Reflections

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The production of this book is a first stage in a journey which sixteen authors and a wider group of co-workers in Advanced Mathematical Thinking have shared. It is pertinent, given the nature of the thinking processes that we have unfolded, to reflect upon what we have done with the spiral of conceptual development in mind. First one begins with a problem which may not be well-defined. Then one uses what tools are available to attack the problem as it progressively becomes clearer, with all the false starts and hard-won minor advances that are inevitable ingredients of the struggle. And now there is a calm after the storm to reflect, to see what gains have been made and what remains to be done.

It would be good to be able to look back on the definitive book on *Advanced Mathematical Thinking*, with all the resolutions of all the problems that occur and a coherent theory that explains what it is and how to help others achieve it. This task is not yet complete, certainly not in a definition-theorem-application format that a mathematician might require of a theory. What has been done is to set out on a journey, on which the reader has been encouraged to participate, to consider the way in which advanced mathematical thinking functions, to understand what makes some thinkers successful and to help others on their journey to greater success. “The journey is the reward”. And at this time we can look back on the pathways we have taken to see what problems have been well-formulated and what solutions have appeared as we move on to the next stage of the journey.

For me, as editor, it has been a fascinating study to see the development of various parts of a theory, to see consonances and dissonances, some of which have been resolved whilst others remain suspended in the ether. At the beginning of the journey I saw through a glass darkly. I have yet to see face to face.

But now there are clearer avenues to follow, beginning with a more focused picture of the nature of the advanced mathematical thinking and moving towards pertinent questions and partial answers. First we must highlight the different ways in which individual mathematicians may think successfully. In particular, the need for all of us, successful in our various ways, to give space to others to help them use their own particular talents to build up their mathematical thinking processes. Then there is the realization of the thorny nature of the full path of mathematical thinking, so much more demanding and rewarding than the undoubted aesthetic beauty of the final edifice of formal definition, theorem and proof.

This is the final chapter of the book *Advanced Mathematical Thinking*, (ed. David Tall), Kluwer: Holland, 251–259 (1991).

It is clear that the formal presentation of material to students in university mathematics courses – including mathematics majors, but even more for those who take mathematics as a service subject – involves conceptual obstacles that make the pathway very difficult for them to travel successfully. And the changes in technology, that render routine tasks less needful of labour, suggest that the time for turning out students whose major achievement is in reproducing algorithms in appropriate circumstances is fast passing and such an approach needs to move to one which attempts to develop much more productive thinking.

It is therefore no longer viable, if indeed it ever was, to lay the burden of failure of our students on their supposed stupidity, when now the reasons behind their difficulties may be seen to be in part to be due to the epistemological nature of mathematics and in part to misconceptions by mathematicians of how students learn. We often teach certain skills because we know that these will bring visible, albeit limited, success, but we now know, somewhat furtively, that the acquiring of those skills may develop concept imagery that contains the seeds of future conflict. We have evidence that a formal approach, which appeals to the sophisticated expert may be cognitively totally inappropriate for the naïve learner and demands new forms of teaching to pass through the transition from elementary mathematics to a point where the economy and structure of modern mathematics is seen as a meaningful goal.

It seems incredible that our list of references is largely dominated by papers written in the last decade with only a few honourable exceptions before the early eighties. What has emerged from a meeting of individuals over a five year period, to reflect on this newly developing area of concern, is a clearer understanding of the full cycle of mathematical thinking: the need to begin with conjectures and debate, the need to construct meaning, the need to reflect on formal definitions to construct the abstract object whose properties are those, and only those, which can be deduced from the definition. Advanced mathematics, *by its very nature*, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal re-construction to build the cognitive apparatus to handle them effectively. It involves a struggle which virtually every author in this book, both severally and individually, sees in terms of a reflection on personal knowledge and a direct confrontation with the inevitable conflicts which require resolution and reconstruction.

College professors see this conflict daily in individual students as they struggle to come to terms with new ideas. In the past they have often tried to help by providing clearer lectures, making the transitions as simple as possible, presenting the ideas in a way which reduces the strain. This may even lead to the successful professor being lauded by his or her students for the clarity of their exposition, but the acid test is *what do the students learn*? And this needs to be assessed in a wider sense than just which algorithms closely related to their course they can carry out, or which definitions and proofs can they correctly reproduce.

Our cognitive studies have shown the manifold differences between the formal definitions of concepts and the images we use in our minds to work with these concepts. They show how the complexity of the subject demands a "chunking" of information in an efficient way so that it can be easily handled, and this is linked to the appropriate use of symbolism for a given context and the appropriate meaning which the individual links to that symbolism.

We have seen a divergence between the visualisers and verbalisers amongst us, just as there appears to be a time-honoured difference between the mental processes of the mathematical giants of the past. In recent months, as I have interacted with the various authors in an attempt to either come to an agreement or to hone our differences into explicit focus, I have been privileged to gain some additional insights.

It is clear that mathematics without process to give results is of little value, in other words, visualising an idea without being able to bring it to fruition is virtually useless. I emphasise this fact even though a major thrust of my work is in the use of visualisation. On the other hand, simply to be able to carry out procedures in a narrow way, without being able to see the overall connections, is also grossly limiting. For me this has led to a belief in a versatile form of thinking which complements the procedural with the global overview. However, we have evidence of mathematicians, such as Hermite, steeped in the logic of their subject who develop a powerful intuition of the processes and their symbolism in such a way as to render visualisation – for them – redundant. We also have evidence of successful students (such as the case of Terence Tao, Clements, 1984) who vastly prefer the power of logical deduction. We therefore need to cater for different types of minds.

Recently, however, in a very different context, I was able to obtain an insight which may prove helpful in this apparent dichotomy. Mathematics – according to the Oxford Dictionary – is said to be "the Science of Space and Number". In recent months I have been reflecting on the fundamental differences between these two different forms of mathematics and the manner in which they develop cognitively. Space, through the study of geometry, begins with gestalts – "that is a triangle", "this is a straight line", "that there is a rectangle" and "this is a square". The child learns to recognise these visual gestalts from examples and non-examples. "Yes, that is a square, but don't think it is not a rectangle, because a square is a special kind of rectangle". Through exploration and interaction with others, the child learns to discriminate between these various gestalts and to isolate some of their properties: "a rectangle has four right-angles and opposite pairs of sides equal", "a square has four right-angles and all four sides equal” and to begin to see relationships “an isosceles triangle has two equal angles and two equal sides”. From here the relationships begin to build into deductions “*if* a figure is a square, *then* it is a rectangle”, “*if* a triangle has two equal sides, *then* it has two equal angles”, definitions begin to be isolated and, finally, these can be formulated in an axiomatic way to give the framework for logical deduction. Indeed, what I have just described in outline was formulated about the development of geometry more clearly as a hierarchy over thirty years ago by Van Hiele (1959).

Number on the other hand is a very different animal. It begins with imitation of the number names recited in sequence, “one, two, three, ...”, perhaps imperfectly at first, “... four, five, nine, seven, ...”, then with more confidence, until the routine of pointing at objects and reciting the number names in proper sequence leads to the concept of counting. This is an encapsulation. The *process* of counting leads to the *concept* of number. By various further strategies of process, “counting all” of two sets (a coordination of two processes), or “counting on” (combining the concept of number of the first set with the process of counting the second) leads from the process of counting to the concept of “sum”. A vital phenomenon occurs here in that the symbolism 4+3 represents to the user both the process of counting and the product of that process, the sum. The rest of the “number” part of mathematics proceeds, in the same way, by encapsulating processes as concepts, *often using the same symbolism for both process and concept*. Thus the process of “repeated addition”, “five threes” becomes the concept of “product”, “5 times 3” – both written as 5x3. The process of “repeated multiplication” becomes the concept of “power” and so on. Of course this prescription is exactly parallel to the discussion of Dubinsky on reflective abstraction. It is a phenomenon known to Piaget and to many an observant teacher since time began – except that there is an amazing simplicity about what is being done. In the number side of mathematics the mathematician makes progress *by being ambiguous* about notation. (S)he uses the same notation for process and product *deliberately*, so that (s)he can powerfully use whichever is appropriate for a given task. To *calculate* means to use the *process*, to *manipulate* is easier with a single object which involves using the product.

As whole number generalizes to signed integer, the symbols +2 and –7 also have dual roles as process and the product: “shift two units right on the number line”, “the integer plus two”, “shift seven units left”, “the number minus two”. The same happens with fractions: “3/4” is both “divide three by four” and the product of the process: “three-quarters”. It is the same with trigonometry, where sine = opposite/hypotenuse

is both an instruction to calculate and a symbolism for the result.

Algebra too exhibits this same dualism of notation where $2+3x$ means both the process of adding two to the product of three and x and also the result of that process.

In chapter 4 Hanna remarks on the irony that in a "discipline touted as precise, the student must develop a tolerance for ambiguity". Instead of being defensive about this state of affairs, it is more appropriate to note that the successful mathematician is the individual who sees the duality of this kind of notation as process and product and who uses the ambiguity in a flexible way. Given the importance of a concept which is both process and product, I find it somewhat amazing that it has no name. So I coined the portmanteau term "procept" for a process which is symbolized by the same symbols as the product. It seems that the whole of number and algebra is built on procepts, so a theory of procepts and their use in mathematics has a vast potential domain of application.

Yet space and geometry are different. They seem to be built on gestalts whose properties are only slowly teased out and put into coherent relationships, then definitions and deductions.

There are therefore (at least) two different kinds of mathematics. One builds from gestalts, through identification of properties and their coherence, on to definition and deduction at advanced levels of mathematical thinking. The other continually encapsulates processes as concepts, to build up arithmetic, then generalizes these ideas in algebra before formalizing them as definitions and deductive theorems in the advanced mathematics of abstract algebra.

If we look at the discussion of Vinner in chapter 5, we find his theories originally began with *geometry*, and his examples include "car", "table", "house", "green", "nice", etc. *None of these are procepts*. However, if we look at the discussion of Dubinsky in chapter 7, we find his examples include "commutativity of addition", "number", "trajectory" (as a coordination of successive displacements), "see-saw" (as the balancing of two objects), "multiplication", "fluid levels" (as a 'variation of variations'). All these are *processes* which become encapsulated as *concepts*. As they stand, they are not all procepts within the narrow meaning of the term just defined. However, they all involve manipulation of quantities, or balancing of quantities, or variation of quantities, and this in turn involves number, which brings us back to proceptual ideas where symbolism is used both to represent a process of manipulation and the result of that process.

The fascinating thing is that, by the time we reach the level of formalism in advanced mathematics, these two different strands move to a similar formulation: the *definition* of concepts and the *deduction* of properties of those concepts.

I believe that the major catastrophe of the new mathematics movement was due to the unproven assumption that "if only the concepts are properly defined, then everything will be OK". The need for clear definitions and deductions caused mathematicians to be covert about the power of their ambiguous use of procepts. This move served our students badly because it failed to acknowledge the methods of the working mathematician. The power in mathematics is not given through unique and precise meaning to symbolism – "a function is a set of ordered pairs such that …" – but through a *duality* which gains *flexibility* through *ambiguity*[1] – a function is both a *process* (to be able to calculate) and a *concept* (which can be manipulated). It is as simple as that. We cheated our students because we did not tell the truth about the way mathematics works, possibly because we sought the Holy Grail of mathematical precision, possibly because we rarely reflected on, and therefore never realised, the true ways in which mathematicians operate.

The evidence which we are collecting with a wide range of ability of much younger children is that the most able naturally use this flexibility (Gray, 1991). In arithmetic they soon learn a few facts then, when they are faced with a new arithmetic problem, they are often able to relate it to one they know and derive new facts from old. The more able therefore have *a built-in knowledge generator* that develops new arithmetical knowledge from old. Once they grasp this, they realise that they do not need to remember so much because they can soon derive what they want to know. They have a flexible *proceptual* knowledge in which number problems such as 4+5 can be decomposed as the process 4+5, which might be seen as 4+4+1 and (if they know 4+4=8) can be reassembled as 8+1=9. Thus the procept 4+5 is decomposed into process and parts of this are recomposed back to derive the concept, or result, 4+5=9.

Meanwhile the lower ability children remember few facts and continue to use the process of counting to add numbers together. If asked 8+4, they faithfully count on four to get "nine, ten, eleven, twelve" but this is rarely remember as a known fact and, instead of having a knowledge generator, *they have an unencapsulated process which produces answers which are not manipulable objects*. Thus there grows a "proceptual divide" between the more able, using proceptual flexibility, and the less able, locked in process.

The same proceptual divide occurs with algebra. The child who sees algebraic notation only as process, is faced with a nightmare, for how can (s)he conceive of $2+3x$ as a process when, without knowing x, it is a process which cannot be carried out. And if x is known, why is it necessary to use algebra anyway?

[1] I am grateful to my colleague Eddie Gray for this phrase, which comes the title of a joint paper (Gray & Tall, 1991) based on his work with the number processes of younger children.

Only the child who can give meaning to the symbolism as a conceptual entity can begin to manipulate more complex expressions meaningfully in the sense of Harel and Kaput in chapter 6.

This same division between those who conceptualise process as product and those locked in process occurs again at higher levels. The limit concept $\lim_{n\to\infty} a_n$ is again a *procept*. The same notation represents both the *process* of tending to the limit, and also the *value* of the limit. But this phenomenon is very different from procepts met in elementary mathematics. There the process could be used to *calculate* the product. Now we have the phenomenon that Cornu identified as an obstacle in chapter 10 understanding the dynamics of the process does not lead directly to the calculation of the limit. Instead indirect alternative methods of computation must be devised.

Just as with arithmetic, the theory of limits has a structure for devising new facts from old. But in arithmetic the new facts are derived from old using the calculation processes of arithmetic and the new facts have the same status as the old: they can be calculated by the processes of arithmetic in the same way. In the case of the theory of limits, the "known facts" are one or two "elementary" deductions from the definition: that $\lim_{n\to\infty} 1/n$ is zero, or that a constant function and the identity function are continuous. All three of these "elementary" facts are derived from the definitions in singularly peculiar ways which can cause initial confusion. The fact that $1/n$ tends to zero might be deduced from Archimedes axiom, or perhaps by some heuristic appeal to the fact that :"I can make $1/n$ smaller than $\varepsilon{-1}$ by making n bigger than the integer part of ε plus one", both of which are strange ways of asserting $1/n$ gets small as n gets large – the student *knows* that anyway! To establish the fact that a constant function is continuous is just "tell me ε and I will tell you δ, in fact you can take any $\delta>0$ you like, say $\delta=1066$". It is a joke that few students have the experience to find funny. The continuity of the identity function is equally enigmatic "OK, take $\delta=\varepsilon$, then, when $|x-x_0|<\delta$, we have $|f(x)-f(x_0)|<\varepsilon$, because $x=f(x)$, can't you see, you dummy?" Unlike arithmetic, once these few "elementary facts" are deduced, few, if any, other such "facts" are calculated directly. Instead the "algebra of limits" is proved, using the coordination of the "unencapsulated definition of the limit" as reported in chapter 10, which is at, or beyond, the zone of competence of most students. The result is that the derived facts are "proved" (any polynomial is "continuous" by an induction argument combining sums and products of constant functions and the identity) yet the actual definition is no longer used because the calculations become horrendous.

Thus it is that the procepts in advanced mathematics work in a totally different and completely enigmatic way compared with the procepts in elementary mathematics. It is no wonder that, faced with this confusion, so many students end up conceiving the limit either as an (unencapsulated) process or in terms of meaningless rote-learned symbol pushing.

Likewise the gestalt geometric concepts work differently in advanced mathematics too. Instead of being “described” and having coherent relationships, they are “defined” and other properties must be “deduced” from the definitions. Again, given the conflict between the elementary ideas where the facts are known and the abstract ideas where they need to be deduced, confusion as discussed by Vinner (chapter 5), is almost inevitable.

So what is the solution? First it should be noted that the chapters of this book nowhere give methods that will produce guarantied success. There is no dispute that, for the most able, a formal presentation may be sufficient to show the structure of the subject which they may appreciate and build into a deductive system. But for the vast majority of students, the way ahead is stony and littered with cognitive obstacles which, if not addressed, will only be isolated in the mind in such a way that they lie there ready to cause conflict in future times – if they do not cause outright confusion already.

The evidence is that students of a wide range of abilities prosper when they can give meaning to the ideas. This does *not* mean that they must always relate the concepts back to some concrete foundation that has physical meaning. Just as the child who counts objects successfully moves on at a very early age to mentally manipulate number symbols in arithmetic, so successive layers of encapsulation of process into procept only need refer to the level of the previous proceptual layer. In fact, once the encapsulation has occurred, the use of the same symbolism causes the process and concept to coalesce into a single level. Thus the so-called hierarchy of concepts, which is an obstacle to learning, becomes, to the successful encapsulator, a single level in which process and concept are dually represented, with the complexity disguised by the simplicity of the symbolism.

The question to be addressed is: if this is the way of success for the more able, what should we do with, or for, or to, the vast majority of our students? The evidence in this book is, that to give them a sense of the full range of advanced mathematical thinking, it is essential to help them reflect on the nature of the concepts and the need for mental reconstruction in an overt and explicit way, and to give them opportunities in which they can learn to conjecture and debate, so that they may participate in mathematical thinking, not just learn to reproduce mathematical thought.

This is not going to be an easy task. What stands against it is, in many cases, fear. Fear of professional mathematicians for the unknown when they leave their neatly planned course structures of theorem-proof-application and give open-ended opportunities for problem-solving. Fear of the increased time that this will take and that they will not “get through the course”. Fear that “standards will drop” because students will not be able to exhibit the ability to carry out all the processes that need to be taught in an “honours degree”. Fear that they dare not make any changes whilst other institutions maintain the traditional standards.

In recent years the fast changes in society are causing all of the well-established truths to be reassessed. In Britain through the Institute of Physics, university departments of physics have mutually agreed to reduce the content of the three year physics course by one third to give more room for understanding what is actually taught. In mathematics a step in similar direction might not be out of place. It is not necessary to change the whole of the approach in a single step. Given a modest reduction in content, a new flexibility could allow, say, a single course in problem-solving, of a general nature, to be introduced early in the course, to encourage creativity in mathematical thinking, even though it introduced no new content, but compensated in terms of reflection on higher processes. For ten years I have run such a problem-solving course and I know the way it changes students’ perceptions of themselves and builds up confidence through success in small things that steadily grow more complex. They learn to *talk* to each other, to *verbalise* mathematics, to *speak* coherently. They even learn to enjoy interchanging information and helping each other, whereas before they had often believed that good students only do mathematics for themselves, on their own.

Given a modest reduction in content, it might be possible to allow time for students to explore their own conjectures in a specific subject area. In my own analysis lectures I regularly set up a problem scenario and leave the students to work in groups to try to solve the problem. “OK, so the intermediate value theorem seems obvious, but suppose you knew f was continuous between a, b and that f(a) and f(b) had opposite signs – how would *you* prove that it is zero in between?” Setting this as homework does not have the same effect as encouraging students to talk together in class time, and the best way to do this is for the instructor to make sure that there is a good topic for investigation and then leave the room. Some of my best teaching occurs when I am somewhere else drinking coffee and getting paid for it! A return to the classroom after an appropriate passage of time may find that the students have not solved the problem, but they often have experiences on which a proof can then be constructed through a mutual dialogue. In this way they learn to participate in the construction of mathematical knowledge rather than just remembering and repeating it.

Viewing the third part of this book – the review of the literature – we see authors adopting very different stances. Robert and Schwarzenberger highlight the difficulties of the transition from school to university. Eisenberg begins with a catalogue of failure in the teaching of the function concept. Cornu is fascinated by the processes of knowledge creation and the parallel between the epistemological obstacles in the past and the cognitive obstacles of students. Artigue continues the study to further levels of mathematical analysis and certain avenues of hope begin to appear. Tirosh reviews the cognitive conflicts inherent in contemplating the infinite and gives a detailed report of a single experiment exemplifying how students may be taught to reflect on their knowledge and actively participate in its reconstruction. Alibert and Thomas look at the process of proof and show the difficulties of the formalism and how it might be tackled through debate. Finally Dubinsky and I look forward to the use of the computer and the way in which this may change the nature of mathematics and provide an environment for learning. Despite the different tone of some chapters, the message of hope for reflective reconstruction of knowledge is there in all of them.

My recent thinking has led me to realise that the computer can be used in a very special way in learning – to carry out the processes, so that the user can concentrate on the product. This is the essence of a spread-sheet, a graph-plotter, a symbol manipulator, and so on. In other words, the computer allows a change in the encapsulation procedure from process to object. Instead, of forcing the student first to learn and interiorise the process, the computer can carry out the process and allow the user to focus on the properties of the product. In this way there can be a shift of attention away from the process (in which the less able may become trapped) and towards the mathematical objects, and their relationships at a higher level. Instead of just learning the processes of *solving* differential equations, students may first appreciate the *existence* and *uniqueness* of solutions, and construct them in a meaningful, quasi-physical way, building an approximate solution curve by putting together short straight-line segments of the appropriate gradient.

Thus the final plank in the new charter of advanced mathematical thinking in the information age is what I have termed the *principle of selective construction of knowledge*, in which the learner is allowed, even encouraged, to separately focus on the processes of mathematics and the procepts produced by those processes. It is now possible to get a computer to carry out the algorithms so that the student can concentrate on the properties of the product. In this way the student can be encouraged to construct the properties and relationships enjoyed by the product whilst supressing consideration of the process which is constructed internally by the computer. The student may at one time selectively concentrate purely on the process and at another on the higher level relationships. *Both* activities remain essential, for the process is needed to be able to *do* mathematics and the higher level relationships are essential to fit it together in a meaningful way. The interesting factor is that the focus on the process need not always precede the construction of the properties of the product. The intuitive idea of existence and uniqueness of differential equations can be investigated before formulating any symbolic solution. In this way the use of the computer gives new teaching and learning strategies in advanced mathematics.

We therefore arrive at a possible new synthesis in teaching and learning advanced mathematics which offers a more complete cycle of advanced mathematical thinking to students, even those of more modest abilities. The active participation in thinking is essential for the personal construction of meaningful concepts. Students need to be challenged to face the cognitive reconstruction explicitly, through conjecture and debate, through problem-solving, and they may be assisted in the acquisition of insights at higher levels by selectively sharing the construction with the computer. This does not remove the need to pass on information in the theorem-proof-application mode, for this is the crowning glory of advanced mathematics. But students need to be assisted through a transition to a stage where they see the necessity and economy of such an approach. Therefore, step by step, through professors being given a little space to experiment, initially as part of a traditional curriculum, a new balance may be struck, between the shining edifice of advanced mathematics that is the rightful pride of the mathematical community and the fuller range of advanced mathematical thinking that gave rise to its construction.