Inconsistencies in the Learning of Calculus and Analysis

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1.Research Background

In the mid-seventies, as a lecturer in mathematics rather than a researcher in mathematics education, I first became involved in the difficulties of students learning the calculus and analysis. At that time my belief, which I suspect remains common amongst professional mathematicians, was that the difficulties could be eased by preparing materials in a logical and coherent way for students to understand. Exploratory investigations into students‘ conceptions revealed the inadequacy of this viewpoint as the inherent cognitive conflict in many of the concepts was exposed.

Inconsistencies identified in research

In Tall (1977) the results of the an investigation into student's beliefs were reported, based on written responses to questionnaires by a population of 36 mathematics students in their first week of study at university.

One item made the request:

> If you know the definition of the limit of a sequence, write it down: $s_n \to s$ as $n \to \infty$ means:

A later one asked:

> Is $0{\cdot}\dot{9}$ (nought point nine recurring) equal to one, or is it just less than one? Explain the reason behind your answer.

Only 10 out of 36 claimed to know a precise definition and only seven were able to formulate a definition that was

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mathematically acceptable. Of these seven, only *one* responded that $0{\cdot}\dot{9} = 1$.

Thirteen of the thirty six held apparently conflicting views, asserting that $0{\cdot}\dot{9}$ was less than 1, whilst elsewhere stating that

$$\lim_{n\to\infty} \left(1+\frac{9}{10}+\frac{9}{10^2}+\ldots+\frac{9}{10^n}\right) = 2.$$

A week later the students were asked to write down various decimals as fractions, including

0·25
0·05
0·3
0·333...
$0{\cdot}\dot{9} = 0{\cdot}999\ldots$

Two thirds of the students (24) now said that $0{\cdot}\dot{9} = 1$ (or 1/1), including 13 who had previously affirmed the result was less. Their written answers also exhibited the conflict in terms of crossings out and added comments.

Subsequent research by one of my Ph.D. students, Monaghan (1986), working with 16/17 year-olds studying the calculus, showed that ”recurring decimals are perceived as dynamic, not static, entities and are not *proper* numbers. Similar attitudes exist towards infinitesimals when they are seen to exist“. Comparing students taking a calculus course with students of a similar ability who were not, he concluded that ”the first year of a calculus course has a negligible effect on students’ conceptions of limits, infinity and real numbers“.

Other investigations reveal a wide range of difficulties with the meaning of concepts in calculus and analysis (for example, Schwarzenberger & Tall 1978, Tall 1979a,b, 1980 a,b).

Cognitive conflict associated with limits of sequences, limits of functions and continuity is considered in Tall & Vinner (1981). The phenomena are here interpreted in terms of the theory of *concept image* and *concept definition*, defined as follows:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. ... As the concept image develops it need not be coherent at all times. ... We shall the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion.

On the other hand:

> The *concept definition* [is] a form of words used to specify that concept.
>
> (Tall & Vinner 1981, page 152)

The paper considers the curriculum studied earlier by pupils in an attempt to postulate reasons for the mismatch between students’ evoked concept images and their knowledge of concept definitions. For example, the students had a concept image of a continuous function, which could have come from a variety of sources, not least being the colloquial meaning of the term in phrases such as "it rained continuously all day" (meaning there was no break in the rainfall). A questionnaire administered to 41 first year university mathematics students included the question:

Which of the following functions are continuous?

If possible, give reason for your answer.

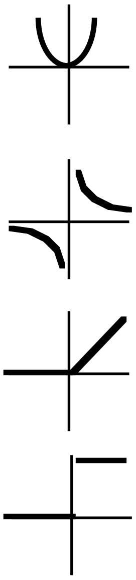
$f_1(x) = x^2$

$f_2(x) = 1/x \ (x \neq 0)$

$f_3(x) = \begin{cases} 0 & (x \leq 0) \\ x & (x \geq 0) \end{cases}$

$f_4(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$

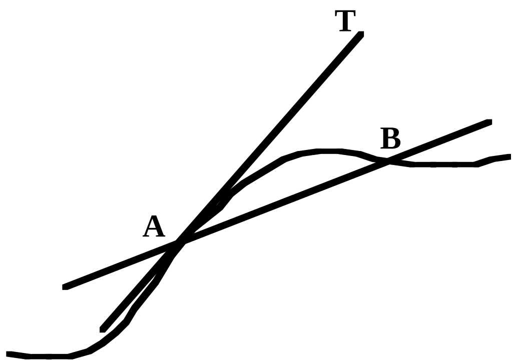
$f_5(x) = \begin{cases} 0 & (\text{rational}) \\ 1 & (\text{irrational}) \end{cases}$



Mathematically f_1, f_2 and f_3 are continuous, whilst f_4 and f_5 are not. But conceptual imagery suggests strongly that f_2 is *not* continuous (because its graph has a break) and other evoked images have a coercive effect on student responses. For example, f_3 is considered by some to be ”not continuous“ because ”it is not given by a single formula“, or because ”it does not change smoothly at the origin“.

In questionnaires it is quite possible for students to give ”right answers for wrong reasons“, or even ”wrong answers for right reasons“. Tall (1986a) asked students to respond in writing whether they thought the following statement to be true or false:

As B→A the line through AB tends to the tangent AT.
(True/False?)



Of a sample of nine 16 year old students interviewed in depth (as part of a larger project), four said the statement was ”true“ but linked the symbol B→A to vector notation and visualized B as moving to A *along the line BA*, so that the line (segment) BA ”tends“ to the tangent. Meanwhile, one student considered the statement ”false“ for the sensible reason that the lines were infinite so, way off at infinity, the line AB and tangent AT were still a long way apart no matter how close A and B became. For this reason questionnaires alone, without follow-up interviews, may not reveal the full story.

Vinner (1983) observed that students develop individual concept images of a tangent which conflict with the formal definition. In particular, early exposure limited to tangents to circles may cause them to see the tangent as a line which "touches the graph but does not cross it". Tall (1986a, 1987) used the computer to encourage discussion and a more visual approach to the tangent as part of a wider study in the calculus with a (pre- and post-test) comparison between 41 students aged 16/17 using the computer, 65 comparable students following a more traditional approach and 47 university students aged 18 (post-test only).

The computer helped provoke new insight into the notion of the tangent, though new conflicts could arise if not handled carefully (for instance, more students using the computer were now likely to believe that a tangent was a "line through two very close points on the graph" (because the program approximated the tangent in this way)).

The main thrust of the research in Tall (1986a) was to test a ”cognitive approach to the calculus“ using software that enabled the teacher to demonstrate, and the students to explore a wide range of examples of the gradient of a curved graph. There was significant improvement in the experimental students‘ ability to sketch gradients of given graphs, and their conceptualizations transferred to the more general case of the gradient and tangent of a graph at a point where the function was given by different formulae on either side.

2. Position Paper

On considering the evidence, it becomes very clear that the concepts under consideration are highly complex and not necessarily coherently understood even by teachers or professional mathematicians. Several papers (e.g. Tall 1980c, 1981a, 1981b, 1985b) have been devoted to the mathematical and cognitive problems with limits and infinity, including the existence of mathematical subcultures with conflicting views (such as standard and non-standard analysis, one denying the existence of infinitesimals, the other affirming it). The purpose of these papers was to underline the fact that there may not always be one universal mathematical truth which acts as a touchstone by which all others are judged. Furthermore, Monaghan (1986) demonstrated that these formal cultures are different again from students‘ belief structures, for instance, he shows that students‘ conceptions of the continuum ”do not conform to the classical or the non-standard paradigms“.

It was therefore in the early eighties that my attention turned to using the computer to provide a rich environment in which concepts could be demonstrated, explored and discussed. My thesis is that an environment allowing the user to explore both examples and non-examples of a mathematical concept or process can help the user abstract the general properties embodied in the examples and contrasted by the non-examples. An environment designed with this in mind is called a *generic organizer*, and (Tall 1986b) consists of a collection of generic organisers for visualizing calculus concepts. A new approach to the calculus using these organisers is described in a series of six articles in Mathematics Teaching, starting with Tall (1985a).

Computer software can provide a representation or model of the mathematical phenomena, but not always an exact translation. (For instance, pictures are drawn using finite pixels, so that straight lines do not normally look straight on a computer screen.) However, these obvious inconsistencies may be regarded as an *advantage*, not a hinderance. If the student can clearly *see* that the representation is not exact, it is possible to discuss the reasons why, and to begin to build up richer mental models.

It is my belief that we do students a disservice by organising the curriculum so that they are presented only with simple ideas first and given too great an exposure to an environment which contains regularities that do not hold in general. This just sows the seeds for later cognitive conflict. For example, doing geometry of curves only with circles, can give a dangerously limited idea of a tangent, or studying differentiation initially only with polynomials for so long may cause students to abstract general ”rules“ which are not true in a wider context. (For instance that at a maximum the derivative is zero, rather than *if* the derivative exists, *then* it is zero...)

There are two possibilities (which are not mutually exclusive): one is to research the cognitive conflict to be prepared to face it when it occurs, a second is to give a richer conceptualization from the start within which the conflict may later be framed. My approach is to design generic organisers to allow students to explore general ideas through specific examples and non-examples. In this way powerful general ideas can be introduced at the outset (such as differentiable and non-differentiable functions at the beginning of the calculus) and specific examples with pertinent properties can be investigated (such as the tangent to a straight line or at an inflection point) to help students avoid narrow over-generalization.

This will need a radical reform of the curriculum. It applies at all levels of development. Much research, based on pre-computer environments, may be in error because it occurred in a context which may not pertain in future. For example, we teach young children about simple fractions in terms of halving and quartering because this is within their physical and mental capacity. But just because they may be *physically* incapable of dividing a cake into seven equal pieces does not mean that they are *mentally* incapable of visualizing it aided by appropriate software. A rich computer environment allowing children to carry out their mental ideas may give the opportunity to circumvent some of the trivializing introductions that may stunt future growth. We cannot make the complicated concepts more simple, but we can give far richer experiences that enable them to be seen in a wider, and more powerful, context. Recent research shows this to be a promising direction to follow (Tall & Thomas 1988).

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