Using the computer to represent calculus concepts
(Utiliser l’ordinateur pour se représenter des concepts du calcul differentiel et intégral)

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Résumé

En Angleterre on étudie le calcul à trois niveaux:

> 1. Quelques élèves de 15/16 ans étudient la différentiation et l’intégration des polynomes d’une façon intuitive.
>
> 1. Tous les étudiants de mathématiques et sciences de 16/18 ans étudient la différentiation, l’intégration et les équations différentielles d’une façon simple, avec les définitions dynamiques, sans la ε-δ théorie d’analyse formelle.
>
> 3 Les étudiants de mathematiques de 18/21 ans étudient la théorie formelle d’analyse classique à l’université.

Graphic Calculus (Analyse Graphique) est une méthode pour l’enseignement de calcul qui utilise les images dynamiques d’un ordinateur aux niveaux 1 et 2, et donne les idées intuitives à niveau 3.

Il y a plusieurs logiciels pour l’ordinateur de BBC dans *Graphic Calculus*. Par exemple, on peut étudier le concept de la pente d’une courbe avec le logiciel *Gradient*. Le professeur ou l’étudiant peut taper la formule d’une fonction, en utilisant la notation mathématique ordinaire, et l’ordinateur dessine la courbe. Ensuite il dessine la pente numérique de la fonction comme une courbe et on peut taper une formule pour que l’ordinateur dessine sa courbe et on peut voir si les deux courbes sont superposées.

Le logiciel *Area* dessine la courbe et ensuite il dessine l’aire approximative ou la fonction de l’aire approximative avec l’option de superposer une autre courbe pour comparaison.

Il y a les autres logiciels pour les équations différentielles qui dessinent les solutions numériques dans deux et trois dimensions.

La théorie de *Graphic Calculus* est un exemple d’une “cognitive approach” (une approche cognitive). Le logiciel ne montre que les exemples d’un concept, mais ceux-ci aident les étudiants en construisant le concept général. J’appelle le logiciel un “generic organiser” (un organisateur générique) parce qu’on éspère que les étudiants comprendront les exemples spécifiques comme les exemples génériques (typique du concept général).

La conférènce est divisée en quatre parties:

1. Une description brève de l’enseignement du calcul en Angleterre,
2. La théorie d’une approche cognitive avec les organisateurs génériques,
3. Une approche cognitive au calcul: *Graphic Calculus*,
4. Un rapport d’une étude empirique dans la salle de classe.

Plenary lecture, Le IVème École d’Été de Didactique des Mathématiques, Orléans,
Recueil des Textes et Comptes Rendus, (1986), 238–264.

Introduction

The development of a graphic approach to the calculus which I shall describe here is part of a more general learning theory in which the computer is used to enhance learning. In this presentation I shall explain the general theory which gives rise to a *cognitive approach* to learning mathematics. Such an approach is aimed at presenting the concepts to be in a manner appropriate for the current cognitive development of the pupil. The computer is programmed to enable the user to manipulate examples of mathematical processes and to see them *dynamically*. Through experience in this way, pupils may come to see *specific examples* (single entities) as *generic examples* (representatives of a class of examples), which in turn help in the abstraction of the *general concept.*

My presentation will consist of four parts:

1. A brief description of the way calculus is taught in England,
2. A theory of learning using the computer: the notion of *generic organiser*.
3. A cognitive approach to the calculus: *Graphic Calculus*.
4. Report of empirical studies in the classroom.

1. Teaching Calculus in England

In England the calculus is studied at three different stages:

1. A small number of more able pupils aged 15/16 study the techniques of differentiation and integration of polynomials in an intuitive way. (In England the classes in secondary school are numbered from the first form at age 11/12 to the fifth form at age 15/16. The first public exam called O-level (ordinary level) is taken by the most able 20%, to be replaced in 1988 by the GCSE (General Certificate of Secondary Education) to be taken by the most able 60%. Only a small percentage of O-level students study calculus, and even fewer may take it in the new GCSE.)
2. All students taking mathematics and sciences in the school sixth form (aged 16/18) study differentiation, integration, and simple ideas about differential equations. The work is explained in a dynamic way ("as x tends to a" or "$x \to a$") and only a very few may see the ε-δ definitions at a later stage.
3. At university mathematics students take courses in mathematical analysis from a more formal point of view, other students may use the calculus in a more practical way.

The approach to the calculus that I have developed is to give a cognitive foundation for the ideas at the Sixth-Form level (or earlier), that is in groups 1 and 2 above, in such a way that it is:

(a) complete in itself,

(b) a foundation for *either* formal standard *or* non-standard analysis.

Calculus is taught in the English Sixth-Form (group 2 above) in classes of ten to twenty pupils, (though some may be smaller). The general technique is for the teacher to explain the ideas at the beginning of the lesson, perhaps asking questions of the pupils as this is done, and then the pupils carry out written exercises to practice the techniques. In the latter part of the lesson it is possible for the teacher to move around the class, speaking individually to the pupils and helping them with their difficulties.

The philosophy of teaching is something like the French notion of the “didactic triangle” between the pupil, the teacher and the mathematics (figure 1):

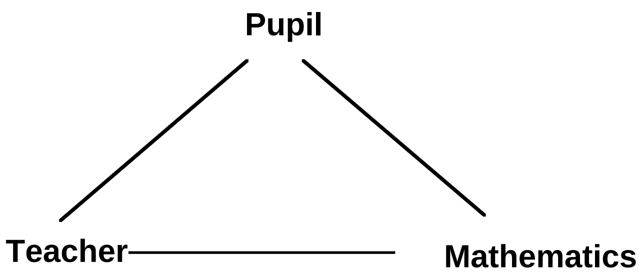

Figure 1 : the “Didactic Triangle”

The mathematics is part of a shared knowledge system, shared by those who have already learnt to understand it. The respresentative of this culture in the classroom is the teacher. The mathematics is *in the mind* of the teacher and the only externalized physical represention s are usually in a text book. Here the mathematics is *static* in fixed words and pictures. The only *dynamic* representation is through the verbal explanation of the teacher and any diagrams (s)he may draw.

Tall & Vinner 1981 demonstrated some of the problems met by students learning the calculus and similar phenomena have also been noted in France (Robert 1982; Cornu 1981, 1983). Students have particular difficulties with the limiting concept and with the interpretation of words whose everyday meaning is different from the technical mathematical meaning.

This problem is made worse by the method which mathematicians use to design more advanced curricula. The calculus is often seen as the beginning of a more formal approach to mathematics, and the topics are often presented in an order which reflects the *formal development* of the ideas. For example, the gradient of a general graph is defined in terms of the derivative, which is itself defined formally as a limit. Therefore the notion of a limit must precede the notion of the gradient of a curved graph in a formal presentation. The problem is that students may initially lack the experience to form the mathematical concept of the limit and instead form their own *concept image* in an idiosyncratic manner. As has been shown by Brousseau this may create an obstacle for later learning.

A possible solution to this problem is to take into account the student's current state of development and attempt to present him/her with experiences that enable them to develop suitable mental images of the mathematical concepts. Instead of a *formal approach*, in which the mathematical pre-requisites are presented in a non-meaningful context before the main concepts are described, one may design a *cognitive approach* in which a global gestalt of the main concept is given in the early stages. For example, instead of beginning the theory of differentiation with a discussion on limits, one may present a global gestalt of the notion of the gradient of the graph. This can be greatly assisted by the use of interactive moving graphics on a computer.

2. A cognitive approach to learning using generic organisers

Various educational psychologists have described principles to organise the learning of mathematics. For example, Ausubel *et al.* 1978 proposes a theory of "meaningful learning" in which the subject matter is presented to the students in a manner which is intended to be "potentially meaningful". To help with this learning, Ausubel postulates the use of an *advance organiser*, which is "introductory material presented in advance of, and at a higher level of generality, inclusiveness and abstraction than the learning task, explicitly related both to existing relevant ideas in the cognitive structure and to the learning task". Such organisers, *by definition*, require the learner to have experience of relevant higher level structure than the task itself. Where the learner is moving into entirely new ground, such as the understanding of the limit concept, a different kind of organising principle may be more appropriate.

In teaching a new domain of knowledge I have found great inspiration in the work of Dienes 1960. Based on Piaget's theory of the concrete operational stage, Dienes designed concrete materials to allow children to play with physical objects that represent abstract mathematical concepts. The same format works extremely well with the computer. The idea is to provide the learner with a computer environment that allows them to explore *examples* of mathematical processes and concepts[1]. By manipulating a number of examples, their common characteristics may be abstracted to give the general concept that is embodied in the examples. One thus arrives at a method of learning a

[1] Mathematical processes are often later encapsulated as concepts: we shall see later that the gradient of a graph may be seen globally as a dynamic process during drawing but, once drawn, the gradient is a static graph, representing a function, and that function may itself be seen as a concept, or a process, of a different kind!

new knowledge domain that works in a complementary fashion to the advance organisers of Ausubel. Instead of looking down on the concepts to be learnt from a higher level one looks at them from below, building up the concept from familiar examples.

An environment that provides the user the facilities of manipulating examples of a concept I term a *generic organiser*. The term "generic" means that the learner's attention is directed at certain aspects of the examples which embody the more abstract concept. Thus the equality 3+2=2+3 may be seen as a *specific* example of arithmetic in which two additions give the same result, or as a *generic* example of the commutative property of addition. The generic example is seen as a representative of a whole class of examples which embody the general property.

The existence of a generic organiser is no guarantee that the student will use it sensibly to abstract the general concept. To help the learner use the system to the best advantage, and to help in the formation of appropriate concept imagery, an external *organising agent* is required, in the shape of guidance from a teacher, a textbook, or appropriate computer teaching material.

The generic organisers in *Graphic Calculus* are all developed with a certain underlying philosophy. First of all the same program is used for teacher demonstration and student exploration. This demands that the programs be flexible in use, so that they are equally suitable for the beginner and for the experienced user. All options available at any given time are specified onscreen (except possibly for little used technical options that might otherwise confuse the beginner), and any routine wrongly entered may be aborted to return to a main list of options. All routines should have variable speeds, with a default speed for the beginner, faster speeds for the expert user, and the ability to slow down, or stop, for the use of a teacher during demonstration.

The Didactic Tetrahedron

The introduction of the computer brings a new dimension into the learning situation. There are now four major components, which may be viewed as forming a tetrahedron in a suitable educational context (figure 2).

It is assumed that the computer has appropriate software available to represent the mathematics, and that this software is designed in a manner that makes the mathematics as *explicit* as possible. It must show the *processes* of the mathematics as well as giving the final results of any calculation.

If the computer software is in the form of a generic organiser, then it may be used in a flexible way. I have seen my own software given to pupils to solve problems without any explanation as to how it should be used. As a challenge, in the right context, this approach can be most effective. My own preference is for an initial element of *teaching* and *discussion* with the teacher using the organiser to demonstrate examples, slowing down the action to explain what is

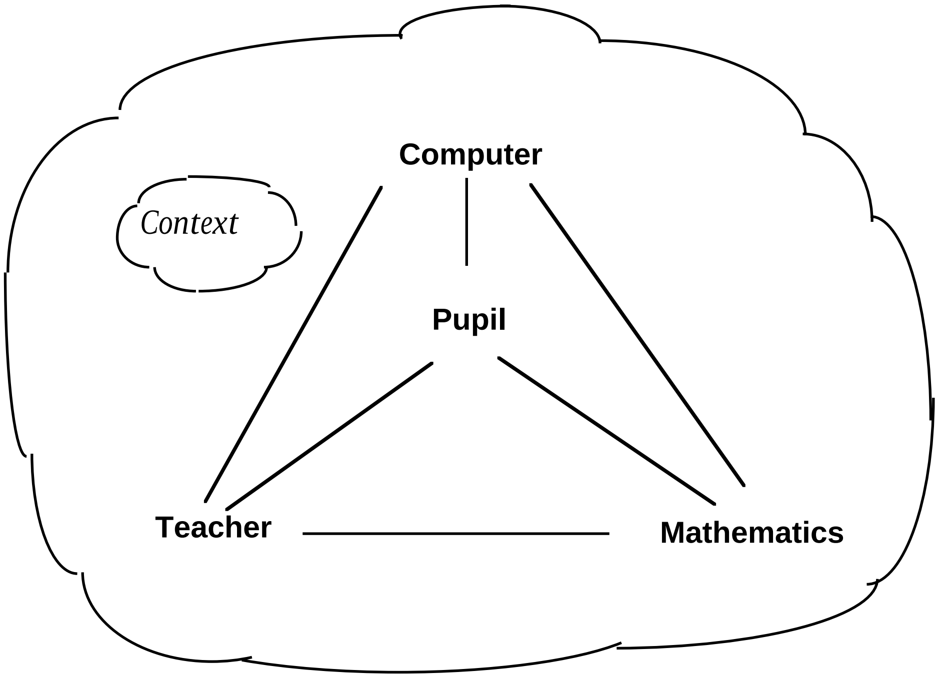

Figure 2 : the "Didactic Tetrahedron"

happening, and pausing on occasion in the middle of a routine when an interesting point is reached that is worthy of further discussion. The intention of discussion at this stage is a *negotiation of meaning*. The idea is to help the students form their own concept images in a way that is likely to agree with the interpretation of the mathematical community. This may be done through a Socratic dialogue between teacher and pupils which is enhanced by the addition of the computer. The mathematics is no longer just in the head of the teacher, or statically recorded in a book. It has an external representation on the computer as a dynamic process, under the control of the user, who may be the teacher, the pupil, or a combination of people working together. Concepts may be built by seeing examples in action, and tested by predicting what will happen on artfully chosen examples before letting the computer carry them out.

The *enhanced Socratic mode* of teaching that I use with my own generic organisers begins with teacher demonstration of the concepts on the computer and dialogue between teacher and pupils in a context that encourages enquiry and cooperation. At this stage only one computer is required for the whole class. At the earliest opportunity, individual pupils will be typing in their own suggestions which arise as a result of the dialogue and, as they gain in confidence, the teacher plays a less central role. There may come a phase of operation in which the pupils are using the generic organiser for their own investigations. This may require more computers to allow all the students access, though I have found that, in a smaller class (say up to ten or twelve pupils) it is possible to organise a system in which the pupils take it in turn to work together in small groups. At this stage the operative part of the didactic tetrahedron is the relationship between the pupil and the mathematical ideas, as represented on the computer by the generic examples. The teacher takes no directive role, being available only to answer questions which may arise in the course of the student investigations. At a later, review stage, further discussion with the pupils is sensible, to probe their ideas and make certain that their concept image is appropriate for the wider mathematical community. In this way the students come to terms with the ideas through experience and build their concepts in a way which is likely to be potentially meaningful.

3. A cognitive approach to the calculus

The first stage of the cognitive approach in *Graphic Calculus* (Tall, 1986b) is to establish the idea of the gradient of a curved graph. In a formal development this is approached through the gradient of the tangent, but in the cognitive approach it is considered to be *the gradient of the graph itself.* The program *Magnify* allows the user to first draw a graph and then select any part of the graph and magnify it. Figure 3 shows the graph of $f(x)=x^2$ drawn from $x=-2$ to 2, and a small part of the curve centred on $x=1$ magnified. The magnified portion looks (almost) straight and has gradient 2.

If students are left to magnify graphs of their choice, they will soon discover that almost all functions they are able to type into the computer have the property that small parts of their graphs magnify to look nearly straight. Without intervention from the teacher they may believe this to be a general property of graphs and form an inadequate concept image. They may never have drawn graphs that do not have this property other than, perhaps, $f(x)= |x|$ (which must be typed into the computer as abs(x)). However, using the idea of the absolute value leads to more interesting graphs, such as $f(x)$=abs(sin$x$), which magnifies at the origin to show two different gradients to the left and right. (Figure 4).

Further examples that may be drawn using the program are graphs such as $f(x)$=sin$x$+sin(100x)/100 (which looks like sin$x$ to normal scale, but has tiny oscillations showing up on magnification), or $f(x)$=sin$x$+abs(sin(100x))/100, which has many corners that cannot be seen on a picture drawn from, say, $x=-5$ to 5. Further examples, which could not be contemplated without the computer, may be drawn and discussed, including $f(x)=x\sin(1/x)$, $f(x)=x^2\sin(1/x)$, f(x)=(x+abs(x))sin(1/x), $f(x)=xe^{1/x}$, and the comparison of $f(x)$=abs(x), $f(x)$=sqr(x^2), $f(x)$=sqr(x^2+0.0001). All of these (except the last!) are smooth except for isolated places where they go wrong.

Again, the concept image of the students would be too narrow if it were left here. *Graphic Calculus* includes a single program that draws a recursively defined fractal graph, which I call the “blancmange function” (after an English jelly which it resembles in a picture)[1]. The blancmange function $y=bl(x)$ is so

[1] Paradoxically, the English refer to this jelly using the French term “blancmange”, but the French refer to it using the English name “le pudding”.

wrinkled, that nowhere will it magnify to look like a straight line (see Tall, 1985, or 1986b). It is possible to prove, using an argument depending on magnification, that the function is nowhere differentiable (Tall, 1982). Thus the *cognitive* idea that a differentiable graph will magnify to look straight can be turned into a *mathematical* proof of the existence of a nowhere differentiable function.

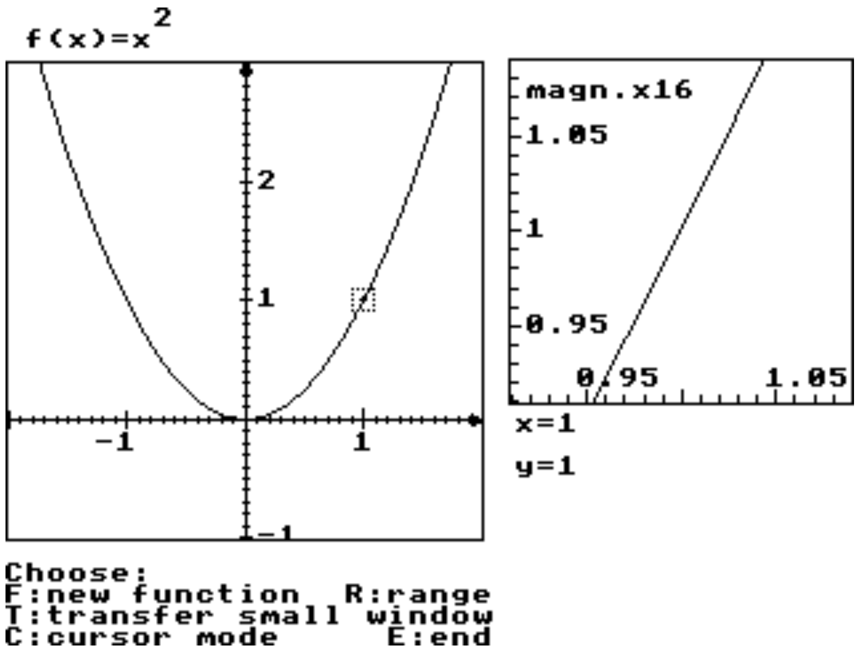

Figure 3 : a “locally straight” graph

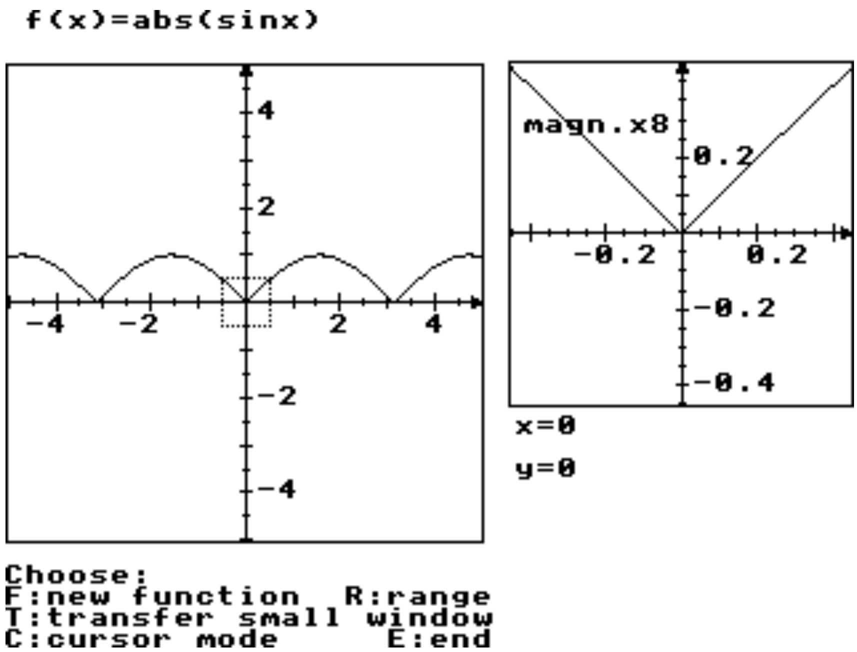

Figure 4 : a graph with a “corner”

If a new function is formed by dividing the blancmange function by a very large number, say $c(x)=bl(x)/10^{10}$, then $c(x)$ is so small that its graph is indistinguishable from the x-axis when drawn to a normal scale. The pictures of $f(x)=x^2$ and $g(x)=x^2+c(x)$ differ by only a tiny amount and they look the same on a computer screen. In fact the BBC computer has only nine digit accuracy and so the difference between $f(x)$ and $g(x)$ is so small, the computer cannot detect it. Yet one graph is everywhere differentiable and the other is nowhere differentiable....

It is a matter of taste as to whether one wishes to introduce these ideas to students early on in the calculus. My own preference is to do so as a valuable source of discussion and an opportunity to develop a broader concept image. However, they are available to introduce whenever the teacher considers appropriate.

Once one is aware that many standard graphs magnify up to look locally straight, it becomes an easy matter to cast one's eye along the graph and mentally magnify it to see its gradient changing. The program *Gradient* includes a routine that moves in steps along a graph, drawing the extended chord through the points over x, $x+c$ on the graph (for fixed c) as x increases, simultaneously plotting the gradient of the chord as it proceeds. Figure 5 shows the gradient of the graph of sin$x$ being built up. It clearly approximates to the graph of cos$x$.

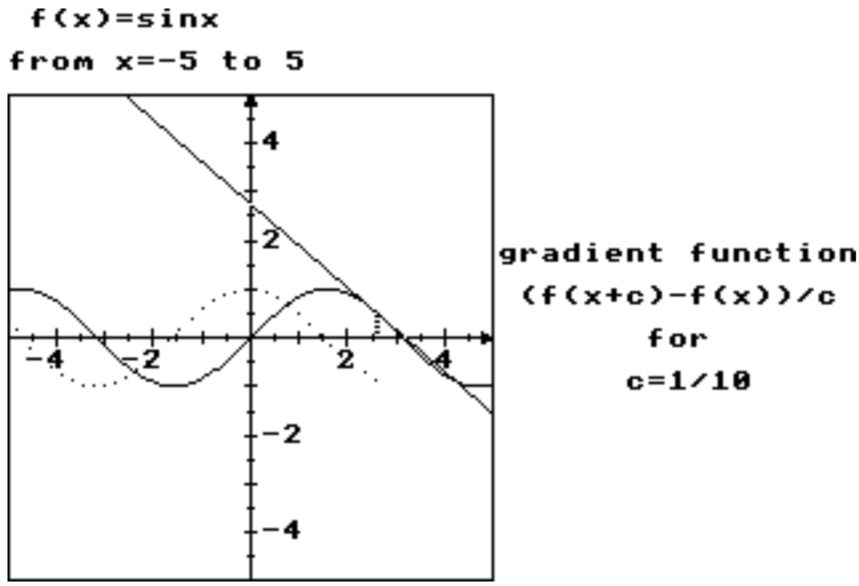

Figure 5 : the gradient function of sin$x$

The program allows the student to type in a formula which they think describes the gradient graph and to compare it with the numerically calculated gradient. In this way they get a good feeling of *why* the gradient of the sine graph should be the cosine graph, even though they may not yet have the knowledge or ability to perform the symbolic manipulation to see that the limit of

$$\frac{\sin(x+h)-\sin x}{h}$$

as h tends to zero is $\cos x$. It is also possible to go direct to the global image of the gradient of the graph without going through the formal limit process at a point. Thus the global gestalt of the gradient function can arise early in the development of the subject.

In practice in the classroom I build up the concept of the gradient of the graph by starting with $y=x^2$ and handling successively in three ways:

(1) the *numerical limit* of $\dfrac{f(x+h)-f(x)}{h}$ as h gets small at a numerical value of x (calculated by the computer as in figure 6),

(2) The *symbolic limit*:

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h} = \frac{2xh+h^2}{h} = 2x+h, \text{ (for } h\neq 0),$$

and so the gradient gets close to $2x$ as h gets small.

(3) The *global gradient* drawn by the computer, compared with the graph of $y=2x$.

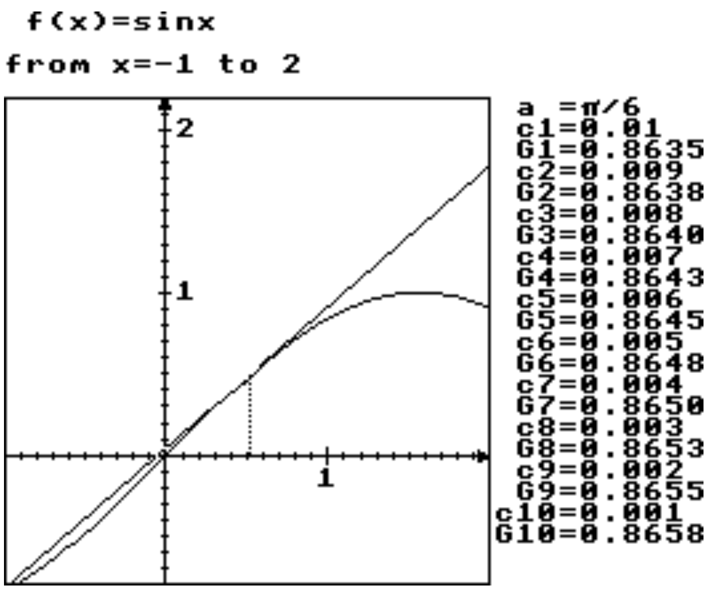

Figure 6 : Calculating a gradient at a point

This is done by teacher explanation and discussion. After this it is a relatively simple matter to allow the pupils to play with the computer and to guess the gradient of x^3 using method (3). Working in small groups, they guess the formula $3x^2$ with little difficulty (although it may take a few preliminary guesses, such as x^4 or $2x^2$ before getting a satisfactory picture). From here they invariably guess the general formula that the derivative of x^n is nx^{n-1} and are able to go on to test it in other cases, such as n=33, –1, –2, 1/2, or π, checking to see if the correct picture is drawn and where there might be difficulties (such as x negative when n is not an integer).

Other functions attacked easily by graphical methods include sin$x$, cos$x$, ln$x$, exp$x$. Less obvious ones include ln(abs(x)), whose gradient everywhere (except zero) is $1/x$, and tan$x$, which I did not think that students would be able to guess, but those with experience of drawing the graphs of trigonometric functions are often successful. (Figure 7.)

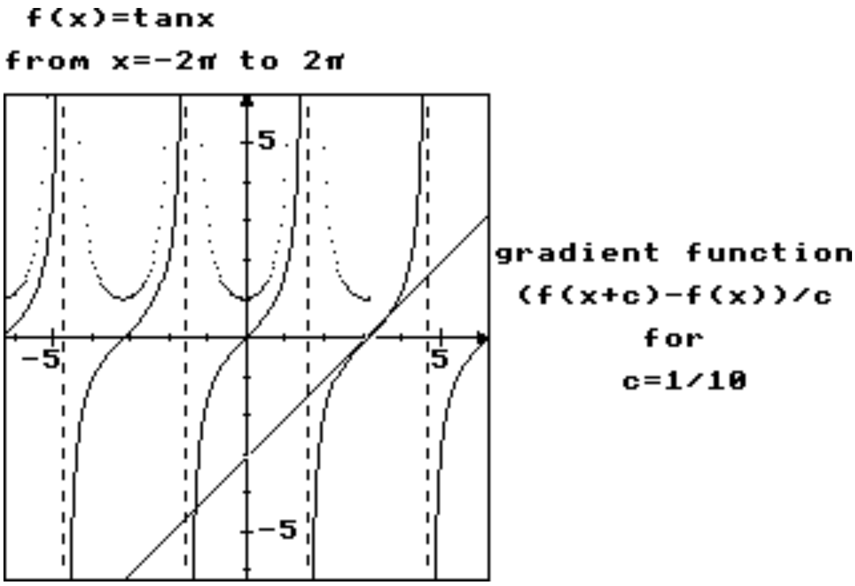

Figure 7 : the gradient of tan$x$

Students say that the gradient graph (the dotted one in the picture) is the kind of shape one might get from squaring the tan$x$ graph (to make it everywhere positive), but then this would be zero where x=0, where the dotted graph seems to have y=1. Thus the gradient graph of tan$x$ might be $\tan^2 x+1$...

Another investigation is prompted by noting that the gradient of 2^x is the same shape, but lower, whilst the gradient of 3^x is the same shape, but higher. Somewhere between 2 and 3 there might be a number e such that the gradient of e^x is again e^x. This is a way in which students may see that the number e arises in a very natural way.

The *Gradient* program is available in the classroom throughout the first introduction of the theory of differentiation, but it is not used all the time. There is much time devoted to the usual development of the formulae for differentiation, with the computer available to draw a picture whenever it is appropriate. For example, during a session calculating derivatives by formal methods, the students may use the computer to check that their formulae give the right graphs. They may be quite surprised to see that a small error in calculation, such as a minus sign where there should be a plus sign, can give a totally wrong graph. This underlines the necessity to take care over algebraic manipulation.

Integration

Just as differentiation may benefit by introducing the global gestalt of the gradient function using the generic organiser *Gradient*, so integration may benefit from introducing the global gestalt of the area function at an early stage. The calculation of the approximate area under a graph could be done by a programmable calculater (see Neill & Shuard, 1982), but the numbers that arise may not be very enlightening unless the students are given considerable help. For example, the approximations to the area under $f(x)=x^2$ from 0 to 1 using 10 strips are:

lower sum: 0.285
upper sum: 0.385,

which are hardly helpful in suggesting that the true area is 1/3. Even with 1000 strips the results:

lower sum: 0.3328335
upper sum: 0.3338335

are only a little more suggestive. Psychologically, one can be far better convinced by approximate pictures than by accurate numerical calculations.

Using the computer with a group of students, my approach is first to discuss these area calculations, supported by pictures. The students are asked to suggest what they think the area is under the curve $f(x)=x^2$ from x=0 to x=1. They write down their guess. Various values are given, from about a quarter to a third, with other guesses such as 0.3, 0.325, and so on. Then we use the program AREA to calculate an approximate value, with the choice of first ordinate, mid-ordinate or last ordinate. In this case the first and last ordinates give upper and lower sums. But the students are not interested in these. They prefer the mid-ordinate because they guess, rightly, that it will give a more accurate answer. (Figure 8.)

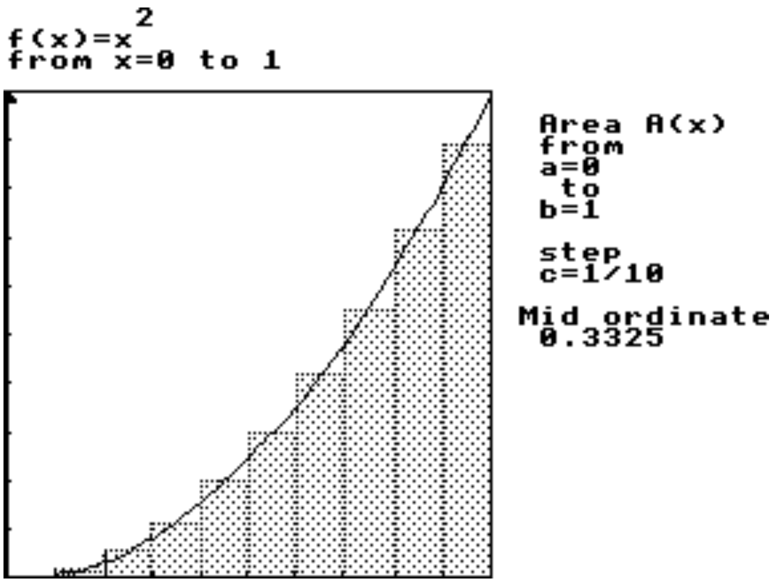

Figure 8 : the area under $y=x^2$

They are not impressed with the idea of using upper and lower sums to give an over-estimate and an under-estimate of the result because the calculations are so inaccurate (as we saw earlier).

The students are also right to distrust this method of introducing the concept because it wastes a great deal of information. It fails to use all the intermediate calculations. If these intermediate calculations are plotted for the area from x=0 to x=5, the resulting graph gives a far more useful global gestalt: the area function. Figure 9 shows a very cleverly selected picture. The graph is that of the function $f(x)=x^2$, and the individual points represent the approximate area function from 0 to x. (As a static picture in a paper it may not be very clear, but plotted in real time the growth of the area function is easier to understand.)

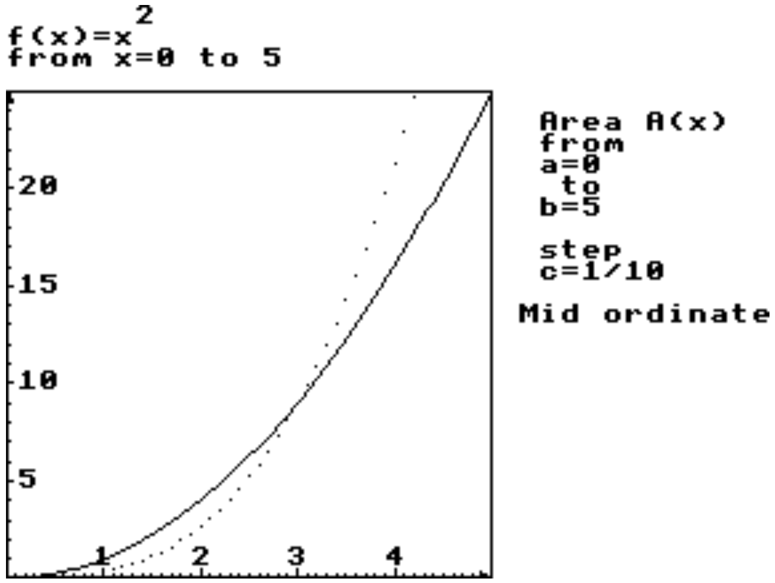

Figure 9 : the area function under $y=x^2$

Students with a little experience of graph-sketching can see that the dotted graph (the approximate area function) looks like a higher power of x than x^2. A reasonable guess might be x^3. But it is not x^3 because it crosses the original graph where x=3, and here $y=x^2$=9. As 3^3 is 27, a better guess for the area function might be $x^3/3$. Thus, *in one intuitive leap,* many students can *see* the idea behind the global area function. From here I go on to discuss the *sign* of the area, when the value of y is positive or negative, and when the step is positive or negative. *The students do not find this idea difficult.* The graph is again drawn *dynamically* and, in a picture such as figure 10, they will see the area build up from right to left, with a negative step, and they will see the signs of ordinate and step being combined to give the sign of the area calculation.

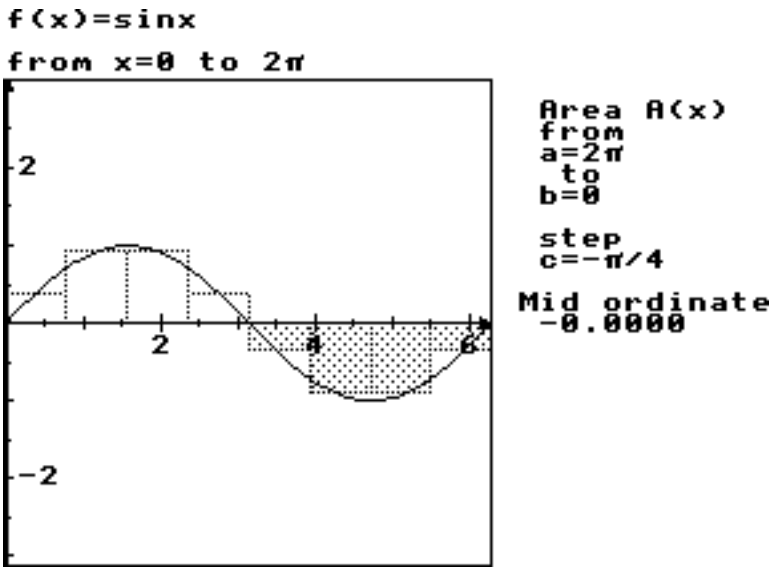

Figure 10 : the sign of the area using a *negative* step is positive below and negative above

From here, I redraw the area function for $f(x)=x^2$, calculated from 0 to x, first for x positive using a positive step 0.1, and then I challenge the students to sketch the graph of the area function for a negative step –0.1. There is usually an interesting discussion. They see that for x negative the ordinate is *positive*, so they conjecture a negative step times a positive ordinate gives a negative calculation that will give an area graph below the axis for negative x. It will also be symmetrical to reflect the symmetry of the graph, so the resulting sketch of the graph is as in figure 11.

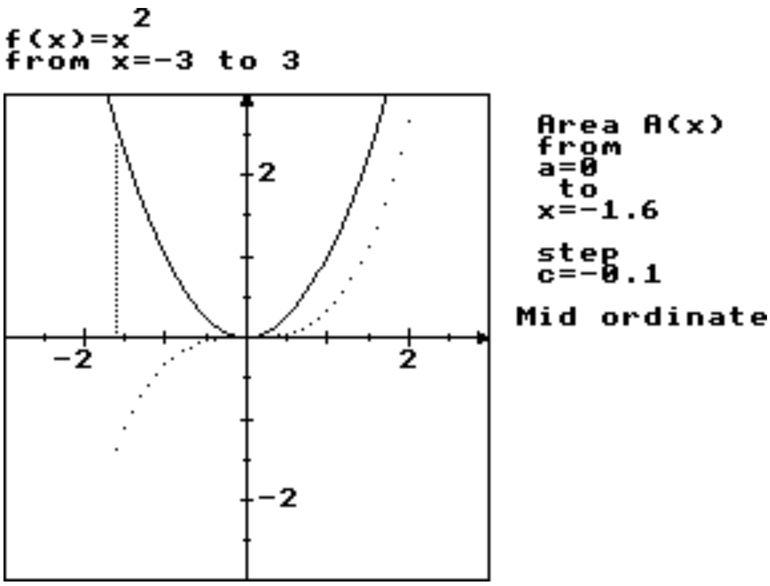

Figure 11 : the area from x=0 using a negative step has a negative value

Of course it is! The area graph is conjectured to be $f(x)=x^3/3$, and this is the shape one would expect… . The program includes a routine to superimpose the graph of $y=x^3/3$ to see how it compares with the graph of the approximate area. To this degree of accuracy the graphs are indistinguishable.

As there are only about 200 pixels horizontally and vertically on the screen, an accuracy of 1 in 200 (only 0.5%) gives a reasonable picture, an accuracy that would be unacceptable in a numerical calculation. So, using a pictorial approach, not only does one get a dynamic feeling for the growing area, and an insight into positive and negative calculations, this is also done with reasonable arithmetic and uses all the intermediate calculations to advantage. Again, one or two more examples: the area function from the origin for x, x^3 and possibly x^0, and the students are ready to conjecture that the area under $f(x)=x^n$ is $x^{n+1}/(n+1)$.

In Tall 1986a, I demonstrate how the program *Area* can suggest the idea for the fundamental theorem of calculus, by stretching the graph horizontally, leaving the vertical scale unchanged. The graph $y=f(x)$ looks horizontal and the area from x to $x+h$ is approximately $f(x)*h$. But this is also $A(x+h)–A(x)$, giving the approximation:

$$A(x+h)-A(x) \approx f(x)*h,$$

and a reason why

$$\frac{A(x+h)-A(x)}{h} \approx f(x),$$

leading to the fundamental theorem:

$$A'(x)=f(x).$$

One may wonder why an *equality* can be deduced by taking the limit of an *inequality*. The reason is seen pictorially because the more one pulls the graph horizontally, the flatter it gets and the better the approximation becomes. In fact, here one can introduce the standard ε–δ definition of continuity. Pictorially, a graph is continuous at $x=a$ if, given a pixel width $\pm\varepsilon$ (for a fixed y-scale), there exists a small interval $a\pm\delta$ such that when the graph is drawn on the screen with the x-range stretched from actual coordinates $a-\delta$ to $a+\delta$, then the graph lies in a pixel height $f(a)\pm\varepsilon$. Thus continuity arises from the need to make the fundamental theorem precise. It need not be introduced without reason (to the student) at an earlier stage. This is yet another example of *didactic inversion* in the sense described by Freudenthal. A cognitive development need not follow the same sequence as a logical development.

Differential Equations

The student usually meets differential equations in the simple form of knowing the gradient $dy/dx=f(x)$, and seeking the graph $y=I(x)$ with this gradient. This information may be represented graphically by drawing an array of short line segments through points (x,y) with gradient $f(x)$, as in figure 12. A solution of the differential equation may then be visualized as a curve which follows the direction lines. A formal solution satisfies $I'(x)=f(x)$.

The gradient direction is a function of x alone, so the solution curves will clearly differ by a constant. Or do they? Closer inspection will show that this is only true for a connected component of the domain. The numerical method

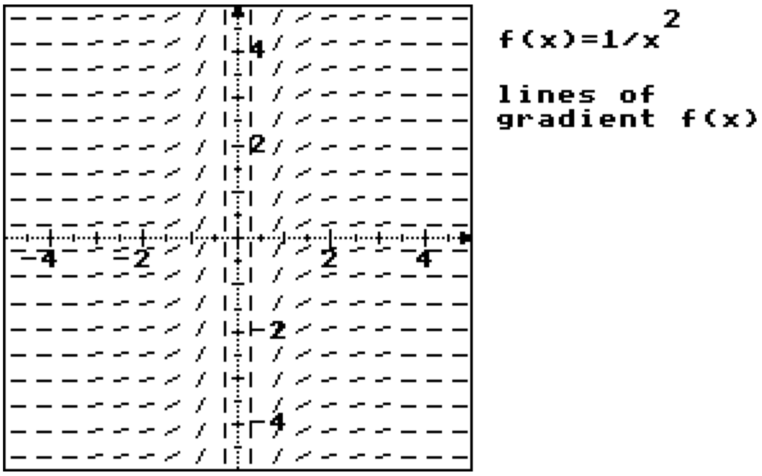

Figure 12 : line segments (through each point (x,y)) of gradient $1/x^2$

of plotting a solution curve by following the direction lines uses a constant step along the graph rather than a fixed x-step. Thus a solution curve of $dy/dx=1/x^2$ starting to the left of the origin will always stay on the left, whilst a solution to the right will always stay to the right. Thus two different solution curves may differ by a constant only over each connected component of the domain. They differ not by a constant, but by a "locally constant function"...

A generalization of this idea is when the gradient dy/dx is a function of both x and y:

$$dy/dx=f(x,y).$$

The idea of a solution is basically the same: draw a direction diagram and trace a solution by following the given directions.

For instance, the differential equation:

$$dy/dx=-x/y$$

has a direction diagram as in figure 13:

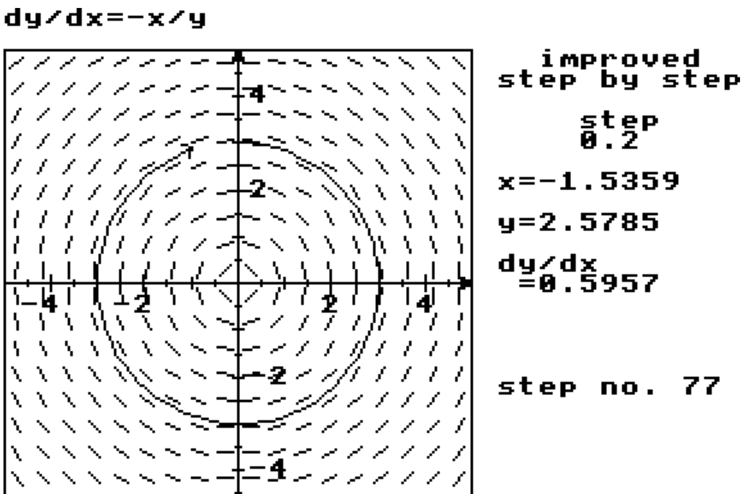

Figure 13 : a solution of $dy/dx = -x/y$

The solutions of this equation are *implicit*, in the form:

x^2+y^2=constant.

The "first order differential equation program" in *Graphic Calculus* is programmed to trace the solution curves numberically, using a choice of an x-step or y-step, in such a way that it can trace round closed loops. At points where the tangent is vertical the interpretation of dy/dx as a real function fails, but one may regard the tangent vector as the vector (dx,dy), which allows dx=0, dy non-zero to represent the vertical direction. Thus a first order differential equation may sometimes be better regarded as giving information about the direction (dx,dy) of the tangent to an implicit solution curve, rather than a formula for the derivative of an explicit function of x.

Thus one may usefully distinguish between *differential* equations (which specify the direction of the tangent) and *derivative* equations (which specify the derivative of an explicit function). The concept of a differential equation is more general than that of a derivative equation, which it includes as a special case.

Using pictures to visualize the direction of the solution curves can often cruelly expose the limitations and downright misrepresentations about differential equations in many elementary textbooks.

Graphic Calculus also includes a program to draw solutions of second order differential equations in the form:

$$d^2y/dx^2 = f(x,y,dy/dx).$$

Even a relatively simple equation such as

$$d^2y/dx^2 = -x$$

does not have a direction field in the x-y plane. There are an infinite number of solutions through each point, one for each starting direction. Figure 14 shows solutions drawn numerically starting from the origin and moving away in various directions.

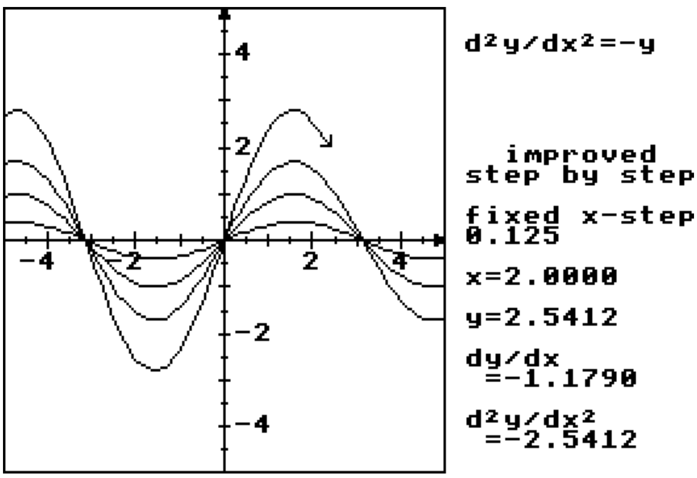

Figure 14 : several solutions of $d^2y/dx^2 = -x$ passing through the origin

Solution of such equations are often attacked by introducing a new variable $v=dy/dx$, and writing the original equation in terms of two first-order equations:

$dy/dx = v$

$dv/dx = -x$.

These equations tell us that at every point in three dimensional space (x,y,v). there is a uniquely defined direction:

$(dx,dy,dv) = (dx,vdx,-xdx)$

which is in the direction $(1,v,-x)$. Once more the solution may be visualized in terms of following the given direction field, but this time in *three* dimensions, not two.

A further program in *Graphic Calculus* draws solutions of such simultaneous differential equations in three dimensions, building up the solution numerically by following the direction at each point, with the projections of the curve in three-space being shown simultaneously in the (x,y) and (x,v) planes. The understanding of the nature of the solution is greatly aided by seeing it evolve dynamically in space (figure 15). The graphics on the BBC computer are colour-coded to aid in the visualization, producing a mental image that is very difficult to intimate in a static black and white picture.

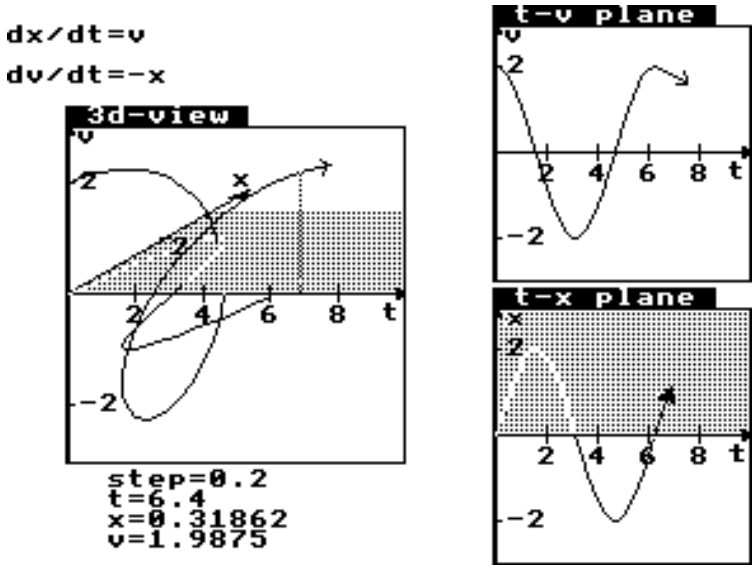

Figure 15 : a three-dimensional picture of a solution of $dx/dt=v$, $dv/dt=-x$

A more extended approach to differential equations using dynamic interactive computer graphics on the Macintosh computer have been used extensively by Hubbard and West at Cornell University, and will feature in Hubbard &West (to appear). Further details are given in Tall & West, 1986.

4. Empirical Studies in the Classroom

To test the theory of a cognitive approach using generic organisers, the differentiation programs of *Graphic Calculus* were used in three classes in two different schools and the progress of the students compared using matched pairs selected from five other classes in the same schools. Some of the students had studied the calculus before and it was possible to consider matched pairs both with, and without calculus experience.

In each school the control and experimental groups used an agreed textbook, with the experimental group given additional support from the generic organisers in *Graphic Calculus*. Data was also collected from a third school using the computer programs, where the organisers functioned less well, and from university students who had not used a computer in the learning of calculus.

In one experimental class the teacher allowed me to take those parts of the lessons where the computer was to be used, in the other two classes the teachers were given an overall plan to follow, but were left to interpret this in their own way. It was intended that the experimental groups should be taught in an enhanced Socratic mode (though this terminology was not given to the teachers at the time). The plan was for the teachers to introduce the notion of gradient of a graph using the programs *Magnify* and *Gradient*, and to lead class discussions. They were to encourage the pupils to use the programs for investigations and to check their formal differentiation by using the program to compare the numerical global gradient function with the derivatives thay had calculated. The teachers in all groups kept a detailed diary of their activities and, by and large, the three experimental classes followed the agreed programme. However, those using the computer in the third school did not follow the programme as specified, with disastrous consequences.

The pre-test included a question on calculating the “rate of change” between two points on a curved graph which was used as part of the information to select matched pairs on the pre-test (figure 16).

Find the average rate of change between the following points on the graph:
(Note: the "average rate of change" from P to Q means the gradient of PQ)

(i) from C to D .....
(ii) from D to E .....
(iii) from A to B .....
(iv) from B to C .....
(v) from C to E .....
(vi) from D to C .....

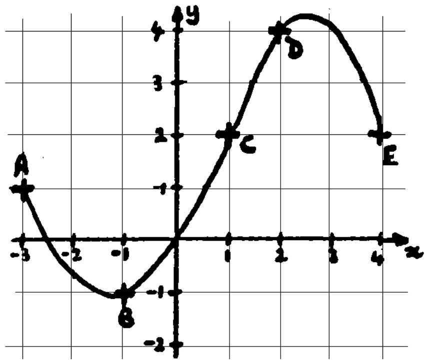

Figure 16 : identifying the gradient of a curve through various points

The interesting question here is part (vi). The line DC is sloping *up*, suggesting a positive result, but the actual *y*-direction from D to C is *down*, suggesting a negative result. This imposes a conflict situation, in which 41% changed their response from pre-test to post-test, with almost equal numbers in each direction. Only 26% give the correct response (with positive sign) on both occasions and 33% get it wrong both times, mainly due to an error in sign. Although the matched pairs with previous calculus experience show a significant improvement on this question (at the 2.5% level), this should not be taken too seriously, as both control and experimental results are subject to conflicting changes on part (vi). The diaries of the teachers show that this situation was not discussed during the lessons and it would be an interesting one to consider in future research.

The idea of the tangent as the limiting position of the extended chord proves to have difficulties for both experimental and control students. On pre-test and post-test they were asked the question in figure 17 (inspired by a question in *Cornu 1983*).

(i) Write down the gradient of the straight line through *A*,*B*...

(ii) Write down the gradient of *AT*... Explain how you might find the gradient of *AT* from first principles.

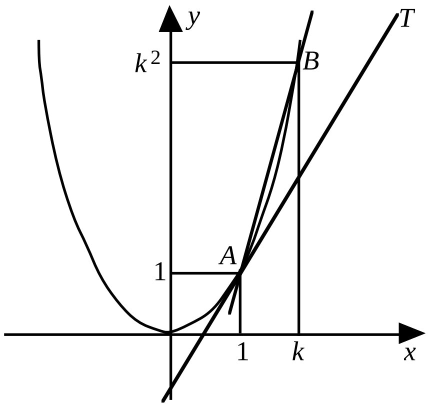

Figure 17 : the relationship between the gradient of a chord and of a tangent

It was hoped that some of the students might calculate the gradient of the chord as

$$\frac{k^2-1}{k-1} = k+1 \text{ (for } k\neq 1),$$

give the gradient of the tangent at (1,1) as 2, and then note that, as k tends to 1, so the chord gradient $k+1$ tends to 2. On the pre-test 16 students (10% of those tested) obtained the value $k+1$ for the gradient of the chord and 2 for the gradient of the tangent, but only *one* suggested a limiting argument. The evidence is not conclusive, but it does not give much support to the idea that the limiting argument is an *intuitive* notion, in the sense that it might be evoked spontaneously by the student meeting the idea for the first time.

In every classroom the derivative was discussed in terms of the limit of the gradient of the chord through two points as one tended to the other. One might expect this to make some impact on the students. There were four questions on the post-test that might be expected to produce an explanation using a limiting argument. They were the question above on “first principles” in figure 13 and three open-ended questions asking respectively for an explanation of the “gradient of a graph”, the “tangent to a graph” and the “derivative of a function”. In table 18 are the numbers of students responding with a limiting notion from the matched pairs and from the students at university. The matched pairs are broken down into those without previous calculus experience (Exp and Contr) and those with previous experience (Exp* and Contr*):

	Exp (*N*=12)	Contr (*N*=12)	Exp* (*N*=27)	Contr* (*N*=27)	University (*N*=44)
1st principles	4	1	6	6	14
gradient	1	0	1	0	8
tangent	2	0	1	0	5
derivative	0	0	0	0	3

Table 18 : Students responding with the limit concept

The low level of responses indicate the high cognitive demand of this general concept, reinforcing the opinion that, although the notion of a limit is the natural foundation of a *mathematical* development of the calculus, it is not a natural starting point for a *cognitive* development.

The limit notion, as explained in the classroom, is a *dynamic* notion, using the terminology “x tends to a” or “$x \to a$”. In addition the students sometimes responded with a *pre-dynamic* concept, where points on the graph are described as being “very close”, or even “infinitely close”, without any indication of a limiting argument. Table 19 shows the significance that the experimental students give more dynamic/pre-dynamic responses than the control students, using a Wilcoxon matched pairs test. Column 1 are the 12 matched pairs without previous calculus experience and column 2 are the 27 matched pairs with previous experience:

	Without previous calculus	With previous calculus
1st principles	n.s.*	5%
gradient	0.5%	1%
tangent	5%	1%

Table 19 : significance that experimental students are more likely to give a dynamic response to the limit concept

(Here n.s. means “not significant” and n.s.* means “not significant at the 5% level, but significant at 10%.)

The question concerning the meaning of the derivative of a function could have had a limiting response, but none of the experimental or control students responded in this way. However, the experimental students without calculus experience were more likely to describe the derivative in terms of the gradient of the graph (table 20).

	Without previous calculus	With previous calculus
gradient	1%	n.s.*
gradient function	5%	n.s.

Table 20 : significance that experimental students are more likely to describe the derivative in terms of the gradient (function)

It seems here that the experimental students who are meeting the derivative for the first time are more likely to see it as a "gradient function" than the corresponding control students, but the control students meeting the calculus for the second time are growing to understand the concept better.

However, this interpretation must be treated with caution. The calculus students with previous experience were using a text-book that described the derivative as the gradient function, and hence this response is to be expected. These control students do not see the derivative *pictorially* as a gradient function, as is shown by the following question from the post-test (figure 21):

Sketch the derivatives of the following graphs:

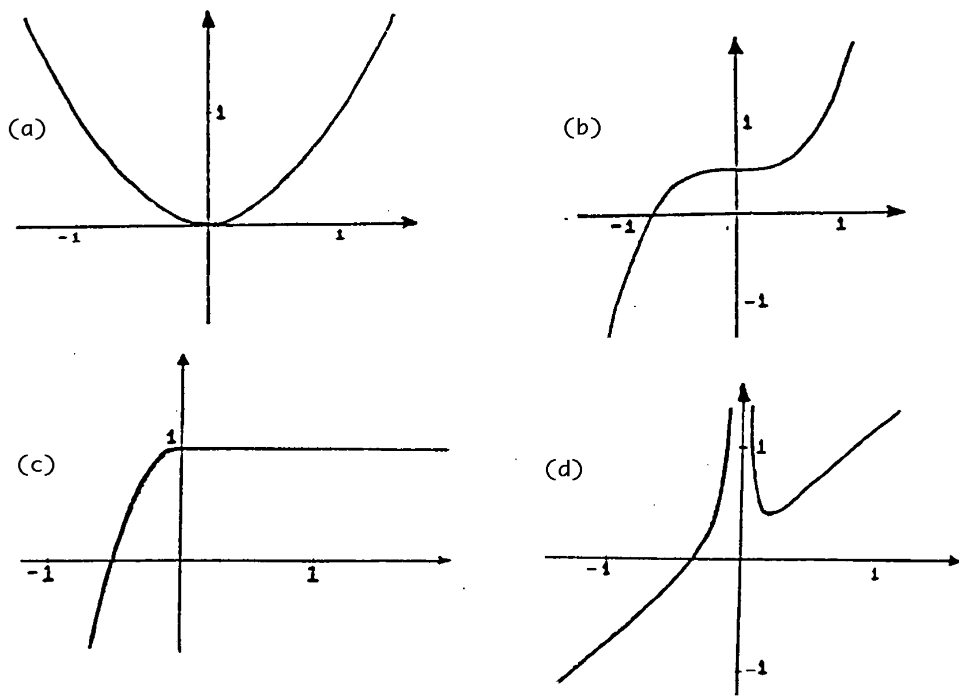

Figure 21 : Sketching the gradient graphs for given graphs

The performances of the three experimental classes (KE, BE1, BE2) are visibly better than the control classes (KC and BC1 to BC4), and on a level comparable with the university students (U). The other school (CE), which also used the generic organisers, failed dismally (table 22):

Graph	(a)	(b)	(c)	(d)	Total
Maximum	5	5	5	5	20
KE	4.93	4.50	4.36	4.03	17.86
BE2	4.62	4.25	4.44	3.94	17.25
BE1	5.00	3.92	3.50	2.00	15.83
BC4	4.55	3.18	2.82	2.18	12.73
BC2	4.11	3.78	1.56	0.94	10.32
KC	3.67	4.00	0.78	0.00	8.44
BC3	2.36	1.71	0.14	0.07	4.29
BC1	1.93	0.87	0.00	0.20	3.00
CE	0.73	0.24	0.08	0.04	1.08
U	4.89	4.55	4.45	3.84	17.73

Table 22 : Student responses to sketching derivative graphs

The reason for the poor performance of the CE group is two-fold: first the students obtained lower marks than any other group on the pre-test (level with BC1 and marginally below BE1 on the first question in figure 12). Second, the students in CE used the computer on just one occasion, using it, on average, less than once each. Threequarters of the CE students complained that the class used it “too little” or “far too little” and of twelve comments from the CE group about unhelpful aspects, nine mentioned either “lack of time”, “confusion as to the aims of the program”, or that it “kept going wrong”. The evidence strongly suggested a weakness in the use of the software.

By contrast, the three experimental groups KE (with whom I worked), BE1 and BE2 were far more positive in their attitudes and almost any statistical test would show the improvement in their scores. Dividing them into their matched pairs once more, and using a one-tail Wilcoxon test to compare their performances on each individual question, shows the experimental students performing significantly better than the control students on all questions, except for the pairs without calculus experience on question 1. Even here the experimental students performed better, but not at the 5% level of significance.

These results show that, although the control students with more experience use the term “gradient function” for the derivative, they are not all able to sketch the gradient function as a graph.

A further question (again taken from Cornu, 1983) showed a graph which was the derivative of one of three others. The students had to say which, and give a reason for their opinion (figure 23).

Graph 1 is the derivative y=f'(x) of a function y=f(x) defined for 0≤x≤8.

Which of the graphs 2, 3, 4 could be the original graph y=f(x)?

Give the reasons for your choice:

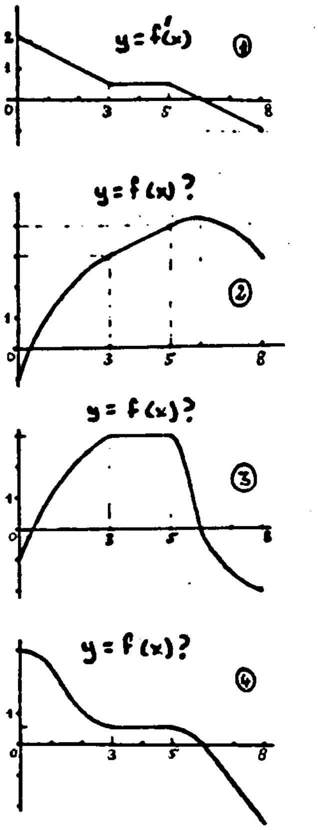

Figure 23 : identifying the graph with a given gradient

The performance of the groups followed a similar pattern to table 18. 67% of the experimental students gave the correct response (b) with a correct reason, a similar level of success as those at university (68%), whilst only 8% of the control students were able to do the same.

None of the group CE managed to give a correct response together with a sensible reason, and more than half of them failed to respond.

If one classifies a student to “perform well” by obtaining 15 out of 20 on table 18 together with a correct response (with reason) to the question in figure 19, then there are 26 out of 42 experimental students in this category (62%) but only 2 out of 72 control students. The probability of such an extreme distribution occurring by chance is less than 1 in 10^9 !

Further tests are described in Tall, 1986d. They show that there is no significant difference between the performances of control and experimental groups on formal differentiation, but that the experimental groups are significantly better at sketching gradients, recognising gradient functions and defining non-differentiable functions (though the latter is cognitively more demanding and fewer students are successful at this task). They also show that the experimental students are far more likely to see the derivative in dynamic or pre-dynamic terms.

One phenomenon that caused me some concern was that a significant number of experimental students were likely to regard the tangent to a curve in pre-dynamic terms as a line through “two very close points”. However, closer analysis showed that there is an even greater problem with the control students seeing the tangent as a line which “touches the curve at one point only” with the possible additional property that it “does not cross the curve”.

It is my conclusion that we can develop a cognitive approach to the tangent by defining a “practical tangent” to be a line through two very close points on a curve. This can prove to be very useful in a pre-calculus course. It is an *operative* definition, which can be used to draw a very good approximation to a tangent before calculus is discussed. The informal definition in terms of “touching” and “not crossing” is only useful for rough sketches and gives a concept image that causes obstacles to learning at a later stage. The practical definition is useful for calculations at an early stage and can be used to *define* the “theoretical tangent” in terms of the limiting process when the students have developed sufficient sophistication to be able to cope with the idea.

Once more, empirical research has demonstrated a process of *didactic inversion* that gives an alternative cognitive approach to mathematics. In this case the cognitive approach, in the shape of the practical tangent, proves to be surprisingly good mathematics!

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