Duality, Ambiguity and Flexibility in Successful Mathematical Thinking

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In this paper we consider the ***duality between process and concept*** *in mathematics, in particular* ***using the same symbolism*** *to represent both a process (such as the addition of two numbers 3+2) and the product of that process (the sum 3+2). The* ***ambiguity of notation*** *allows the successful thinker the* ***flexibility in thought*** *to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema. We hypothesize that the successful mathematical thinker uses a mental structure which is an amalgam of process and concept which we call a* **procept**. *We give empirical evidence to show that this leads to a qualitatively different kind of mathematical thought between the more able and the less able, causing a divergence in performance between success and failure.*

The *duality* of process and concept

The passage from process to concept has long been a focus of research in mathematical education. Piaget speaks of the *encapsulation* of a process as a mental object when

> ... a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower.(Beth & Piaget 1966, p. 247).

Dienes uses a grammatical metaphor to describe how a predicate (or action) becomes the subject of a further predicate, which may in turn becomes the subject of another. He claims that

> People who are good at taming predicates and reducing them to a state of subjection are good mathematicians. (Dienes, 1960, p.21)

In an analogous way, Greeno (1983) defines a "conceptual entity" as a cognitive object which can be manipulated as the input to a mental procedure. The cognitive process of forming a (static) conceptual entity from a (dynamic) process has variously been called "encapsulation" (after Piaget), "entification" (Kaput, 1982), and "reification" (Sfard, 1989). It is seen as operating on successively higher levels so that:

> ... the whole of mathematics may therefore be thought of in terms of the construction of structures,... mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures. (Piaget 1972, p. 70).

The *ambiguity* of symbolism for process and concept

The encapsulation of process as object is seen as a difficult mental activity for "How can anything be a process and an object at the same time?" Sfard (1989). Our observation is that

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this is achieved by the simple device of *using the same notation to represent both a process and the product of that process*. Examples pervade the whole of mathematics.

- The process of *counting* and the concept of *number* (where the number 7 evokes both a counting process and the number produced by that counting)
- The process of *counting all* or *counting on* and the concept of *addition* (5+4 evokes both the counting on process and its result, 9),
- The process of multiplication as repeated addition and its product (4x5 is both 5 groups of 4 and the product 4x5=20),
- The process of *division* of whole numbers and the concept of *fraction* (e.g. $\frac{3}{4}$),
- The process of adding (or shifting) numbers on a number line and the concept of signed number ("*+2*" is both process and concept),
- The process of "*subtract two*" and the concept of "*–2*",
- The trigonometric ratio $\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$,
- The expression $3x+2$ representing both the process of adding 2 to $3x$ and the resulting sum,
- The process of *tending to a limit* and the concept of the *value of the limit* both represented by the same notation such as $\lim_{x \to a} f(x)$.

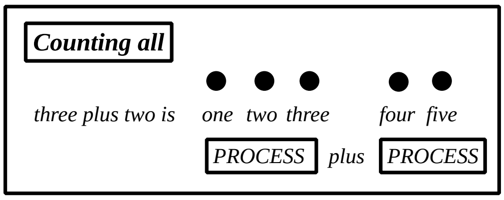
It is through using the notation to represent either process or product, whichever is convenient at the time, that the mathematician manages to encompass both – neatly sidestepping the problem. We believe that this ambiguity is at the root of successful mathematical thinking. It enables the processes of mathematics to be tamed into a state of subjection.

The *flexible* notion of procept

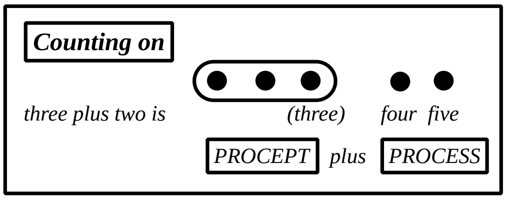
We define a *procept* to be the amalgam of process and concept in which process and product is represented by the same symbolism. Thus the symbol for a procept can evoke either process or concept. For instance, *number* is a procept, in which a number such as "three" represents both the process of counting "one, two, three" and the concept which is the outcome of that process.

As an example of the use of the notion of procept to produce a new theoretical synthesis of the development of mathematical concepts, we consider the development of concept of addition, (see, for example, Carpenter et al, 1981, 1982, Fuson, 1982).

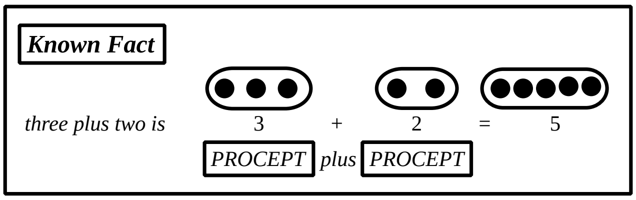
The sum of two numbers, say 3+2, is a procept, first conceived as the process of "counting all" or "counting on". COUNTING ALL therefore may be viewed as PROCESS & PROCESS:



"Counting on" is a more subtle procedure in which the first number (three) is already seen as a whole, and the counting on process counts on two more numbers ("four", "five"). Thus COUNTING ON consists of PROCEPT & PROCESS:



Finally we come to PROCEPT & PROCEPT as embodied in a KNOWN FACT:



A proceptual known fact should be distinguished from a rote learned fact by virtue of its rich inner structure which may be decomposed and recomposed to produce *derived facts*. For instance, faced with 4+5, a child might see 5 as "one more than 4" and might know the double 4+4=8 to derive the fact that 4+5 is "one more", namely 9. For the proceptual thinker this gives a powerful feedback loop which uses proceptual known facts to derive new known facts.

Such proceptual facts may develop great flexibility, where 2+3=5 may be seen equivalently as 3+2=5, 5–3=2, 5–2=3, allowing subtraction to be seen as directly related to addition at the proceptual level, giving a fluent and easy way to develop subtraction facts.

The weakness of unencapsulated process

Meanwhile, the less able child who sees addition only as a process, is faced with a far more difficult task. We have observed that those children who perceive addition as count all or count on, often do so in time, so that although they may produce the right answer (8+4 is 9, 10, 11, *12*), by the time they reach the end of the process they may have forgotten the beginning, and so the sum 8+4=12 is not available as a new fact. Instead they seek security in getting the correct answer by developing strategies for counting, often using real or imagined objects or assigned parts of their body to represent larger numbers (Gray, in press).

For a child whose concept of addition is mainly "count on", or "count all", the strategies of subtraction can only be in terms of a reversal of these processes. "Take away" involves counting the total set, removing the number to be subtracted and counting the remainder. "Count up" relies on counting the subset to be taken away, then counting up to the total. "Count back” relies on counting back the number to be taken away from the total. All of these involve sophisticated double-counting procedures which invariably need concrete or imagined props, such as a number line or ruler, or assigned parts of the body to support the counting process. The corresponding abstract processes prove difficult to carry out if such props are absent.

The less able child thus not only has a weak grasp of known facts as a foundation for knowledge, but also uses more complex procedures with more possibility of error. When the concept of place value is introduced later and the child meets two and three digit addition and subtraction problems, the difficulties are compounded. What might be a simple combination of proceptual ideas for the more able becomes the coordination of several complex processes for the less able, leading to intolerable difficulties and a high probability of failure.

Empirical Evidence

Seventy two children were selected by their teachers in two “typical” schools to represent the chronological ages 7+ to 12+, with each school providing three pairs of children in each year to represent the below average, average, and above average attainers. These children were interviewed individually for half an hour on at least two separate occasions a week apart, and in each session were asked to solve between eighteen and twenty arithmetic problems at various levels of difficulty. Figure 1 (taken from Gray, to appear) illustrates the different strategies used

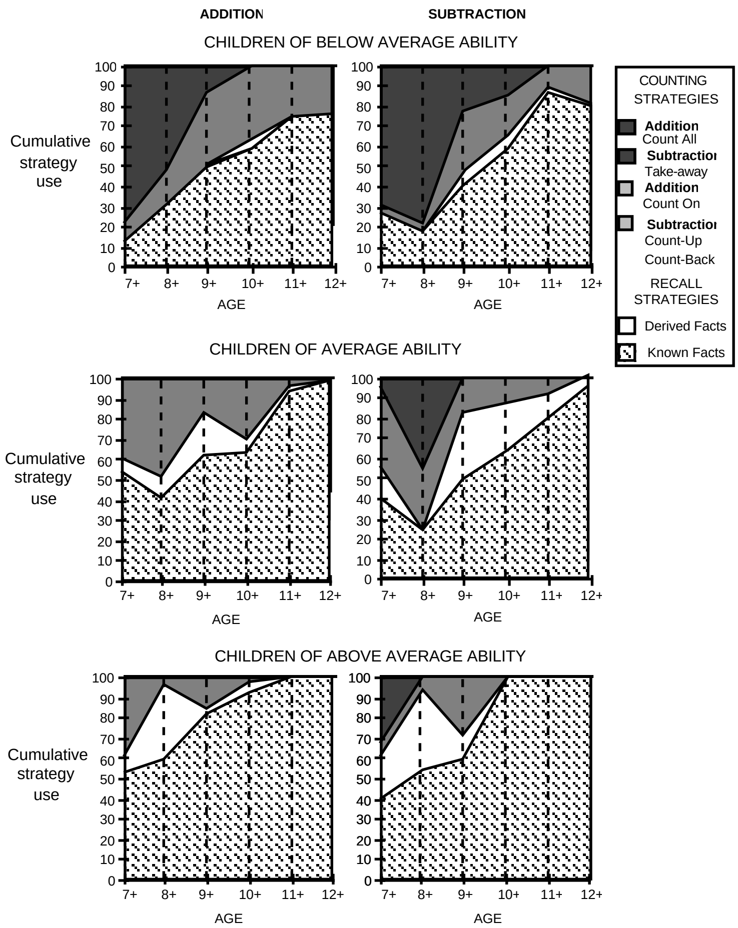


figure 1 : Strategies for solving addition and subtraction involving numbers up to ten

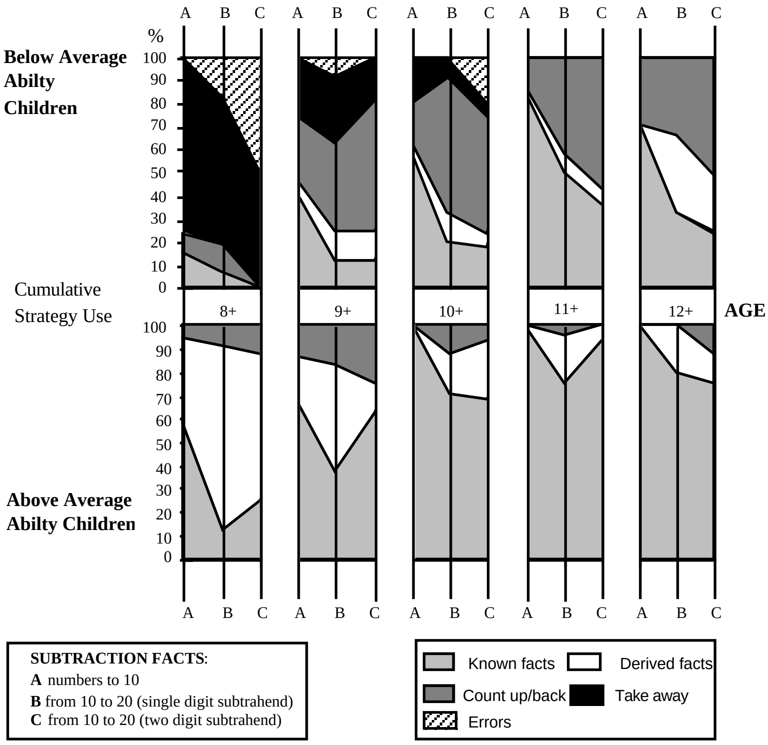


Figure 2 : Strategies for subtraction by below and above average children

by children of differing abilities in solving single-digit addition and subtraction problems.

Note the almost complete absence of derived facts in the less able (particularly in addition), whereas the average and above average start with a high proportion of known facts and use derived facts to generate other facts. As the ages of the children increase, the proportion of known facts increase, but to a lesser extent in the less able.

Figure 2 concentrates on the performances of the below and above average ability groups on three different levels of subtraction problem:

A single digit subtraction (e.g. 8-2),

B subtraction of a single digit number from one between 10 & 20 (e.g. 16-3, 15-9),

C subtraction of one number between 10 and 20 from another (e.g. 16-10, 19-17).

Note the absence of derived facts for any of these levels in the 8+ below average children and the high incidence of derived facts in the same age above average children. See how the above average have 100% known facts in category A by the age of 10+ whilst the below average even tail off in performance at 12+ as the effects of attempting to cope with more complicated arithmetic begins to affect their competence in performance at this level.

The proceptual divide

We have seen that the more able have a proceptual structure available to them with built-in feedback loop. Gray, (in press) has observed that more able children initially build up an increasing array of known facts to support their arithmetic, but then realise that their new ability to derive facts removes the burden of needing to remember them all. *For the more able, arithmetic eventually becomes increasingly simple.* Meanwhile, the less able become trapped in long sequential processes which increase the burden upon an already stressed cognitive structure so that, *for the less able, the arithmetic becomes increasingly more difficult.*

This lack of a proceptual structure provokes a major tragedy for the less able which we call the *proceptual divide*. We believe it to be a major contributory factor to widespread failure in mathematics. It is as though the less able are deceived by a conjuring trick that the more able have learned to use. They are all initially given processes to carry out mathematical tasks but success eventually only comes not through being good at those processes, but by encapsulating them as part of a procept which solves the tasks in a more flexible way.

Figure 3 shows the total range of strategies used by more able and less able children in the ages 8+ to 12+ for specific problems whose answer is not a known fact.

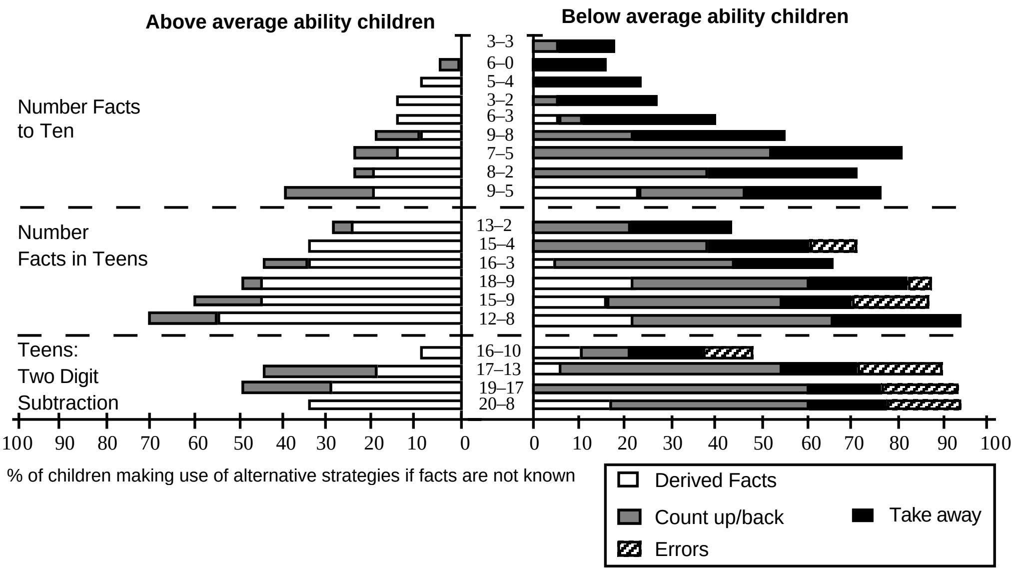


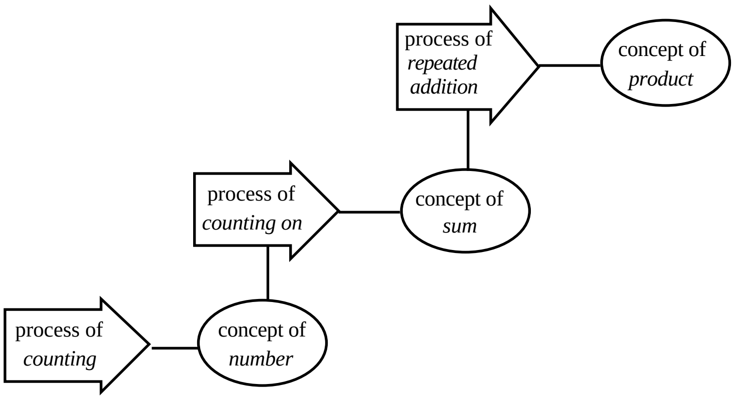
Figure 3 : Strategies for solving problems whose answer is not immediately known

The left hand side shows the above average children using almost all derived facts and a few examples of counting, whilst the right hand side shows few derived facts and a large percentage of counting, take away and errors. The proceptual divide is clearly shown.

The cumulative effect of the proceptual divide

Proceptual encapsulation occurs at various stages throughout mathematics: repeated counting becoming addition, repeated addition becoming multiplication and so on, giving what are usually considered by mathematics educators as a complex hierarchy of relationships:

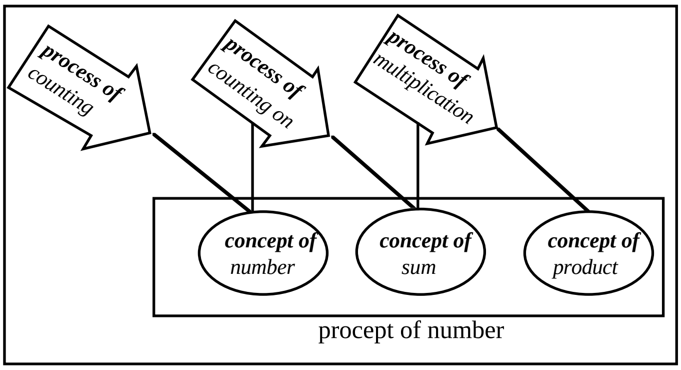
The less able child who is fixed in process can only solve problems at the next level up by coordinating sequential processes. This is, for them, an extremely difficult process. If they are faced with a problem two levels up, then the structure will almost certainly be too burdensome for them to support.



Higher order encapsulations

The more able, proceptual thinker is faced with a different problem. The symbols for sum and product again represent *numbers*. Thus counting, addition and multiplication are operating on the same procept which can be decomposed into process for calculation purposes whenever desired. A proceptual view which amalgamates process and concept through the use of the same notation therefore *collapses the hierarchy* into a single level in which arithmetic operations (processes) act on numbers (procepts).

We hypothesise that this is the development by which the more able thinker develops a flexible relational understanding in mathematics, which is seen as a meaningful relationship between notions at the same level, whilst the less able are faced with a hierarchical ladder which is more difficult to climb. It also provides an insight into why the practicing expert sees mathematics as such a simple subject and may find it difficult to appreciate the difficulties faced by the novice.



Collapse of hierarchy into operations on numbers

Examples from other areas of mathematics

The examples given in simple arithmetic by no means exhaust the possibilities in the mathematics curriculum. We have evidence that the lack of formation of the procept for an algebraic expression causes difficulties for pupils who see the symbolism representing only a process: indeed a process such as $2+3x$ which they are not able to carry out because they do not know the value of x (Tall and Thomas, to appear). We have evidence that the conception of a trigonometric ratio only as a process of calculation (opposite over hypotenuse) and not a flexible procept causes difficulties in trigonometry (Blackett 1990). In both of these cases we have evidence that the use of the computer to carry out the process, and so enable the learner to concentrate on the product, significantly improves the learning experience.

The case of the function concept, where f(x) in traditional mathematics represents both the process of calculating the value for a specific value of x and the concept of function for general x, is another example where the modern method of conceiving a function as an encapsulated object causes great difficulty (Sfard, 1989).

We therefore are confident that the notion of procept allows a more insightful analysis of the process of learning mathematics, in which the precision of definition of modern mathematics ("a function is a set of ordered pairs such that ...") caused seemingly inexplicable difficulties to the student. The ambiguity of process and product represented by the notion of a procept provides a more natural cognitive development which gives enormous power to the more able. It exhibits the proceptual chasm faced by the less able in attempting to grasp what is – for them – the spiralling complexity of the subject.

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