Versatile Learning & the Computer

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Introduction

The activities that are prized in mathematics are usually symbolic and logical. Far less often do we emphasize the visual and holistic. In this paper we discuss the use of the computer to encourage a more versatile approach to learning involving both types of mental activity. Empirical evidence is drawn from three studies: the relationship between equations and the graphs of straight lines [Blackett 1987], the introduction of the gradient of a more general function in the initial stages of the calculus [Tall 1986a] and the introduction of algebraic symbolism [Thomas 1988]. In each case we find that traditional approaches lead to a narrow symbolic interpretation, yet the use of the computer gives a visual framework for the mental manipulation of higher order concepts.

Many researchers have identified two distinct learning strategies (see, for example, Brumby 1982). The first style has been labelled analytic or serialist, whilst the second has been called global, holist or intuitive. The essential characteristics of the two learning styles have been listed as:

(i) Immediately breaking a problem or task into its component parts, and studying them step by step, as discrete entities, in isolation from each other and their surroundings.

(ii) An overall view, or seeing the topic/task as a whole, integrating and relating its various subcomponents, and seeing them in the context of their surroundings. [Brumby 1982, p.244]

Brumby's observations suggested three distinct groups of students: those who consistently used only serialist/analytic strategies, those who used only global/holistic strategies, and those who used a combination of both, whom she described as *versatile learners.* Overall 42% of her sample maintained a serialist/analytic style, 8% were global/holistic and 50% were versatile.

Much mathematical instruction tends to be serialistic in style, so that, in isolated learning activities, serialistic thinkers may not seem to be disadvantaged. However, versatile learners are more likely to be successful in mathematics at the higher levels where the ability to switch one's viewpoint of a problem from a local analytical one to a global one, in order to be able to place the details as part of a structured whole, is of vital importance:

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> ... pupils are expected to do more than simply reproduce items of knowledge, as they have been taught. They must, for example, also be able to restructure bodies of knowledge in ways appropriate to different problems - a difficult task for the serialists because of their inclination to learn sequentially, without necessarily forming an overall picture of the relationships involved. ... whilst holists are busy speculating about relationships, and discovering the connections between initially disjoint areas of mathematics, it may not even occur to serialists to begin to look for such links.[Scott-Hodgetts,1986, page 73]

Thus teaching activities that encourage the development of holistic thinking patterns, linking them to sequential, deductive thinking, may be of benefit in aiding students to obtain a better overall performance in mathematics.

A cognitive approach to the curriculum

Structuring the knowledge domain in a manner appropriate for learning is different from organising the component skills in a logical sequence. A sequence which is seen to be logical by the expert may not be cognitively appropriate for the learner. All the research described here investigates a *cognitive approach* to the curriculum using software that is designed to aid the learner to develop in a versatile manner. The software is specifically designed to provide an environment which has the potential to enable the user to grasp a gestalt for a whole concept at an intuitive level. It is designed to enable the user to manipulate *examples* of a specific mathematical concept or a related system of concepts. Such programs are called *generic organisers*. They are intended to aid the learner in the abstraction of the more general concept embodied by the examples, through being directed towards the generic properties of the examples and differentiating these from the non–examples. This abstraction is a *dynamic* process. Attributes of the concept are first seen *in a single exemplar*; the concept itself being successively expanded and refined by looking at a succession of exemplars.

The teacher is a vital agent in this process, acting as a mentor in guiding the pupils to see the generic properties of examples, demonstrating the use of the generic organiser, and encouraging the pupils to explore the software, both in a directed manner to gain insight into specific aspects, and also in free exploration to fill out their own personal conceptions. This mode of teaching is called the *enhanced Socratic Mode*. It is an extension of the Socratic mode in which the teacher uses the computer software to discuss ideas with the pupils and to assist them to modify and enrich their conceptions. Generic organisers furnish external representations of the abstract mathematical concepts which act in a cybernetic manner, responding in a pre-programmed way to any input by the user, enabling both teacher and pupil to conjecture what will happen if a certain sequence of operations is set in motion, and then to carry out the sequence to see if the prediction is correct.

The gradient of a graph

The standard teaching of the calculus often follows a logical framework, with informal notions of limits, and the pictorial idea of the gradient of a moving chord approaching the gradient of the tangent. Though logically impeccable, such an approach is known to have subtle difficulties for pupils [see, for example, Cornu 1981]. A cognitive approach was designed in [Tall 1986a] using two pieces of graph-drawing software from the Graphic Calculus Pack [Tall 1986b]. The first enables the user to magnify a small portion of any graph, thus being able to investigate the property of “local straightness”. The idea is to use the software as a generic organiser to explore examples and non-examples of locally straight functions; non-examples including graphs with ‘corners’ such as y=|x| or y=|sinx|, with oscillations, such as y=xsin(1/x), and more bizarre graphs such as the blancmange function, which is so wrinkled, it never magnifies to look straight.

The second program offers two generic organisers. One represents the limiting process, visually and numerically, as the secant through (x,f(x)), (x+h,f(x+h)) changes as h decreases in successively smaller steps. Here the secant can be seen to be moving into a position that looks indistinguishable from the tangent, whilst the numerical value of the gradient settles down to a fixed value when displayed to a selected number of decimal places. It allows the limiting process to be visualized in a dynamic sense, linking the geometric picture to the numerical process, and generates discussion about the accuracy of the numerical computer operations and the sense of “getting within a specified accuracy” given the limitations of computer arithmetic. The second organiser plots the secant through the two points (x,f(x)), (x+c,f(x+c)), for fixed c, as x increases, simultaneously plotting the value of the gradient as a point. This process causes the secant to step along the curve, changing in gradient as it does so, leaving behind a trace of points representing the gradient of a graph. It encourages the gradient to be seen in a global sense, helping the pupil to see a static picture of a graph as a dynamic entity, scanning it by eye, and seeing the changing gradient.

A teaching experiment was performed to compare the learning in three experimental classes of 16/17 year olds, taught using a computer in the enhanced Socratic mode, with five similar control classes taught by traditional methods. A pre-test matched their abilities on knowledge of gradients of straight lines and possible understanding of initial calculus concepts. The post-test included the pictures of four graphs for which the students were asked to sketch the derivatives, such as those in figure 1.

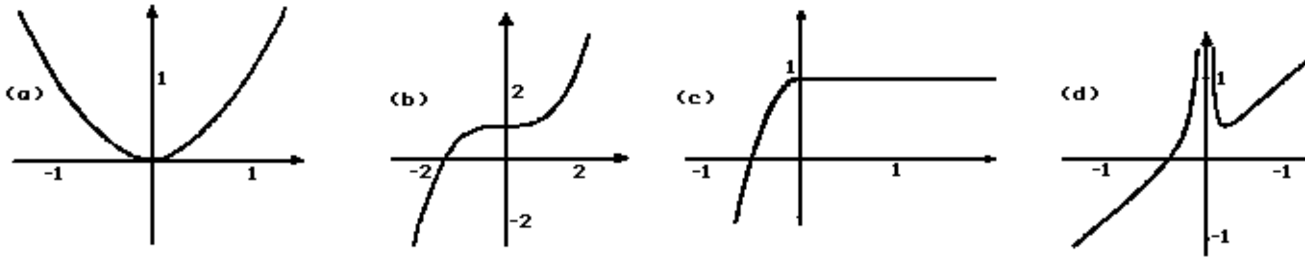

Figure 1

Most of the students with no computer experience were faced with a sequence of three separate tasks: first to guess the formula of the given graph, then to perform symbolic differentiation, then to sketch the graph of the derivative. The students with computer experience had developed a global/holistic view which enabled them to perform the task in a single dynamic process by looking along the graph to visualise the changing gradient. This task is essentially easier than the three stage serialist task. On the post-test the experimental students scored approximately 80% on this topic (a similar score to that obtained by a group of more highly qualified first year university mathematics students) whereas the control groups averaged only around 30%.

Using matched pairs selected on the pre-test, experimental students scored at a statistically significant higher level than the controls on global/holistic skills, including sketching derivatives, recognising graphs of derivatives, relating the derivative of a function to its gradient function, and in explaining the notions of gradient, tangent and differentiation from first principles from a geometrical viewpoint. At the same time, more traditional logical/sequential tasks, such as explaining symbolic differentiation from first principles or routine differentiation of polynomials and powers, showed no significant difference.

The experimental students were found to be performing in global/holistic terms at a level comparable with more highly qualified university students with two years additional experience of traditional calculus.

Visualising properties of straight lines

Blackett [1987] used a graph plotter to introduce the notions of straight line graphs to pupils aged 14-15. The software could draw graphs of expressions such as y=mx+c, giving the possibility of investigating the relationship between the algebraic expression and its geometric representation without having to carry out any intervening calculations.

In the first experiment he used the enhanced Socratic mode of teacher directed class discussion using the computer followed by pupil investigations in smaller groups, and compared this with the progress of comparable groups taught by colleagues using a more traditional pencil and paper approach. He taught three experimental classes of low, average and above average ability, who were matched with corresponding control classes each with at least as good a performance on a pre-test.

The experimental pupils began using paper and pencil plotting to investigate properties of pairs of coordinates that would lead to straight line equations. The introduction of the computer allowed them to tackle activities that might be too demanding using paper and pencil. For example, the task might be to to draw $y=x+20$ and $y=2x+10$ on the same axes to include their point of intersection (requiring some investigation to determine appropriate scales). An extension might be to challenge the pupils to think of an equation which would produce a line to cut the y-axis halfway between the two points where the first two lines crossed the y-axis, to meet the two lines at their point of intersection.

The software included a scaling option, leading naturally to discussions of conceptually demanding concepts such as the relationship between the numerical gradient and its pictorial representation when the scales are changed.

The post-test showed an the overall improvement in performance in all the groups, with the experimental peforming significantly better, except for one control class taught by a teacher who used the pre-test to teach the pupils specific tasks likely to arise in the post-test. Analysing the responses in detail revealed that, where the questions on the post-test were very similar to the pre-test, these control students scored higher than the corresponding experimental students, but where there were small conceptual differences they scored considerably lower. Blackett reports that:

> Pupils who had been taught to answer specific questions rather than the underlying concepts experienced difficulty whenever new questions varied, even slightly, from those they had met previously. These results appear to highlight the effects of encouraging instrumental as opposed to relational understanding.[Blackett 1987, p.93]

He also noted substantial discrepancies between the children’s performance on the post-test and traditional school tests. In particular there were a number of children who performed badly on traditional serialist/analytic questions yet performed well on global/holistic tasks.

Blackett took these exceptional children from the lowest ability class and added them to the highest ability class for an introduction to the idea of the gradient of “locally linear” graph, using the “Graphic Calculus” software. His colleagues were sceptical of his chances of success with the less able pupils because of their perceived lack of ability measured on standard school tests. However, Blackett found them well able to cope and concluded that:

> Students who had achieved a clear understanding of the straight line and its equation, particularly the significance of the gradient, can, with the aid of suitable computer graphics, develop an equally clear understanding of locally linear graphs and the curves associated with polynomial equations. ... There were pupils unable to handle number work successfully but were nevertheless able to demonstrate an understanding of advanced concepts presented in a visual form requiring either a visual interpretation or a drawing, rather than a calculation, as an answer.
>
> [Blackett 1987, pp. 127, 128]

His experiment indicated that these pupils, aged 14/15, were able to perform the task of sketching a global derivative at a level comparable with the 16/17 year old experimental students in the previous section.

Introducing Algebra

Mental images need not be only pictures in the traditional sense. The ideas we are discussing apply equally well to such things as symbolic concepts, provided that they are capable of being visualised and mentally manipulated. In Tall & Thomas [1986], a “dynamic approach” to algebra is described which involves a combination of activities designed to give children the ability to visualise the concept of an algebraic variable. This involves programming activities (in BASIC), such as:

```
A = 3
PRINT A +1
```

to see what the effect will be. Again the computer acts in a reasonable and predictable fashion; if `A=3`, then `PRINT A+1` will print `4`. Coupled with these activities are games involving the physical storage of a number in a box drawn on card and marked with a letter, so that a variable may be visualised as a placeholder for a number, and software which enables mathematical formulae to be evaluated for given numerical values of the letters involved. Matched pairs of 11/12 year olds were selected; one set studied the dynamic algebra module whilst their matched pairs did other (non-algebraic) work. The results, as one might expect, showed the experimental students performing at a higher conceptual level where algebraic notation was concerned. For example the successful responses to the question:

improved from 0% to 42.9% (as compared to 27% of 15 year olds in Hart, [1981]).

In order to test the long-term effects of the approach, a further algebra test was given over one year after the initial experiment. A summary of the results and a comparison with their previous results are given in table 2. This demonstrates that, more than one year after their work on basic concepts of algebra in a computer environment, they were still performing significantly better.

Test	Experim. Mean	Control Mean	p (1-tail)
Post Test (N=21)	41.2 %	25.3 %	<0.0005
Delayed Post Test (N=20)	43.9 %	32.6 %	<0.005
one year later (N=10)	55.8 %	47.3 %	<0.025

Table 2

This lends strong support to the idea that the introduction of a module of work such as the dynamic algebra package, with its emphasis on conceptualization and use of mental images rather than skill acquisition, can provide significant long-term conceptual benefits.

Skills and Higher Order Concepts

A second teaching experiment was held in which a dynamic algebra approach using the computer was compared with more traditional teaching methods [Thomas 1988]. The subjects of this second experiment were children taken from six mixed ability forms in the first year of a 12-plus entry comprehensive school. On the basis of an algebra pre-test it was possible to organise 57 matched pairs covering the full ability range.

In the first stage of the comparison the experimental group used the dynamic algebra module during their normal mathematics periods, using computers in small groups over a three week period. At the same time the control group used a traditional skill-based module developed in the school over some years, covering basic simplification of expressions and elementary equation solving in one unknown. Immediately following the work they were given a post-test identical to the pre-test.

The results superficially showed that there was no significant difference in overall performance, but on skill-based questions related to the content of the traditional module, the control group performed significantly better, whilst on questions traditionally considered to be conceptually more demanding, related to the experimental material, the experimental group performed better. For instance the question “simplify 3b-b+2a” was answered correctly by 61% of the control group, but only 29% of the experimental group. Meanwhile the question “for what values of a is a+3>7” was answered correctly by 31% of the experimental group and only 12% of the control group. The experimental group also performed better on more general problems less related to the taught subject matter, for example 50% of them responded correctly to the question “What is the perimeter of a rectangle D by 4” compared to 27% of the control.

Some six months later the pupils were all given the same traditional revision course on their earlier algebra, without any use of the computer. Both groups were re–tested and a comparison of matched pairs now showed the experimental students performing significantly better than the control students. In traditional manipulative skills they made up their deficit, whilst retaining a measure of superiority in conceptually higher order tasks.

Manipulative & Higher order skills	Experim. Mean	Control Mean	p (1-tail)
Manipulative Skills : Post Test	(N=48)	65.1 % 68.7%	n.s.
Delayed Post Test (N=47)	75.3%	72.4 %	n.s.
Higher order Skills : Post Test	(N=48)	35.7% 29.9 %	<0.025
Delayed Post Test (N=47)	42.2%	36.3%	<0.025

Table 3

The differential effects of the two treatments could be considered as a manifestation of the skills versus conceptual understanding dichotomy, in terms of the levels of understanding defined by Küchemann [1981]. His level 1 involves purely numerical skills or simple structures using letters as objects, level 2 involves items of increased complexity but not letters as specific unknowns. Level 3 requires an understanding of letters as specific unknowns; level 4 requires an understanding of letters as generalized numbers or variables. It is only at levels 3 and 4 that children are really involved in algebraic thinking rather than arithmetic. In the experiment the control pupils outperform the experimental pupils at levels 1 and 2, whilst the experimental pupils outperform the control pupils at the higher levels.

This suggests that there are differential effects from the two approaches in respect of surface algebraic skills (in which the control students have greater facility) and deeper conceptual understanding (in which the experimental students perform better). An alternative and, we suggest, more viable explanation is that the traditional levels of difficulty depend on the approach to the curriculum and may be altered by the use of the computer to encourage versatile learning.

Conclusions

All three experiments show a change in the nature of learning when an enhanced Socratic mode of teaching is employed using the computer. In each case there was a significant improvement in conceptual understanding of concepts traditionally considered to be of higher order, and evidence of a more versatile form of thinking related to the computer experiences. This gives support for the hypothesis that the computer can be used in the enhanced Socratic mode to provide experiences to encourage versatile learning.

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