Constructing the Concept Image of a Tangent

David Tall
Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL
U.K.

When a learner meets a new mathematical concept, it may be invested with implicit properties arising from the context, producing an idiosyncratic concept image which may cause cognitive conflict at a later stage. The purpose of this empirical research is to test the hypothesis that interactive computer programs, encouraging teacher demonstration and pupil investigation of a wide variety of examples and non-examples, may be used to help students develop a richer concept image capable of responding more appropriately to new situations. Three experimental classes of sixteen year-olds were taught using computer packages capable of magnifying graphs to see if they "looked straight", and to draw a line through two close points on a graph. These formed the basis of class discussion and small group investigation to encourage the formation of a coherent relationship between the concepts of gradient and tangent. For comparison, five other classes were taught by more traditional methods. Two questionnaires administered during the course confirmed that the experimental students were able to respond more appropriately in new situations, for example in the case where a function is given by a different formula to the left and right. However, the notion of a "generic tangent" – an imagined line touching the graph at only one point (even where this is inappropriate) – persisted in both groups, though significantly less amongst the experimental students.

Building and testing Mathematical Concepts

The computer introduces a new factor into the classroom relationship between the pupils, the teacher and the mathematical concepts to be considered. It enables aspects of the mathematics to be externalised and manipulated on the computer VDU. In terms of Skemp's three modes of building and testing mathematical concepts (Skemp 1979), it offers a direct (mode 1) method of building and testing using the computer software, in addition to discussion with the teacher (mode 2) and internal consistency of the mathematics in the mind of the learner (mode 3):

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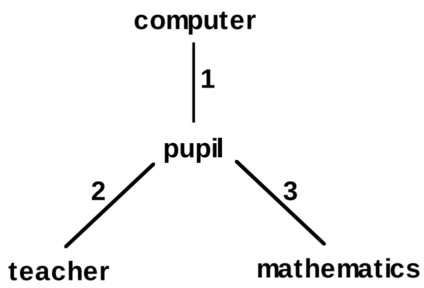


This more immediate mode of building and testing can be highly advantageous in introducing new concepts that previously have seemed extremely abstract to pupils. However, there may be a danger that the computer introduces inappropriate factors that may cause difficulties of their own. For example, a "straight line" on a computer VDU is a coarse sequence of high-lighted pixels that, at best, may only look fairly straight. A highly magnified picture of a small portion of a curved graph might look "nearly straight", and the superimposed tangent at a point on the graph might look almost the same,differing by a pixel or so here and there. Such difficulties require careful handling by the teacher. However, the differences between the practical (and inaccurate) computer picture and the theoretical ideas can also provoke a great deal of discussion that can be most rewarding for the pupils. As Hart has observed (1983, page 52):

> The brain was designed by evolution to deal with *natural complexity*, not neat "logical simplicities".

Mathematicians analyse concepts in a formal manner, producing a hierarchical development that may be inappropriate for the developing learner. Instead of clean, formal definitions, it may be better for the learner to meet moderately complicated situations which require the abstraction of essential points through handling appropriate examples and non-examples. Such complexity requires discussion and "negotiation of meaning" between teacher and pupils.

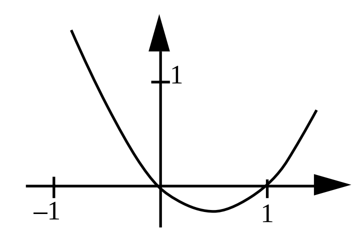
The notion of a tangent is a complex concept which causes difficulties when it is met in extreme circumstances. Vinner (1982) has observed that early experiences of the tangent in circle geometry introduces a belief that the tangent is a line that touches the graph at one point and does not cross it; this produces a concept image that causes cognitive conflict when more extreme cases are considered, such as the tangent at a point at inflection, where it does cross the curve, or the case of a tangent at a cusp which is slightly more contentious.

Classroom activities

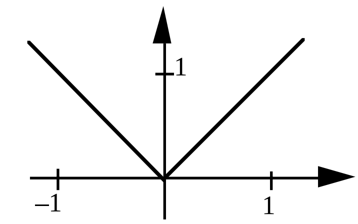
In three experimental classes, of 12, 14 and 16 pupils, the aim was to negotiate the meaning of the tangent concept through using the computer to draw a line through two very close points on the graph as part of a broader introduction to the idea of gradient of a graph in the calculus. This was to be demonstrated by the teacher leading a discussion centred on the computer, before encouraging the students to work with the computer in small groups. It was part of the brief for the experimental students to consider cases, such as $y=|\sin x|$, which have “corners” where they have neither gradient, tangent or derivative, though they visibly have different left and right gradients. One of the programs used purported to draw a “tangent”, when it actually drew the straight line through $(x, f(x))$, $(x+h, f(x+h))$ for $h=0.0001$. This seemed to draw a “tangent” to $y=|\sin x|$ at the origin, providing a rich source of discussion. They were also given experiences of more complex graphs such as $y=x\sin(1/x)$ at the origin. The researcher took an active part in the experimental group of 14 pupils, suggesting activities to be followed by the other two groups, whilst the five control classes followed a more traditional strategy assuming an intuitive knowledge of the meaning of a tangent. All teachers kept diaries of their activities.

The test investigations

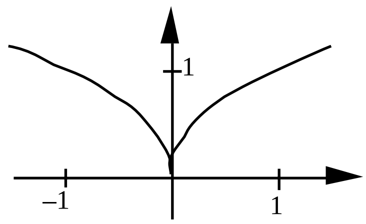
Two brief tests were administered to the students during the course of their work. The first followed immediately after they had studied the notion of the gradient of a graph at a point, the second after they had studied the notion of a tangent in greater detail. Both involved the same sequence of graphs:



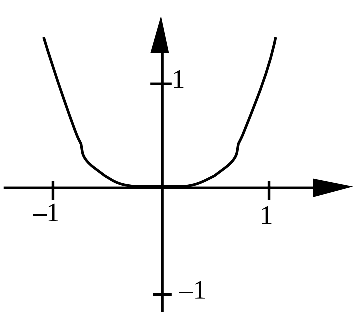
(1) $y=x^2-x$



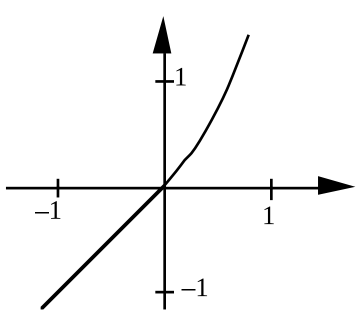
(2) $y=\mathrm{abs}(x)$



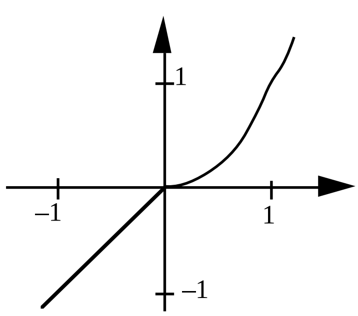
(3) $y=\sqrt{(\mathrm{abs}(x))}$



(4) $y=\mathrm{abs}(x^3)$



(5) $y=\begin{cases} x & (x \le 0) \\ x+x^2 & (x \ge 0) \end{cases}$



(6) $y=\begin{cases} x & (x \le 0) \\ x^2 & (x \ge 0) \end{cases}$

In the GRADIENT INVESTIGATION,for each graph the students were asked:

Can you calculate the gradient at x=0 ? YES/NO
If YES, what is the gradient, if NO, why not ?

In the TANGENT INVESTIGATION they were asked:

Does the graph have a tangent at x=0 ? YES/NO
If YES, please sketch the tangent, if NO, why not?

In each case the first question was to establish a base-line of performance, it being hoped that virtually all students would be able to answer the question correctly. The second question tested the concepts of gradient/tangent at a point with different left and right gradients, (where the experimental students would expect to have an advantage). The third tested the concept of gradient/tangent at a cusp (and here mathematicians may fail to agree over whether there is a tangent or not!) The fourth involved a function for which the students did not know the formula for the derivative, so they could not easily solve the problem in either case by differentiation. The fifth and sixth cause difficulties because there are different formulae on either side of the point under consideration. The fifth has the additional difficulty that it *does* have a tangent at the origin but, to the left, the tangent coincides with the graph and so causes conflict with those students who believe that a tangent touches the graph at one point only. The last two questions, in particular, would test a students' concept image in a broader context than they had previously encountered.

The tests were also administered to a group of first year university mathematics students, who are more highly qualified than the students in either control or experimental groups.

In the limited space at the disposal of this paper I shall report the total responses of the control and experimental students. In Tall (1986) there is a deeper consideration of matched pairs of students (matched on a pre-test not given here) which supports the same conclusions and gives more detailed information about students with and without previous calculus experience.

In each table the "correct" response will be given in **bold type** (though its "correctness" is sometimes a matter of opinion). Other responses will be subdivided wherever appropriate and the reasons for the subdivisions will be discussed. Where "statistical significance" is quoted, this will always be using a one-tailed χ^2-test, sub-dividing the responses of experimental and control groups into "correct" and "all other" responses, with the hypothesis that there will be more correct responses from the experimental group. The experimental students usually perform at least as well as the university group. Unless explicit mention is made, it may be assumed that the differences between the experimental students and the university students is not statistically significant.

Graph (1): $y=x^2-x$

	gradient			tangent		
	yes	no	nr	**yes**	**no**	nr
Experimental (N=41)	**40**	1	0	**41**	**0**	0
Control (N=65)	**57**	6	0	**64**	**1**	0
University (N=47)	**47**	0	0	**47**	**0**	0

Although marginally more control students gave incorrect responses to the gradient question, this is not statistically significant.

Graph (2): $y=\mathrm{abs}(x)$

	gradient						tangent						
	yes				**no**	nr	yes					**no**	nr
	0	1	±1	other			many	two	left	right	balance		
Experimental (N=41)	2	0	0	1	**38**	0	2	5	0	0	0	**32**	0
Control (N=65)	23	14	4	3	**21**	0	8	9	0	2	17	**29**	0
University (N=47)	7	1	2	1	**36**	0	2	0	0	0	8	**37**	0

More experimental students give NO responses than control ($\chi^2=34.73$, $p<0.000001$), and more say there is no tangent ($\chi^2=9.70$, $p<0.01$). The experimental students NO responses are significantly higher than those at university ($\chi^2=3.12$, $p<0.05$) whilst the numbers responding with no tangent are not significantly different ($\chi^2=0.03$).

The control students use their concept images to put forward a number of reasonable hypotheses, such as noting that the gradient has the two values ±1, or averaging the two values to get zero, or calculating the derivative of abs(x) to get abs(1), or simply ignoring the *abs* symbol altogether to obtain the derivative 1. Several control students showed insight into the problem, asserting that there was no gradient with comments such as:

> “no, because the line is going in two directions at 90 degrees”.

Note that five experimental students assert there are two tangents, almost certainly the legacy of discussion about “left” and “right” tangents.

Graph (3): ***y*=√(abs(*x*))**

	gradient						tangent						
	yes			no		nr	yes					**no**	nr
	no	0	other	∞	**no**		many	two	*vert*	balance	other		
Experimental (N=41)	*8*	1	0	*5*	**27**	0	1	0	*3*	1	0	**36**	0
Control (N=65)	*10*	11	16	*6*	**20**	2	3	1	*23*	10	3	**24**	1
University (N=47)	*14*	5	4	*6*	**18**	0	2	1	*23*	2	0	**19**	2

This question is a difficult one to answer, for it even provokes debate amongst mathematicians. As it does not magnify to look straight at the origin (with two superimposed half-lines), a theoretical case can be made for no tangent and no gradient (noted in **bold** type in the above table). Some would argue that there is a vertical (undirected?) tangent, with infinite gradient (noted in *italics*). A minority of control students draw a "balance" tangent along the *x*-axis that touches the cusp symmetrically.

Significantly more experimental students respond NO to the gradient than control (χ^2=11.16, p<0.01), and more experimental students than those at university (χ^2=5.16, p<0.05). Grouping those who respond NO or give the gradient as infinity (YES or NO), shows significantly more experimental students than control (χ^2=20.46, p<0.001), and more experimental than university (χ^2=4.52, p<0.05).

Significantly more experimental students than control say that there is no tangent (χ^2=17.71, p<0.0001), and more experimental than university (χ^2=7.43, p<0.01). Likewise, grouping "no tangent" with "vertical tangent", there are significantly more responses in the combined category from experimental students than from control (χ^2=10.79, p<0.01).

Graph (4): ***y*=abs(*x*^{3})**

	gradient					tangent			
	yes			no	nr	yes		**no**	nr
	0	0(?)	other			horizontal	other		
Experimental (N=41)	**35**	5	0	10	0	**39**	2	0	0
Control (N=65)	**36**	22	5	20	0	**46**	0	1	0
University (N=47)	**35**	9	1	20	0	**46**	0	1	0

There is an ambiguity in interpreting the gradient calculation in this case. Although most students obtain the result 0, a significant number, especially amongst the control students, carry out the calculation through an erroneous differentiation, explicitly noting the derivative of abs(x^3) to be either abs($3x^2$) or $3x^2$ (a correct formula being $3x$(abs(x))...) These are denoted in the table in the column headed 0(?). There may be other pupils, giving the response 0, carrying out the calculation in a similar way, without noting down the incorrect formula, although there is a significantly larger number of experimental students in this category (χ^2=8.90, p<0.01). There is no significant difference between experimental students and university students and no significant difference in the drawing of the tangent at the origin between any of the groups.

Graph (5): $y=x$ ($x \le 0$), $y=x+x^2$ ($x \ge 0$)

	gradient				tangent				
	yes		no	nr	yes			**no**	nr
	1	other			**standard**	generic	other		
Experimental (N=41)	**39**	1	0	0	**31**	8	0	2	0
Control (N=65)	**32**	8	24	1	**22**	30	2	15	1
University (N=47)	**45**	0	2	0	**29**	14	0	4	0

This is the most interesting example of all. The experimental students are very successful at calculating the gradient of the curve at the origin, even though all functions considered in the course were given as single formulae. The control students, however, find difficulties because they calculate the gradient by differentiation and are confused by the different formulae on either side of the origin. Comments include:

> "The line changes its characteristics - it is two graphs."
>
> "Because at *x*=0 is where two functions meet."

Significantly more experimental students than control give the gradient as 1. (χ^2=21.91, p<0.0001.)

The tangent produces another difficulty because it coincides with the graph itself to the left of the origin. Coerced by their belief that a tangent touches the graph at one point only, many students draw the tangent a little off the curve, so that it seems to touch only once. This is termed a *generic tangent* in the table, a generic concept being defined as one abstracted as being common to a whole class of previous experiences. Even a minority of the experimental students draw the generic tangent including some saying the gradient is 1. However, the number drawing a correct tangent is significantly higher amongst experimental than control (χ^2=15.91, p<0.0001.)

Graph (6): $y=x$ ($x\leq 0$), $y=x^2$ ($x\geq 0$)

	Gradient			Tangent							
	yes	**no**	nr	Yes: many	Yes: two	Yes: left	Yes: right	Yes: balance	Yes: other	**no**	nr
Experimental (N=41)	36	**5**	0	0	3	0	0	1	0	**37**	0
Control (N=65)	39	**24**	1	3	4	2	4	7	4	**30**	3
University (N=47)	43	**3**	1	1	4	0	1	2	0	**38**	1

A significantly higher number of experimental students respond correctly to the gradient question (χ^2=8.09, p<0.01). The tangent question has a wide variety of responses, with some seeing "many" or an "infinite number" of tangents touching the corner on the graph, others seeing two, or one (either left, right, or a line balancing at a rakish angle on the corner). Once again, significantly more experimental students explain that there is "no tangent" at the origin. (χ^2=10.79, p<0.01.)

Conclusions

The research emphasises the difficulties embodied in the tangent concept, but suggests that the experiences of the experimental group helped them to develop a more coherent concept image, with an enhanced ability to transfer this knowledge to a new context. For example, they were better able to interpret the tangent/gradient at a point where the formulae changed but left and right gradients were the same. However, potential conflicts remained, with a significant number of students retaining the notion of a "generic tangent" which "touches the graph at a single point", giving difficulties when the tangent coincides with part of the graph.

At the general level the research lends support to the theory that the computer may be used to focus on essential properties of a new concept by providing software that enables the user to manipulate examples and non-examples of the concept in a moderately complex context. This allows a curriculum development to be more appropriate cognitively by giving students general ideas of concepts at an early stage, to encourage discussion and the active construction of a shared meaning.

References

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