4) and (5) show that the majority of students see $y=3/x$, $xy=5$ as functions, the major obstacle for the first being that it is not defined for $x=0$,

and for the second, not only is it not defined for $x=0$, but the expression is not considered a function until it has been manipulated to get “ y as an expression involving x ”. The latter is a common prototype for a function.

Expression (6) shows that the majority of students think that $y = \pm \sqrt{4x-1}$ is a function. This resonates with the “ y equals an expression in x ” prototype. The fact that y is not given uniquely is less significant for the majority, although the minority giving more technical responses show a marked improvement because they are consciously aware that a function must give (at most) one value of y for each value of x .

Expressions (7), (8) and (9) address the problems of defining functions differently on different sub-domains. These do not fit the prototypes familiar to most students. Even so, the correct response to (7) is remarkably high. Experience suggests that students whose function prototypes involve a single formula may consider expression (7) not as one, but as *three* different functions (Vinner 1983). In fact, *no* student made such a comment, indeed, those failing to respond positively were more concerned that the printing of the inequality signs might be ambiguous. Perhaps it helps in this case that each formula on the subdomains is familiar and that the function is everywhere defined. Certainly the fact that (8) is not everywhere defined caused problems because:

“ y is not defined for all x ”,

“doesn’t state what y is if x is not rational”,

“no definition of y if x is irrational”.

The difficulties with (8) and (9) seem also due to the strangeness of these expressions and the fact that they do not fit the students’ mental prototypes.

(8) “is not a function of x , there is no connection mathematically”,

“no real link with x , i.e. not actually applying a function to x , where the answer would be y ”,

“ y is not in proportion to x ”,

“no relation between x and y ”,

“not continuous on the real number line”,

“ $y=0$ is constant”,

“ y doesn’t change as x changes”.

Conflicts with constant functions

Comparing student performance on the *expression* $y=4$ and the *graph* of $y=\text{constant}$, we find only 28% reply correctly in the affirmative to both. 41% respond negatively to both questions, 29% say the graph corresponds to a

function but the algebraic expression does not, with only 3% the other way round (table 2). The percentages for the 38 students giving more technical responses are starred in brackets. Although the percentage of correct responses rises from 28% to 42% for these students, it is still only a minority.

Is $y=\text{const}$ a function?		algebra	
		% yes	% no
g r a p h	% yes	28 (42*)	29 (26*)
	% no	3 (5*)	41 (26*)

Table 2

There is evidence of conflict in a significant number of scripts, as students change their mind when realizing that the algebraic expression clearly does not involve x , but the graph seems more likely to be a function. One student who thought initially that $y=4$ was not a function, then wrote it as $y=4x^0$, hence obtaining “a formula involving x ”. This may very well be related to the description of the relationship between x and y in terms of variables: that the dependent variable y varies as the independent variable x varies. The expression $y=4$ offends this prototype because y does not vary!

The circle as a function

Comparing the responses to the graphic and algebraic representations of a circle, we find that 52% erroneously regard both graph and expression as representing functions, 12% say “yes” to graph and “no” to expression, 10% say “no” to graph and “yes” to expression, and only 25% correctly say “no” to both (table 3). The more technical responses increase the percentage correct from 25% to 47% – still less than half.

Is a circle a function?		algebra	
		% yes	% no
g r a p h	% yes	52 (18*)	12 (24*)
	% no	10 (11*)	25 (47*)

Table 3

The position is worse when we consider which students give a correct response to both questions in algebraic and graphic modes:

Only 11% of all students assert both that $y=\text{constant}$ is a function and a circle is not. The percentage only increases to 29% among the more technical responses.

Thus, even amongst the most able students in the sixth form, the vast majority do not have a coherent concept of function at the end of their A-level studies.

Reflections

Because the general function concept is difficult to discuss in full generality we take the pragmatic route of de-emphasizing theory and emphasizing practical experience. Attempts to teach the formal theory, as in the New Mathematics of the sixties, have proved unsuccessful. But the other side of the coin – teaching the concept through examples, as in the current curriculum – leads to mental prototypes which give erroneous impressions of the general idea of a function. Even amongst the students who receive some training in the notion of a function, only a small minority respond coherently and consistently.

We have described some of the symptoms, but not the cure. The function concept is an extremely complex idea whose wider ramifications took centuries to be made explicit. In the development of the individual student the full implications only become apparent over a period of several years. We therefore believe that there are bound to be conceptual obstacles as the concept matures in the mind. When the function concept is introduced initially, the examples and non-examples which become prototypes for the concept are naturally limited in various ways, producing conflicts with the formal definition. We can attempt to give more general experiences which will improve the situation, but we face a formidable, fundamental obstacle:

The learner cannot construct the abstract concept of function without experiencing examples of the function concept in action, and they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept.

The literature is littered with examples of failure to comprehend the full complexities of the function concept (Dreyfus & Vinner 1982, Vinner 1983, Even 1988, Markovits *et al* 1988, Barnes 1988, Tall 1990). Clearly, if we are to make progress we must attempt to develop an approach which makes the prototypes developed by the students as appropriate as possible. One promising approach is the use of computer programming to encourage the student to construct functions *as processes* through programming the procedures which take an input and process it to give the corresponding output. Successful steps have already been made in this direction (Breidenbach *et al*, to appear).

However, we should continue to be aware of the conflicts which will occur from time to time as the learner has new experiences of sophisticated mathematical concepts. It is the awareness that mental reorganization to cope with increasing complexity is both difficult and necessary that will help us design more appropriate curricula in the future.

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