The Psychology of Advanced Mathematical Thinking

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Introduction

In the opening chapter of *The Psychology of Invention in the Mathematical Field*, the mathematician Jacques Hadamard highlighted the fundamental difficulty in discussing the nature of the psychology of advanced mathematical thinking:

> ... that the subject involves two disciplines, psychology and mathematics, and would require, in order to be treated adequately, that one be both a psychologist and a mathematician. Owing to the lack of this composite equipment, the subject has been investigated by mathematicians on the one side, by psychologists on the other ... [Hadamard 1945, page 1.]

Exponents of the two disciplines are likely to view the subject in different ways - the psychologist to extend psychological theories to thinking processes in a more complex knowledge domain - the mathematician to seek insight into the creative thinking process, perhaps with the hope of improving the quality of teaching or research. Although we will consider the cognitive side from a technical point of view as we discuss the most useful concepts in the psychology of advanced mathematical thinking, our main aim will be to seek insights of value to the mathematician in his professional work.

Different kinds of mathematical mind

Writing in the first decade of this century, the celebrated mathematician Henri Poincaré asserted:

> It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban[1] who pushes on his trenches against the place besieged, leaving nothing to chance. The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advanced guard. [Poincaré, 1913 page 210]

He supported his arguments by contrasting the work of various mathematicians, including the famous German analysts, Weierstrass and Riemann, relating this to the work of students:

[1] Sebastien de Vauban (1633-1707) was a French military engineer who revolutionized the art of siege craft and defensive fortifications.

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> Weierstrass leads everything back to the consideration of series and their analytic transformations; to express it better, he reduces analysis to a sort of prolongation of arithmetic; you may turn through all his books without finding a figure. Riemann, on the contrary, at once calls geometry to his aid; each of his conceptions is an image that no one can forget, once he has caught its meaning.
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> ... Among our students we notice the same differences; some prefer to treat their problems 'by analysis,' others 'by geometry.' The first are incapable of 'seeing in space', the others are quickly tired of long calculations and become perplexed. [Poincaré, 1913, page 212.]

Of course, there are not just two different kinds of mathematical mind, but many. At the turn of the century (at least) three strands of mathematical thinking may be distinguished: the intuitionist view represented by Kronecker, who asserted that ‘God gave us the integers, the rest is the work of man’, the formalist view of Hilbert that mathematics was the meaningful manipulation of meaningless marks written on paper, and the logicist view of Russell, that mathematics consisted of deductions using the laws of logic. Kronecker’s view was such that he caused Cantor’s 1873 paper on the existence of transcendental numbers to be refused publication in Crelle's Journal. His objection was that if one asserts something exists, then it must be exhibited, say by showing how it can be constructed. Cantor’s proof, however, was based on a non-constructive counting argument using cardinal infinities: there are strictly "more" real numbers than algebraic numbers (solutions of polynomial equations with integer coefficients), so there must exist a transcendental (non-algebraic real) number. He did not say specifically what such a transcendental number might be. Even today there are divergences of opinion between mathematicians on what constitutes a proof. Whilst most pragmatic mathematicians allow proof by contradiction, some only allow proof by direct construction (e.g. Bishop [1977]) and others may dispute to what extent logical principles such as the axiom of choice may be permitted

The point about raising these differences at this juncture is that the reader is also part of life's rich tapestry, with a personal view of mathematics that will differ in many ways from the conceptions of others. It may come as a surprise the first time one realizes that other people have radically different thinking processes. It first happened to the author when using pictures to help students visualize ideas in mathematical analysis, at a time when he did not question the implicit belief that such an approach was universally valid. Whilst writing a text book on complex analysis, a colleague in the next room was engaged on a similar enterprise, yet the latter's book had almost no pictures at all. He only included a diagram illustrating the argument of a complex number after a great deal of heart searching. To him a real number was an element of a complete ordered field (satisfying specific axioms) and a complex number was an ordered pair of real numbers. The argument of a complex number (x,y) was defined as a real number α such that $\cos(\alpha)=x/\sqrt{(x^2+y^2)}$, $\sin(\alpha)=y/\sqrt{(x^2+y^2)}$; it did not require a geometrical meaning. He took this hard line to make sure that his arguments were the product of logical deduction and not dependent anywhere on geometric intuition. At the time the author sympathized with his philosophical viewpoint, but considered it the product of a sophisticated development that few students might share. It was not until later that the realization dawned that there were a number of students who genuinely preferred such a formal approach.

Meta-theoretical considerations

The discussion of the preceding session is a salutary reminder that any theory of the psychology of learning mathematics must take into account not only the growing conceptions of the students, but the conceptions of mature mathematicians. Mathematics is a shared culture and there are aspects which are context dependent. For example, an analyst's view of a differential may be very different from that of an applied mathematician, and a given individual may strike up different attitudes to this concept depending on whether it is in an analytic or applied context.

At a far deeper psychological level we all have subtly different ways of viewing a given mathematical concept, depending on our previous experiences. For example, the "completeness axiom" for the real numbers is viewed by some as "filling in all the gaps between the rational numbers to give all the points on the number line". Thus there is "no room" to fit in any more numbers: the line is now "complete". But, to others, the "completion" is only a technical axiom to adjoin the limit points of cauchy sequences of rational numbers. In this case it is perfectly possible to embed the real numbers in a variety of larger number systems, which include infinitesimals and infinite numbers. The latter idea, however, is anathema to many mathematicians, including Cantor, who denied the existence of infinitesimals on the grounds that a full arithmetic (including division) was not possible in his theory of cardinal infinities. Even today many mathematicians are troubled by the infinitesimal ideas of non-standard analysis; often they do not deny its logic, but they may sense a deep seated psychological unease as to its validity.

Thus any theory of the psychology of mathematical thinking must be seen in the wider context of human mental and cultural activity. There is not one true, absolute way of thinking about mathematics, but diverse culturally developed ways of thinking in which various aspects are relative to the context.

Concept image and Concept Definition

In Tall & Vinner [1981], the distinction is made between the individual's way of thinking of a concept and its formal definition, thus distinguishing between mathematics as a mental activity and mathematics as a formal system. This theory applies to expert mathematicians as well as developing students:

> The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure

> logic that gives us insight, nor is it chance that makes us make mistakes... We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. ... As the concept image develops it need not be coherent at all times. The brain does not work that way. Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole. [Tall & Vinner 1981]

In this way it is possible for conflicting views to be held in the mind of a given individual and to be evoked at different times without the individual being aware of the conflict until they are evoked simultaneously.

The mature mathematician is not immune from internal conflicts, but he or she has been able to link together large portions of knowledge into sequences of deductive argument. At this stage it seems so much easier to categorise this knowledge in a logically structured way. Thus it is more likely to for a mature mathematician to consider it helpful to present material to students in a way which highlights the logic of the subject. However, a student without the experience of the teacher may find this approach initially difficult. The easy interpretation of the student difficulties is that the students lack the necessary intellect. It is a comforting viewpoint to take, especially when one is amongst those who share the mathematical understanding. But it is not realistic in the wider context of the needs of the students.

This can easily lead to an impasse between mathematicians and mathematics educators. The mathematicians are concerned with the higher levels of mathematics and may consider that educators have little to offer at this level. The first task therefore is to sensitize the mathematician to the different types of mathematical mind that occur, operating in quite different ways, and to use this knowledge to highlight the different ways that the developing mind may need appropriate experiences to gain insight into higher mathematical processes.

Cognitive Development

There are many competing theories in psychology. Behaviourist theory, built on external observation of stimulus and response, refuses to speculate about the internal workings of the mind. It provides observable and repeatable evidence of the behaviour of animals, including humans, under repeated stimuli, but it has limited application to mathematical thinking beyond the mechanics of routine algorithms. Constructivist psychology, on the other hand, attempts to discuss how mental ideas are created in the mind of each individual. Though this may pose a dialectic problem for the mathematician with a Platonic ideal of mathematics existing independently of the human mind, this approach to the psychology of advanced mathematical thinking can give significant insight into the creative processes of research mathematicians as well as the difficulties experienced by mathematics students.

The great Swiss psychologist Piaget attempted to underpin his theories of genetic epistemology with mathematical ideas. He saw the individual's need to be in dynamic equilibrium with his environment as an underlying theme in his work. To be stable meant that any divergence from equilibrium could be reversed, so that any operation needed a corresponding inverse operation. This triggered off in his mind the metaphor of *group theory* to model stable mental operations, for a mathematical group has an identity element and every element in the group has an inverse element which causes it to return to the identity.

Piaget saw the child grow into the adult through a series of stages of equilibrium, each one richer than the one before. He identified four main stages. The first is the *sensori-motor* stage prior to the development of meaningful speech, followed by a *pre-operational* stage when the young child realises the permanence of objects, which continue to exist even if they are temporarily out of sight. The child then goes through a transition into the period of *concrete operations* where he or she can stably consider concepts which are linked to physical objects, thence passing into a period of *formal operations* in the early teens when the kind of hypothetical 'if-then' becomes possible.

Piagetian stage theory has been extended to higher levels. For instance, Ellerton [1985] suggests that cognitive development proceeds via a series of levels, Level I being Piaget's sensori-motor, pre-operational and concrete stages, Level II, the first of the formal levels, repeating the sensori-motor, pre-operational and concrete stages at a first level of abstraction. Development is pictured as a continuous movement along a spiral. Biggs & Collis [1982] suggest a repetition of formal operations at successively higher levels, each repeating the learning cycle: unistructural, multistructural, relational.

However, this should be seen in the light of research which shows that many college students are not able to perform at the abstract level of formal operations, which Piaget reported occurring in children during their early teens. For example, the American psychologist Ausubel criticizes the stage theory:

> ... because of Piaget's tendency to underestimate the abstract thinking of young children and because such a high percentage of American high school and college students fail to reach this abstract level of cognitive logical operations. [Ausubel et al. 1968, page 230]

> Representative studies have indicated that only 15% of junior high school students ... 13.2% of high school students ... and 22% of college students were at this level. [ibid. page 238]

The concrete/formal distinction proved to be a useful starting point in developing local hierarchies of difficulty in extensive studies such as Hart (ed.) [1981] in the 11 to 16 age range, and in the study of early calculus concepts by Orton [1980]. But it is characteristic of Piaget's original theory that he asserted that the movement from one stage to another cannot be greatly accelerated by the affects of teaching. Differences of cognitive demand have often been used in a *negative* sense to describe students difficulties, but rarely to provide *positive* criteria for designing new approaches to the subject. This aspect of Piagetian stage theory was summed up by Papert [1980]:

> The Piaget of stage theory is essentially conservative, almost reactionary, in emphasising what children cannot do. I strive to uncover a more revolutionary Piaget, one whose epistemological ideas might expand the known bounds of the human mind.

Although one might attempt to develop some kind of stage theory for advanced mathematical thinking, in the hope of applying this to the teaching of students, such an approach is limited for several reasons:

> (1) Stage theory has rarely proved effective in developing positive teaching strategies,
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> (2) The kind of knowledge in higher mathematical thinking may have several different viable routes in its development, so that a single description of stages may simply be inaccurate,
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> (3) Few university mathematics teachers may be interested in teaching based on such a strategy.

There is also a mathematical metaphor which suggests why stage theory may be inappropriate as a curriculum building strategy for more advanced mathematics. Piaget used a group theoretic metaphor to underpin his sense of the dynamic equilibrium of cognitive growth. This even led him to postulate a system called a "grouping" which had only some of the axioms of a group to explain the cognitive operations available in the transition between stages. This has not been very helpful.

On the other hand, a more obvious mathematical metaphor for the disturbance of dynamic equilibrium lies in catastrophe theory. Here a system controlled by continuously varying parameters can suddenly leap from one position of equilibrium to another when the first becomes untenable. Depending on the history of the varying parameters, the transition may be smooth, or it may be discontinuous. Such a metaphor suggests that stage theory may just be a linear trivialization of a far more complex system of change, at least this may be so when the possible routes through a network of ideas become more numerous, as happens in more advanced mathematical thinking.

Transition and mental reconstruction

A far more valuable aspect of Piaget's theory is the process of *transition*from one stage to another. During such a transition, unstable behaviour is possible, with the experience of previous ideas conflicting with new elements. Piaget uses the terms *assimilation*to describe the process by which the individual takes in new data and *accommodation* the process by which the individual's cognitive structure must be modified. He sees assimilation and accommodation as complementary. During a transition much accommodation is required. Skemp [1979] puts similar ideas in a different way by distinguishing between the case where the learning process causes a simple *expansion* of the individual's cognitive structure and the case where there is cognitive conflict, requiring a mental *reconstruction*. It is this process of reconstruction which provokes the difficulties that occur during a transition phase.

Thus students meeting new formal mathematical ideas for the first time may face difficulties in cognitive reconstruction to adapt to the new way of thinking and may need help in this transition phase. Dubinsky [1985] has produced valuable insight into Piagetian theory applied to college level mathematics, not by concentrating on the stage theory, but by looking at the process of *reflective abstraction.* The interest here is the way in which the individual reflects on new knowledge and, through the process of accommodation, modifies his or her existing cognitive structures to be able to make sense of it.

Obstacles

The most serious problem occurs when the new ideas are not satisfactorily accommodated. In this case it may be possible for conflicting ideas to be present in an individual at one and the same time:

> New knowledge often contradicts the old, and effective learning requires strategies to deal with such conflict. Sometimes the conflicting pieces of knowledge can be reconciled, sometimes one or the other must be abandoned, and sometimes the two can both be "kept around" if safely maintained in separate compartments.[Papert, 1980, page 121]

The thesis of Cornu [1983] studies the conceptual development of the limit process from school to university and underlines how the colloquial use of the term "limit" effects the mathematical usage. He discusses the notion of an "obstacle", introduced by Gaston Bachelard in "La Formation de l'Esprit Scientifique", and further discussed by the mathematical educator:

> An obstacle is a piece of knowledge; it is part of the knowledge of the student. This knowledge was at one time generally satisfactory in solving certain problems. It is precisely this satisfactory aspect which has anchored the concept in the mind and made it an obstacle. The knowledge later proves to be inadequate when faced with new problems and this inadequacy may not be obvious.[Cornu 1983, (original in French)]

The obstacles found by Cornu include the problems student face when the determination of limits no longer reduces to simple numerical and algebraic calculations. He discusses how infinity intervenes and is surrounded in mystery, yet the methods "work" without the students understanding why. He demonstrates how students' experiences can lead to belief in the infinitely large and the infinitely small, with "nought point nine recurring" being a number "just less than one" and the symbol ε representing to many students a quantity that is smaller than any positive real number, but not zero. There are implicit assumptions that the limiting process "goes on forever", that the limit "can never be attained".

In Tall [1986], an explanation is given for these phenomena as the *generic extension principle*:

> If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter- examples, the mind assumes the known properties to be implicit in other contexts.

For example, most convergent sequences described to beginning students are of a simple kind given by a formula such as 1/n, which tends to the limit (in this case zero), but the terms never equal the limit. In the absence of any counter-examples students begin to believe that this is always so. The rich experience of colloquial language supports this belief (Schwarzenberger and Tall 1978), with phrases like "gets close to" suggesting that the terms of a sequence can never be coincident with the limit. Thus the implicit belief is slowly formed that a sequence of terms converging to a limit gets closer and closer, but never actually gets there.

Furthermore, if all the terms of a sequence have a certain property, it is natural to believe that the limit has the same property. Thus the sequence 0.9, 0.99, ... has terms all less than 1, so the limit "nought point nine recurring" must also be less than one... This leads to the mental image of a limiting object that is termed a *generic limit* in Tall [1986]. A generic limit need not be a limit in the mathematical sense, but it is the concept of the limit that the individual holds in his or her mind as a result of extrapolating the common properties of the terms of the sequence.

This phenomenon happens not just with sequences of numbers, but sequences of functions and other mathematical objects that share a common property. Historically this is enshrined in the "principle of continuity" of Leibniz:

> In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.
>
> [Leibniz in a letter to Bayle, January 1687.]

It arises even earlier in the work of Nicholas of Cusa (1401-1464) who regarded the circle as a polygon with an infinite number of sides, and inspired Kepler (1571-1630) to formulate a metaphysical "bridge of continuity" in which normal and limiting forms of a figure are characterized under a single definition. Thus Kepler (*Opera Omnia II*, page 595) saw no essential difference between a polygon and a circle, between an ellipse and a circle, between the finite and the infinite, and between an infinitesimal area and a line.

The generic extension principle arises time and again in history. For example, Cauchy's assertion that the limit of continuous functions is continuous and Peacock's "Principle of Algebraic Permanence", in which the properties of extended number systems, such as the real and complex numbers, were based on the principle that the any algebraic law which held in the smaller system also held in the extension. The latter held sway for some time in the nineteenth century until Hamilton invented (discovered?) the quaternions, an extension of the complex numbers whose multiplication is not commutative.

Obstacles arising from deeply held convictions about mathematics are rarely easy to erase from the mind. We all carry with us a mental rag-bag of such beliefs, many of which we suppress, but do not eliminate, when faced with the logic of mathematics. Often the only trace of such an obstacle is through a sense of unease when there is a logical deduction that does not "feel right".

Intuition and rigour

Mathematicians often regard the terms "intuition" and "rigour" as being mutually exclusive by suggesting that an "intuitive" explanation is one that necessarily lacks rigour. There is a grain of truth in this statement, in that an intuition may arrive whole in the mind and it may be difficult to separate the components into a logical deductive order. But the opposition between the two concepts is a false dichotomy as we shall soon see.

In a sense we have not one, but two brains. In attempting to assist patients who had serious epileptic fits, Sperry and his colleagues took the drastic action of partial or total severance of the corpus collosum that links the two hemispheres of the brain and found that each could essentially operate independently, though carrying out totally different functions:

> Though predominantly mute and generally inferior in all performances involving language or linguistic or mathematical reasoning, the minor hemisphere is nevertheless clearly the superior cerebral member for certain types of tasks. If we remember that in the great majority of tests it is the disconnected left hemisphere that is superior and dominant, we can review quickly now some of the kids of exceptional activities in which it is the minor hemisphere that excels. First, of course, as one would predict, these are all non-linguistic non-mathematical functions, largely as they involve the apprehension and processing of spatial patterns, relations and transformations. They seem to be holistic and unitary rather than analytic and fragmentary, and orientational more than focal, and to involve concrete perceptual insight rather than abstract, symbolic sequential reasoning. [Sperry, 1974]

Glennon [1980] summarizes the findings of the activities of the two halves of the brain taken "from many research studies" in the following form:

Left Hemisphere	Right Hemisphere
Verbal	Visuospatial (including gestural communication)
Logical	Analogical, intuitive
Analytic	Synthetic
Linear	Gestalt, holist
Sequential	Simultaneous & multiple processing
Conceptual similarity	Structural similarity

He prefaces the table with the caveat that "we who work in the instructional psychology of mathematics must not presume to be neurologists" and warns of the differences between individuals (for example those who are left-handed). Perhaps we should refer not to the "left" and "right" brains but to the metaphorical activities involving these two different processing modes.It is also pertinent to notice that the left brain is acting essentially as a sequential processor and the right brain acts in parallel, processing visual imagery in a powerful simultaneous way. If there is any basis in fact for these two different kinds of activity then the two columns suggest a possible neurological reason underlying the two different kinds of mathematical mind discussed earlier in quotation from Poincaré.

However, this is not the whole story, for are we to deny the role of intuition in the mind of a mathematician who is more logical in his way of thinking about mathematical concepts? Surely not. Poincaré, speaking of Hermite, said:

> ... his eyes seem to shun contact with the world; it is not without, it is within he seeks the vision of truth. [Poincaré, 1913, page 212]

> ... When one talked to M. Hermite, he never evoked a sensuous image, and yet you soon perceived that the most abstract entities were for him like living beings. He did not see them, but he perceived that they are not an artificial assemblage and that they have some principle of internal unity. [ibid page 220]

The conclusion is inescapable. Intuition is the product of the concept images of the individual. The more educated the individual in logical thinking, the more likely his concept imagery will resonate with a logical response. This is evident in the growth of thinking of students, who pass from initial intuitions based on their pre-formal mathematics, to more refined formal intuitions as their experience grows:

> We then have many kinds of intuition; first, the appeal to the senses and the imagination; next, generalization by induction, copied, so to speak, from the procedures of the experimental sciences; finally we have the intuition of pure number... [Poincaré, 1913, page 215.]

From a psychological viewpoint, Fischbein [1978] comes to similar conclusions, citing two different types of intuition:

> *Primary intuitions* refer to those cognitive beliefs which develop themselves in human beings, in a natural way, before and independently of systematic instruction.

> *Secondary intuitions* are those which are developed as a result of systematic intellectual training ... In the same meaning, Felix Klein (1898) used the term "refined intution": and F. Severi wrote about "second degree intuition" (1951).
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> [Fischbein, 1978, page 161]

Thus one may view the usual mathematician's dichotomy between intuition and rigour as one between the holistic, visual thinking characteristic of the right brain, and the rigour of the sequential, logical thinking characteristic of the left. But the psychologist sees the possibility of more sophisticated (secondary) intuitions arising from refined concept images which can include the mental imagery of logic and deduction. Thus aspects of logic too can be honed to become more "intuitive" to the mathematical mind. The development of this refined logical intuition should be one of the major aims of more advanced mathematical education.

Curriculum Design in Advanced Mathematics Teaching

During the difficult transition from pre-formal mathematics to a more formal understanding of mathematical processes there is a genuine need to help students gain insight into the concepts. A mathematician's logic may here fail him (or her) in designing a teaching schedule. A mathematician often takes a complex mathematical idea and "simplifies" it by breaking it into smaller components ready to teach each component in a logical sequence. From the expert's viewpoint the components may be seen as parts of a whole. But the student may see the pieces as they are presented, in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available.

For example, a mathematical analysis of the notion of the derivative f'(x) requires the notion of the limit of (f(x+h)-f(x))/h as h tends to zero, so mathematically the derivative must be preceded by the discussion of the notion of a limit. To make the process mathematically easier the limit process is initially carried out with x fixed; only at a later stage is x allowed to vary to give the notion of a function. Thus the sequence suggested by a formal mathematical analysis is:

(1) notion of a limit

(2) for fixed x, consider the limit of $(f(x+h)-f(x))/h$ as h tends to zero

(3) call the limit $f'(x)$, then allow x to vary to give the derived function.

However, when the learner is at stage (1), the limit notion is mysterious because it seems “plucked out of the air”, without any real reason. There are already cognitive obstacles here, as observed by Cornu [1983], and others. At stage (2) the limiting process introduces further obstacles, as detailed in Tall & Vinner [1981]. Nor is the passage from (2) to (3) as easy cognitively as it seems mathematically. Many students see (2) as a purely symbolic activity, and do not see the derivative $f'(x)$ as a function, with a graph. Given the graph of $f(x)$, they do not relate it to the graph of $f'(x)$ (Tall [1986]).

The purpose of Tall [1986] is to develop a more appropriate approach that "uses both sides of the brain", complementing a numerical and algebraic approach to the derivative with a global, visual appreciation of the gradient of a graph generated on a computer. More generally, it may be possible to use the complementary power of visualization to give a global gestalt for a mathematical concept, to show its strengths and weaknesses, its properties and non-properties, in a way that makes it a logical necessity to formulate the theory clearly. But now the theory is built up in a way that the student has a global framework within which to develop the formal deductions.

Problem-solving

For many undergraduates, problem-solving means learning the contents of a set of lecture notes and applying this knowledge to specific problems clearly related to the material taught. For research mathematicians, problem-solving is a more creative activity, which includes the formulation of a likely conjecture, a sequence of activities testing, modifying and refining until it is possible to produce a formal proof of a well-specified theorem. Although Polya's archetypal book [1945] has been available for some time, it is only recently the work of mathematics educators such as Mason et al [1980] and Schoenfeld [1985] has made problem-solving a really practical undergraduate activity. In particular, Mason et al concentrate on the different phases of mathematical thinking, essentially built on those of Polya. First there is an entry phase, in which the problem is clarified and the possible tools that might be used to attack it are assembled. Then comes the main attack phase in which various techniques are used creatively to attempt to crack the problem. If this leads to a mental block, a re-entry is called for, otherwise a successful attack will then lead into a review phase, first to check the steps of the solution, then to attempt to refine it and make it logically satisfying. Finally the cycle may be repeated once more in extensions of the problem, with new phases of entry, attack and review.

Using this style of approach, it *is* possible to get undergraduates to develop original ways of solving problems. The solutions may not be found as quickly as they might, given active teaching by the lecturer but the activities can help the students gain in confidence and desire to attack problems that they might previously have been unwilling to attempt. Such problem-solving activities can also help to stimulate reflective thinking and to develop an internal monitor within the student's mind to help keep track on the progress of the solution process and to ring warning bells when the solution may be leading up a blind alley.

Analysis and Synthesis

Poincaré was at pains to show the complementary roles of synthesis and analysis in mathematical thinking. The former is is concerned with building up new ideas and is related to the entry and attack phases of Mason et al., whilst the latter is concerned with breaking down ready formed concepts and making them precise, as in the review phase.

Hadamard considers Poincaré's description of his own personal research activities and notes:

> .. the very observations of Poincaré show us three kinds of inventive work essentially different if considered from our standpoint, viz.,
>
> a. fully conscious work
> b. illumination preceded by incubation
> c. the quite peculiar process of the sleepless night.
>
> [Hadamard, 1945, page 35.]

Here Poincaré reports the necessity of working hard at a new problem, then relaxing to allow the ideas to incubate in his subconcious, during which time he had a sleepless night thinking vigorously about new ideas until suddenly, some time later, a sudden illumination bursts into his consciousness with a solution. After a further time had elapsed, at his leisure, he was able to analyse what had happened and build up a formal justification of his theory.

Once more we see the two complementary sides of the coin. Synthesis begins with the conscious act of an entry phase to begin to put ideas together, followed by a more intuitive activity, in which subconscious interplay between concept images takes place, until a powerful resonance forces the newly linked concepts to erupt once more into consciousness. Analysis, on the other hand, is a much more cool and logical conscious activity which is often fairly routine and mechanical.

Teaching of younger children emphasizes the *synthesis* of knowledge, starting from simple concepts, building up from experience and examples to more general concepts. Now the accent at this level is changing to include more problem solving and open-ended investigations. Teaching at university often emphasizes the other side of the coin: *analysis* of knowledge, beginning with general abstractions and forming chains of deduction from them which may be applied in a wide variety of specific concepts.

Working with much younger children, Dienes [1960] proposed a theory as to how concepts may be built up from concrete examples, yet Dienes & Jeeves [1965] formulates a far more general 'deep-end' principle in which "there is a preference for extrapolation by leaps and interpolation, rather than always by step-by-step". They respond to their own question 'When is it possible to generalize from a simple case to a more general case and when is it better for them to particularize from a more complex case to the simple case?" with the remark that "this is not likely to be answered by a simple positive or negative statement". They suggest that it is more a question of 'the optimum degree of complexity required to start with". This response may be just as valid for teaching and learning at more advanced levels.

Mathematical Proof

Viewed as a problem-solving activity, we see that proof is actually the final stage of activity in which ideas are made precise. Yet again, so much of the teaching in university level mathematics*begins* with proof. In his preface to *The Psychology of Learning Mathematics*, Skemp succinctly refers to this as showing the students the product of mathematical thought, instead of teaching them the process of mathematical thinking. The splendid tomes of Bourbaki are a monument to the intellect of the mathematical mind, and may be used to help the learner appreciate the formal structure of mathematics. But once again, Poincaré has pertinent observations to make:

> To understand the demonstration of a theorem, is that to examine successively each of the syllogisms composing it and to ascertain its correctness, its conformity to the rules of the game? ... For some, yes; when they have done this, they will say: I understand. For the majority, no. Almost all are much more exacting they wish to know not merely whether all the syllogisms of a demonstrations are correct, but why they link together in this order rather than another. In so far as to them they seem engendered by caprice and not byt an intelligence always conscious of the end to be attained, they do not believe that they understand.
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> [Poincaré, 1913, page 431.]

> Perhaps you think I use too many comparisons; yet pardon still another. You have doubtless seen those delicate assemblages of silicious needles which form the skeleton of certain sponges. When the organic matter has disappeared, there remains only a frail and elegant lace-work. True, nothing is there except silica, but what is interesting is the form this silica has taken, and we could not understand it if we did not know the living sponge which has given it precisely this form. Thus it is that the old intuitive notions of our fathers, even when we have abandoned them, still imprint their form upon the logical constructions we have put in their place.
>
> [ibid, p. 219.]

Thus it is that so many mathematicians demand that a proof should not only be logical, but that there should be some over-riding principle that explains*why* the proof works. Thus the proof of the four colour theorem, by exhaustion of all possible configurations using a computer search *seems* logical, yet many professional mathematicians, though keen to see the theorem proved once and for all, are nevertheless sceptical that there may be some subtle flaw in the computer “proof”, because there seems still to be no rhyme or reason to illuminate why it works as it does.

Yet this is not always passed on to students. Sawyer [1987] reports how he tried to teach theorems in functional analysis by referring back to theorems in real variables that he expected them to know, only to find that they had no recollection of them.

> The reason for this was that in their university lectures they had been given formal lectures that had not conveyed any intuitive meaning; they had passed their examinations by last-minute revision and by rote.

He tells how he was shocked to learn of a lecturer who became stuck in the middle of a proof, turned his back on the class to draw a picture to aid him, then erased it and carried on with the formal proof without enlightening the class how he had used his intuition to rebuild it. He observes:

> ... to teach calculus well is a very demanding task. Three things have to be done: first to show by a drawing that some result is extremely plausible; second, to give counter-examples, which indicate the circumstances in which the conjecture would fail; third, to extract from these considerations a formal proof of the result.
>
> These remarks do not apply only to lectures and books for undergraduate. Felix Klein pointed out that in papers for research journals the suppression of intuitive considerations was a common and highly undesirable practice.
>
> [Sawyer, 1987]

Beginning students have even greater difficulty with proof before they attain familiarity with the workings of the mathematical culture. Thus, in a questionnaire, Tall 1979, investigating which proof of the irrationality of $\sqrt{2}$ was more clear, students preferred a proof that showed that the square of any rational must have an even number of prime factors, and therefore such a square could not be 2 because the prime 2 occurs an odd number of times (namely once). They preferred this to the standard proof by contradiction and another more general demonstration taken out of Hardy's *Pure Mathematics*. This was despite the fact that the proof was in fact not a formal proof at all, but a discursive explanation with examples demonstrating what form was taken by the square of a typical rational.

Leron, in a series of papers has demonstrated how proof can be made more understandable to students during this delicate initial phase before the formalities are part of their cognitive structure. His method is, essentially, to properly structure the proof, so that it is clear what is going on at any given time, and to make the proof as direct as possible. Thus contradiction proofs are re-written so that they are initially direct and constructive, with any contradiction being introduced as late as is practicable in the proof.

It is a truism that we can only think with the cognitive structure that we have available to us. Thus it comes as no surprise that students find formal mathematics more difficult than experienced mathematicians may feel is reasonable. By the same token, when we look at the psychology of advanced mathematical thinking, it is no wonder that we each find it easier to use our own knowledge structure to formulate our own theories. As a mathematician entering mathematics education it is no surprise that I first attempted to use catastrophe theory to describe the discontinuities in learning. Likewise those who begin mathematics education with a background of Piagetian theory are likely to attempt to explain things in these terms, those with experience in computer studies are likely to use computer analogies, mathematicians are likely to attempt to use mathematical constructs to aid them in formulating a viewpoint, and so on. In trying to formulate helpful ways of looking at advanced mathematical thinking, it is important that we take a broad view and try to see the illumination that various theories can bring, the useful differences that arise and the common links that hold them together.

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