LONGER-TERM CONCEPTUAL BENEFITS FROM USING A COMPUTER IN ALGEBRA TEACHING

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This paper provides evidence for the longer-term conceptual benefits of a pre-formal algebra module involving directed computer programming, software and other practical activities designed to promote a dynamic view of algebra. The results of the experiments indicate the value of this approach in improving early learners' understanding of higher level algebraic concepts. Our hypothesis is that the improved conceptualisation of algebra resulting from the computer paradigm, with its emphasis on mental imagery and a global/holistic viewpoint, will lead to more versatile learning.

The Background

In a previous paper (Tall and Thomas, 1986) we described the value of a three week "dynamic algebra" module designed to help 11 and 12 year-old algebra novices improve their conceptual understanding of the use of letters in algebra. The activities include programming (in BASIC), coupled with games involving the physical storage of a number in a box drawn on card, marked with a letter, and software which enables mathematical formulae to be evaluated for given numerical values of the letters involved. This paper carries the work further with two experiments that test the nature of the learning and its longer term effects.

Theoretical Considerations

The formal approaches to the early learning of algebra have nearly always considered the topic as a logical and analytical activity with very little, if any, emphasis on the visual and holistic aspects of the subject. Many researchers, however, have identified the existence of two distinct learning strategies, described variously as serialist/analytic and global/holistic respectively. The essential characteristics distinguishing these two styles have been recorded (e.g. Bogen 1969), with the former seen as essentially an approach which breaks a task into

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parts which are then studied step-by-step, in isolation, whereas the latter strategy encourages an overall view which sees tasks as a whole and relates sub-tasks to each other and the whole (Brumby, 1982, p.244). Brumby's study suggests that only about 50% of pupils consistently use both strategies, thus meriting the description of versatile learners. The advantages of versatile thought in mathematics are described by Scott-Hodgetts :

> Versatile learners are more likely to be successful in mathematics at the higher levels where the ability to switch one's viewpoint of a problem from a local analytical one to a global one, in order to be able to place the details as part of a structured whole, is of vital importance. ...whilst holists are busy speculating about relationships, and discovering the connections between initially disjoint areas of mathematics, it may not even occur to serialists to begin to look for such links. [Scott-Hodgetts,1986, page 73]

These observations on learning styles correlate well with a number of physiological studies which indicate that the mind functions in two fundamentally different ways that are complementary but closely linked (see, for example, Sperry et al 1969, Sperry 1974, Popper & Eccles 1977). The model of the activity of the mind suggested by these studies is a unified system of two qualitatively different processors, linked by a rapid flow of data and controlled by a control unit. The one processor, the familiar one, is a sequential processor, considered to be located in the major, left hemisphere of the brain, responsible for logical, linguistic and mathematical activities. The other processor, in the minor, right hemisphere, is a fast parallel processor, responsible for visual and mental imagery, capable of simultaneously processing large quantities of data. The two processors are linked physically via the corpus collosum, and controlled by a unit located in the left hemisphere. This image of the two interlinked systems, one sequential, one parallel, is a powerful metaphor for different aspects of mathematical thinking. Those activities which encourage a global, integrative view of mathematics, may be considered to encourage the metaphorical right brain. Our aim is to integrate the work of the two processors, complementing logical, sequential deduction with an overall view, and we shall use the term *cognitive integration* to denote such an approach, with the production of a versatile learner as its goal (see Thomas 1988 for further details).

The approach to the curriculum described here uses software that is designed to aid the learner to develop in a versatile manner. In particular, the software provides an environment which has the potential to enable the user to grasp a gestalt for a whole concept at an intuitive level. It is designed to enable the user to manipulate *examples* of a specific mathematical concept or a related system of concepts. Such programs are called *generic organisers* (Tall, 1986). They are intended to aid the learner in the abstraction of the more general concept embodied by the examples, through being directed towards the generic properties of the examples and differentiating them from non-generic properties by considering non–examples. This abstraction is a *dynamic* process. Attributes of the concept are first seen *in a single* *exemplar*; the concept itself being successively expanded and refined by looking at a succession of exemplars.

The generic organiser in the algebra work is the "maths machine" which allows input of algebraic formulae in standard mathematical notation and evaluates the formulae for numerical values of the variables. The student may see examples of the notation in action, for example 2+3*4 evaluates to 2+12=14, and not to 5*4=20`. Although this contravenes experience using a calculator, the program acts in a reasonable and predictable manner, making it possible to discuss the meaning of an expression such as 2+3a and to invite prediction of how it evaluates for a numerical value of a. In this way the pupils may gain a coherent concept image for the manner in which algebraic notation works.

The teacher is a vital agent in this process, acting as a mentor in guiding the pupils to see the generic properties of examples, demonstrating the use of the generic organiser, and encouraging the pupils to explore the software, both in a directed manner to gain insight into specific aspects, and also in free exploration to fill out their own personal conceptions. This mode of teaching is called the *enhanced Socratic Mode*. It is an extension of the Socratic mode where the teacher discusses ideas with the pupil and draws out the pupil's conceptions (Tall, 1986). Unlike the original Socratic dialogue, however, the teacher does not simply elicit confirming responses from the pupil. After leading a discussion on the new ideas to point the pupils towards the salient features, the teacher then encourages the pupils to use exactly the same software for their own investigations.

The generic organiser provides an external representation of the abstract mathematical concepts which acts in a cybernetic manner, responding in a pre-programmed way to any input by the user, enabling both teacher and pupil to conjecture what will happen if a certain sequence of operations is set in motion, and then to carry out the sequence to see if the prediction is correct.

The computer provides an ideal medium for manipulating visual images, acting as a model for the mental manipulation of mathematical concepts necessary for versatility. Traditional approaches which start with paper and pencil exercises in manipulating symbols can lead to a narrow symbolic interpretation. Generic organisers on the computer offer anchoring concepts on which concepts of higher order may be built, enabling them to be manipulated mentally in a powerful manner. They can also encourage the development of holistic thinking patterns, with links to sequential, deductive thinking, which may be of benefit in leading to better overall performance in mathematics.

Longer-Term Benefits in Algebra

In order to test the long-term effects of the "dynamic algebra module", a follow-up study was carried out over one year after the initial experiment previously described (Thomas and Tall, 1986). By this time the children were now 13 years old and had transferred to other schools where they had completed a year of secondary education. Eleven of the matched pairs attended the same secondary school and were put into corresponding mathematics sets, so that during their first year (aged 12/13) they received equivalent teaching in algebra. At the end of the year they were all given the algebra test used in the original study. A summary of the results and a comparison with their previous results are given in table 1. This demonstrates that, more than one year after their work on basic concepts of algebra in a computer environment, they were still performing significantly better.

Test	Experim. Mean (max=79)	Control Mean (max =79)	Mean Diff.	S.D.	N	t	df	p
Post test	32.55	19.98	12.57	10.61	21	5.30	20	<0.0005
Delayed Post-test	34.70	25.73	8.47	11.81	20	3.13	19	<0.005
one year later	44.10	37.40	6.70	7.76	10	2.59	9	<0.025

Table 1

This lends strong support to the idea that the introduction of a module of work, such as the dynamic algebra package, with its emphasis on conceptualization and use of mental images rather than skill acquisition, can provide significant long-term conceptual benefits.

Skills and Higher Order Concepts

A second teaching experiment was held in which a dynamic algebra approach using the computer was compared with more traditional teaching methods. The subjects of this second experiment were 12/13 year old children in six mixed ability classes in the first year of a 12-plus entry comprehensive school. The school is divided into two halls with children apportioned to provide identical profiles of pupil ability, but the teaching is done by a unified team of teachers, allowing direct comparisons of different teaching methods. On the basis of an algebra pre-test it was possible to organise 57 matched pairs covering the full ability range in the classes.

In the first stage of the comparison the experimental group used the dynamic algebra module during their normal mathematics periods, using computers in small groups of two or three over a three week period during the autumn term. At the same time the control group used a traditional skill-based module employed in the school over some years, covering basic simplification of expressions and elementary equation solving in one unknown. Immediately following the work they were given a post-test containing the same questions as the pre-test.

The results (given later in table 5) superficially showed that there was no significant difference in overall performance, but analysis of individual questions presented an interesting picture. On skill-based questions related to the content of the traditional module, the control group performed significantly better, whilst on questions traditionally considered to be conceptually more demanding, the experimental group performed better. Table 2 shows typical skill-based questions and the better performance of the control group:

Question	Experimental %	Control %	z	p
Multiply 3c by 5	14	41	3.07	<0.005
Simplify 3a+4b+2a	50	73	2.46	<0.01
Simplify 3b-b+2a	29	61	3.36	<0.0005
Simplify 3a+4+a	38	78	1.60	n.s.
G jigsaws and J jigsaws =?	55	78	2.39	<0.01

Table 2

Table 3 shows the better performance of the experimental group on questions considered to be more demanding in a traditional approach, requiring a higher level of understanding, including the concept of a letter as a generalized number or variable:

Question	Experimental %	Control %	z	p
For what values of a is a+3>7 ?	31	12	2.33	<0.01
For what values of a is 6 > a+3 ?	22	6	2.33	<0.01
a+b=b, always, never, sometimes ... when?	31	17	1.65	<0.05
M+P+N=N+M+R, always, never, sometimes ... when?	38	28	1.08	n.s.
Perimeter of rectangle D by 4	50	27	2.46	<0.025
Perimeter of rectangle 5 by F	50	29	2.24	<0.025
Larger of 2n and n + 2 ?	7	0	1.91	<0.05

Table 3

The differential effects of the two treatments could be considered as a manifestation of the skills versus conceptual understanding dichotomy, in terms of the levels of understanding defined by Küchemann [1981]. His level 1 involves purely numerical skills or simple structures using letters as objects, level 2 involves items of increased complexity but not letters as specific unknowns. Level 3 requires an understanding of letters as specific unknowns; level 4 requires an understanding of letters as generalized numbers or variables. It is important to understood that these levels were not intended to be a hierarchy but rather a description of children's functional ability. However, it is only at levels 3 and 4 that children are really involved in algebraic thinking rather than arithmetic and few children (17% at age 13) attain this level of understanding. Table 2 shows that the control pupils outperform the experimental pupils at levels 1 and 2, whilst table 3 shows that the experimental pupils outperform the control pupils at the higher levels.

This suggests that there are differential effects from the two approaches in respect of surface algebraic skills (in which the control students have a greater facility at this stage) and deeper conceptual understanding (in which the experimental students perform better). An alternative (and, we suggest, more viable) explanation is that the traditional levels of difficulty depend on the approach to the curriculum and may be altered by a new approach using the computer to encourage versatile learning.

Longer-term effects on skills and conceptual ideas

In the summer term, some sixth months later, the pupils were all given the same traditional revision course on their earlier algebra, without any use of the computer. Both groups were re–tested and a comparison of matched pairs was made again. Table 4 shows the pupils performance on the test as a whole. On this occasion the experimental students now performed significantly better than the control students.

Test	Experim. Mean (max=67)	Control Mean (max=67)	Mean Diff.	S.D.	N	t	df	p (1-tail)
Post test	36.0	35.9	0.1	10.46	47	0.06	46	n.s.
Delayed Post-test	42.1	39.3	2.76	8.91	46	2.08	45	<0.025

Table 4

In the conceptually demanding questions of the type mentioned in table 1, the experimental students continued to maintain their overall superiority (table 5).

Test	Experim. Mean (max=26)	Control Mean (max=26)	Mean Diff.	S.D	N	t	df	p (1-tail)
Post test	9.28	7.77	1.50	4.81	48	2.14	47	<0.025
Delayed Post-test	10.97	9.45	1.51	4.73	47	2.17	46	<0.025

Table 5

Meanwhile, on the skill-based questions, the experimental students marginally surpassed the control students, although the difference was not statistically significant.

The effects of Gender

Although the researchers did not set out to look specifically at the relationship between performance and gender, a factor analysis including ability and gender among its variables was included. A random sample of girls and boys was taken and a comparison on pre-test and post-test made. In the sample the girls performed less well than the boys on the pre-test, but made a statistically significant improvement to perform better than the boys on the post-test. The reasons for this are not altogether clear at this stage. It was certainly noticeable that the more able boys, with previous computer experience, were constantly showing their prowess at making the computer print screensful of coloured characters, and some saw the elementary activities as a little beneath their dignity. Meanwhile some of the girls had initial difficulties and took the task extremely seriously, discussing the problem and helping each other in small groups. Thus the experiment was unable to distinguish whether the difference was social or cognitive.

Conclusions

The experiments provide evidence of a more versatile form of thinking related to the computer experiences. Further this improved understanding of concepts usually considered to be of a higher level and difficult to attain by traditional methods, was shown to of a long-term nature. There is also support for the hypothesis that the computer can be used in the enhanced Socratic mode to provide experiences to encourage versatile learning through cognitive integration.

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