Concept Image and Concept Definition

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Student difficulties with formal definitions are not a new phenomenon, it occupied the mind of one of the great mathematicians at the turn of the century:

> What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils. (Poincaré, 1908)

The “New Mathematics” of the 1960s was based on the structural approach to mathematics, assuming that if only we could formulate the mathematical definitions and deductions correctly, then this would improve the learning of mathematics. But even when this was done, difficulties continued to persist. Careful analysis suggest that these difficulties are not stupidity on the part of students, but a natural human phenomenon that is found within all of us.

In the last decade, empirical research has emphasised that individuals build up their mental imagery of a concept in a way that may not always be coherent and consistent, and that previous experiences may colour the meanings of phenomena when they are met in new contexts. This is particularly in evidence in the introduction of more advanced mathematical concepts: For instance, the verbal definition of a limit “$s_n \to s$”, in the form “we can make s_n as close to s as we please, provided that we take n sufficiently large”, induces many individuals the belief that s_n can never equal s (Schwarzenberger & Tall 1978).

Many mathematical terms have an everyday meaning which can subconsciously interfere with the mathematics:

> Within mathematical activity, mathematical notions are not only used according to their formal definition, but also through mental representations which may differ for different people. These ‘individual models’ are elaborated from ‘spontaneous models’ (models which pre-exist, before the learning of the mathematical notion and which originate, for example, in daily experience) interfering with the mathematical definition. We notice that the notion of limit denotes very often a bound you cannot cross over, which can, or cannot, be approached. It is sometimes viewed as reachable, sometimes as unreachable. (Cornu 1981)

Even the way in which we structure the mathematical curriculum can lead to implicit beliefs that may be true in the given context, but later lead to cognitive conflict. For example, Vinner (1983) observed that many students believe that a tangent to a curve touches it, but may not cross it. This is implicitly true in circle geometry. But it was found that when students were asked to draw the tangent to the curve $y=x^3$ at the origin, many drew a line a little to one side which did not pass through the curve.

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During the late seventies and early eighties many examples of such conflicts were noted across a wide range of topics, including secants tending to tangents (Orton 1977), verbal and other difficulties with decimals, (Tall 1977), geometrical concepts (Vinner & Hershkowitz, 1980), the notion of function (Vinner 1983), limits and continuity (Tall & Vinner 1981), convergence of sequences (Robert 1982), limits of functions (Ervynck, 1983) the tangent (Vinner 1983, Tall 1987), the intuition of infinity (Fischbein et al 1979), the meaning of the differential (Artigue 1986), and so on.

To highlight the role played by the individual's conceptual structure, the terms "concept image" and "concept definition" were introduced in Vinner & Hershkowitz (1980) and later described as follows:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. ... As the concept image develops it need not be coherent at all times. ... We shall the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion.
> (Tall & Vinner 1981, p.152)

On the other hand:

> The *concept definition* [is] a form of words used to specify that concept. (ibid.)

What becomes overwhelmingly obvious when a powerful mathematical concept such as the notion of function is analysed is its cognitive complexity. The concept definition of a function may be given in the form “a relation between two sets A and B in which each element of A is related to precisely one element of B“, but the experience of the concept has many other facets. For instance it may be viewed as an *action* that assigns to each element x in A a corresponding element f(x) in B, or as a *graph*, or as a *table of values*. Experiences in a specific context (such as the calculus) may suggest that a function is always given by a *formula*, or perhaps extended a little to allow a finite number of formulae at different parts of the domain.

It is eminently possible for students to be taught to respond correctly to questions involving the formal definition whilst developing a concept image which includes potential conflicts in the following sense:

> ... we shall call a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition a *potential conflict factor*. Such factors need never be evoked in circumstances which cause cognitive conflict, but if they are so evoked, the factors concerned will then be called *cognitive conflict factors*. ... They only become cognitive conflict factors when evoked simultaneously. In certain circumstances cognitive conflict factors may be evoked subconsciously, with the conflict only manifesting itself by a vague feeling of unease.
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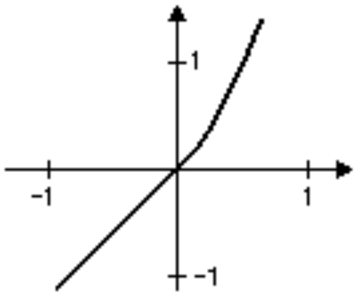
> ... a more serious type of potential conflict factor is one in the concept image which is at variance not with another part of the concept image but with the formal concept definition itself. Such factors can seriously impede the learning of the formal theory, for they cannot become actual cognitive conflict factors unless the

formal concept definition develops a concept image which can then yield a cognitive conflict. Students who have such a potential conflict factor in their concept image may be secure in their own intepretations of the notions concerned and simply regard the formal theory as inoperative and superfluous.
(Tall & Vinner 1981)

When students meet an old concept in a new context, it is the concept image, with all the implicit assumptions abstracted from earlier contexts, that responds to the task. If the image is built on experiences that conflict with the formal definition, this can lead to responses which are at variance with the formal theory. For instance, (Tall 1986) asked 16 year old students to draw the tangent to the following graph at the origin:



$$y = \begin{cases} x & (x \le 0) \\ x + x^2 & (x \ge 0) \end{cases}$$

Only 22 out of 65 students (34%) in a control group studying traditional mathematics drew the correct tangent y=x. Some were not able to cope with the task at all, asserting that the tangent could not exist because

> The graph is two separate functions, and there is not a tangent at x=0

or:

> ... because the tangent should touch the line at one specific point but this tangent would touch it constantly.

30 students (46%) drew the tangent moved round a little so that it looked as if it touched the curve at only one point. In Tall (1986) this is called a *generic* tangent: it embodies the generic property of "touching at one point only" shared by many examples previously experienced. Another image evoked by some students is the observation that the graph is beginning to turn at the origin, so this is reflected by moving the tangent round a little...

These potential conflicts can be reduced by discussion at an earlier stage, provided that this occurs in an environment that is meaningful to the students. For instance, the concept of the tangent was discussed in Tall (1986) with three experimental groups using a computer representation of the concept, drawing a line through two very close points on the graph to give a practical approximation. This allowed investigations of graphs with corners and the tangent to graphs at an inflection point. It did not entirely resolve the problem of the concept of a tangent "touching but not crossing", but the improvement in performance was significant: 31 out of 41 (76%) now responded with the correctly drawn tangent whilst only 8 (20%) sketched a generic tangent. (Using a χ^2 test, the improvement is significant at the 0.01 % level.)

Howson and Austin (1980) make the observation that

> In England the tradition is to rely on the formal definition in higher education and informal or ostensive definitions in primary or secondary schools.

The transition stage between the two, the upper age range in the secondary school, is a time when definitions are beginning to be used in a more technical sense, though the tradition varies in different countries. What the research on cognitive conflict suggests is that it is not sensible to expect students to be able to argue logically from concept definitions without expecting interference from their individual concept images.

Referring to the concept definition of a function, Vinner (1983) claims:

> (1) In order to handle concepts one needs a concept image and not a concept definition.
>
> (2) Concept definitions (where the concept was introduced by means of a definition) will remain inactive or even be forgotten. In thinking, almost always the concept image will be evoked.

In the light of such an observation, we might ask how our insight into the variety of concept images may suggest a way ahead in the teaching and learning of mathematics? It suggests that a purely structural approach to the subject is unlikely to succeed unless the students have the concept imagery to be able to deal with the formalities, a fact all too well known following the experience with the "new mathematics".

The sheer variety of individual concept imagery suggests that it is not simply a case of passing on mathematical knowledge in a formal way. The alternative is to give the students richer experiences so that they are able to form a more coherent concept. The latter is not as easy as it sounds, as it involves a balance between the variety of examples and non-examples necessary to gain a coherent image and the complexity which may increase the cognitive demand to unacceptable levels.

As Euclid is reputed to have said to Ptolemy "there is no Royal road to geometry", and nor is there a guaranteed easy route to success in other areas of mathematics. However, experiments so far indicate that it is possible to use the computer in an imaginative way to provide a rich context for discussing and developing more appropriate cognitive images. It gives us a hopeful way to occupy our time until the next revolution.

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