New Cognitive Obstacles in a Technological Paradigm

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We stand at a point in history which has all the makings of a change in paradigm, in the sense of Kuhn. This change is being caused by a major innovation in technology: the computer. Interestingly this change is also occurring at a time when several different schools of thought are putting forward consonant theories for the difficulties of learning mathematics which may fruitfully be pulled together. Here I refer to such ideas as the theory of cognitive obstacles initiated in science by Bachelard (1938) and now taken on by many contemporary French mathematics educators, the theory of cognitive "frames" featured in Davis (1984), the theory of concept definition and concept image in Tall & Vinner (1981) and Kaput's 'complex web of mental representations' (mentioned in his article for this project).

At this point in the presentation of the Research Agenda Project we are reviewing the research of the past before turning to the possible effects of the new technology. The paper under consideration was limited by the organisers to the pre-computer curriculum, but in this reaction I intend to discuss the two aspects, the internal one which looks at the nature of the review, and the external one which looks forward to its implications in the new paradigm. It is therefore significant that the author selects as his focus of attention the notion of "cognitive obstacle" which features throughout the pre-computer papers considered here and will no doubt continue to be a major feature of research in the future.

The papers discussed are mainly concerned with gathering and analysing information as to the cognitive obstacles found in the current curriculum, concentrating on three main topics: equations in two variables, graphs of equations in two variables and the notion of function. The reviewer takes a standard school curriculum view of algebra, leavened with the wisdom of his experience of mathematical education. His viewpoint is essentially constructivist, (though the term is not actually mentioned in the article) and he chooses to interpret the notion of cognitive obstacle in terms of Piagetian theory, where the learner is confronted with new ideas that cannot be fitted into the learner's existing cognition, leading to an inability to cope adequately with the new information. Some, but not all, of the research discussed has an explicit Piagetian foundation, in particular the data found by the researchers is sometimes interpreted in terms of quasi-Piagetian stages, representing increasing levels of complexity which not all of the pupils manage to reach. In

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others there is simply a questionnaire presented, seeking likely phenomena which are then classified when the data is known.

The author chooses to present the main results from each paper in a commendably coherent personal interpretation. However, he does not enter into a detailed discussion of the various theories underlying some of the papers, nor does he have the space to refer to related papers about topics other than algebra in which relevant theory may be found. For example he does not refer to the theory of concept image which Vinner uses to underpins his approach, and which gives a reasonable explanation for the nature of cognitive obstacles, nor does he refer to any work from the information processing school who see cognitive obstacles in terms of mental programs with 'bugs' in them, an analogy conjured up by a computer science environment. (See the 'gentle criticism' of the 'buggy model' in Thompson's article later in this agenda discussion.) It would be a valuable activity for the Research Agenda Project to take this complex of related theories and to analyse in what ways they agree or are complementary, and in what ways they differ.

The major problem with most of the investigations considered is that they are linked to the curriculum *as it is at present* and one must consider to what degree the conclusions may remain relevant in a new computer paradigm. This consideration is intimately linked to the nature of two essentially different types of cognitive obstacle.

The nature of a cognitive obstacle

The notion of a "cognitive obstacle" was first introduced in the realms of science by Bachelard (1938) and highlighted in mathematical education by Brousseau (198?). In their terms an obstacle is a 'piece of knowledge of the student that has in general been satisfactory for a time for solving certain problems, and so becomes anchored in the mind, but subsequently, when faced with new problems, it proves to be inadequate and difficult to adapt'. The implication of Piagetian stage theory is that there are certain *fundamental* obstacles that occur for us all. If such universal obstacles exist, they would therefore also apply in a new paradigm.

I postulate that the reason for the belief in fundamental obstacles arises from the fact that certain concepts have a degree of complexity that makes it necessary to be acquainted with them in a certain order. For example, fractions are, of necessity, more complicated than whole numbers, and experience with operations on whole numbers leads to the implicit property that 'multiplication makes bigger', which leads to a cognitive obstacle when the individual meets the multiplication of fractions less than one.

However, some topics, traditionally taught in a certain order, may not have the *a priori* property that one concept is essentially more complex than the other. For instance, fractions are usually met in traditional syllabuses before negative numbers, but there is no reason why, given an appropriate context, the two topics should not be taught in the reverse order. Indeed, given the advocacy of John Thorpe's opening paper in this agenda project, and the response by Joan Leitzel, one might make a case for a considerable reduction of effort on fractions in the curriculum of the future.

One may hypothesise that cognitive obstacles are a product of the student's previous experience and their internal processing of these experiences. Granted this hypothesis, it would follow that an alternative sequencing for the curriculum (where practicable) may change the nature of understanding and the type of cognitive obstacle that may arise.

For example, empirical research shows that the problem 'multiply $3c$ by 5' is at a lower conceptual level than 'for what values of a is $a+3>7$?' (Küchemann 1981). However, this is based on a traditional approach to algebra in which manipulation skills are often taught before seemingly deeper conceptual skills. In an experiment Thomas (1988) used the computer to give a conceptual understanding of the notion of a variable to experimental groups and compared this with a parallel control groups studying a traditional algebra sequence. This showed the experimental pupils reversing the traditionally accepted levels of difficulty (table 1).

Question	Experimental % correct	Control % correct
Multiply 3c by 5	14	41
For what values of a is $a+3>7$?	31	12

Table 1

Thus the computer is likely to challenge many fondly held beliefs concerning the comparative difficulties of algebraic concepts. It may also help mathematics educators to sequence the curriculum in a manner more appropriate for cognitive development.

I contend that the very way in which we logically sequence the curriculum, limiting the child initially to simple cases for a substantial period before passing on to more complex cases, is bound to set up cognitive obstacles.

Thus simple cases of functions, limited to those given by simple formulae, will lead to the impression exhibited by Vinner and Dreyfus, that a function 'cannot have two rules of correspondence', or that the graphs are always 'continuous' because the examples encountered by the student always have had this property.

According to Hart (1983),

> ... the brain is by nature's design, an amazingly subtle and sensitive *pattern-detecting* apparatus (p.60) ... designed by evolution to deal with *natural complexity*, not neat 'logical simplicities' ... (p.76).

Thus our curricula, designed to present ideas in their logically simplest form, may simply *cause* cognitive obstacles, as for example, in the 'intuitive function concepts' wherein the student is asked if the change of literal symbol may induce a change in the values of a function table. Here one recalls Brousseau's notion of a 'didactical contract', the implicit unspoken agreement between teacher and pupil as to the nature of the tasks to be carried out in the classroom. Looking through this review one is often struck by the difficulty of understanding the nature of the didactical contract in some of the questions as posed. Clearly the students do not understand the nature of the game that is being played and the meanings of the algebraic symbols.

Already curricula are being designed which incorporate more complex problem-solving tasks from the outset, allowing the student to perform in a human way by abstracting relevant information from a rich context. The nature of cognitive obstacles produced by this type of approach is, once again, a matter for research.

The computer will allow topics to be approached in a richer variety of ways, allowing new sequences of presentation of ideas to avoid known cognitive obstacles, though they are very likely to introduce new obtacles of their own. Witness, for example, the work of Nachmias and Linn (1986) who show that a significant proportion of students interpret computer representations literally. In a physics experiment the students inserted a probe into a cooling liquid and the temperature was represented as a function of time as a graph on a computer screen. Unfortunately, the large pixel size onscreen made the graph appear jagged rather than smooth, which about one-third of the students interpreted as a true representation of cooling: they thought the liquid remained at a constant temperature for a while, then suddenly dropped a little (although this was actually given by the onscreen fall to a lower pixel level). The 'authority of the computer' may therefore be an impediment in learning, especially in the early stages though, on the other hand, the predictability of computer software may also be a powerful learning tool to help the student form meaningful mental representations of concepts currently known to provoke difficulties. For example, programming in both Logo (Sutherland 1987) and BASIC (Tall & Thomas 1986) gives students the possibility of a meaningful conceptualization of a variable which, in their different ways, help them to perform better at standard tests involving the meaning of the concept of a variable. This suggests that the cognitive obstacles that currently arise in the meaning of algebraic notation may occur to a different (and one may hope a lesser) extent in the new computer paradigm.

Avenues for future research

The review has concentrated on describing the literature to date which does not explicitly look to the new technology. However, there are pointers to interesting avenues for future research. First of all, in a changing paradigm, we need to ask the overall question:

(1) What role will algebra play in the new technological paradigm?

For example the manipulation of algebra to solve equations will be less important for that class of problems for which a numerical solution is appropriate, for which simple numerical algorithms on the computer will suffice. Leitzel and Demana (1988) report how the early stages of algebra can be replaced for effective solution of many real problems by using a calculator, and this approach includes problems which are very difficult to solve algebraically. Likewise the existence of symbolic manipulators may affect the need to spend excessive time in the classroom learning techniques which can now be carried out by a computer (though anyone who has used MuMath, or the new symbolic calculator, the HP28C, will know that it is vital to understand the essential algebraic principles, even if the computer is used to carry them out at a greater level of complexity than the individual may care to do by hand). We must genuinely rethink the role that algebra will play in the mathematics of the future and this will be no easy task.

The second fundamental question related to cognitive obstacles is to initiate new research to discover the effects of different kinds of computer experiences on students conceptualizations:

(2) How does the computer environment change the nature of the mathematical concepts, the development of students' conceptualizations and the related cognitive obstacles?

In view of the growing realization of the complexity of the mathematical concepts, which cannot be explained to the learner purely in terms of mathematical definitions and logical development, we must also ask:

(3) How we can encourage students to participate *actively* in the construction of appropriate meanings, some of which will be very different in the future paradigm?

This latter question brings me to a point that is not explicitly germane to this review, yet rarely discussed in papers presented at the Research Agenda Meeting: the question of *programming*. I make a plea for its consideration here because the review is implicitly about the manner in which individuals construct mathematical meaning for the concepts of equations, graphs and functions, and the cognitive obstacles which arise in the process. A most valuable way of building and testing algebraic concepts is through programming, even though this is bound to be bestowed with meanings different from those found in pencil and paper algebra. An elementary understanding of how the computer works through programming may provide insight into how variables are handled and how graphs are drawn, so that some of the cognitive obstacles mentioned so far may be confronted and discussed openly, enabling the student to construct a richer and more coherent conceptualization.

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