Investigating Graphs and the Calculus in the Sixth Form

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The microcomputer presents unrivalled opportunities to help students understand mathematical concepts with its fast numerical processes, moving graphics and interactive facilities. However, the reality of the mathematical classroom is that too few micros are currently available to allow students adequate access. The present article describes the use of a single BBC computer in a lower sixth classroom for drawing graphs and providing insight into the calculus. The micro was available for most of the mathematics periods during the year and used whenever it seemed appropriate.

A large part of an A-level pure mathematics course consists of:

(i) An introduction to real functions and investigations into their behaviour, usually incorporating a pictorial approach,

(ii) Practising algebraic techniques and manipulation of trigonometric identities,

(iii) Introductory calculus with particular emphasis on derived functions and antiderivatives of combinations of polynomials, trigonometric functions, exponentials and logarithms,

(iv) Elementary ideas of proof.

We were concerned to see how a graphical approach could contribute to effective teaching in each of these areas. Much of the initial work appeals to pictures of functions and pictorial illustrations of finding the derivative. but because of the limitations of blackboard and chalk and static pictures in text-books this quickly gives way to algebraic processes. Our idea was to use the computer to give an understanding of the geometric ideas. For this purpose we developed and tested the graph-drawing program “Supergraph” and the suite of programs “Graphic Calculus”. The plan was to provide facilities for graph drawing that were so flexible that the computer could be switched on whenever we felt the urge to ask a “what if” question in terms of drawing a graph.

The use of the computer proved to have a profound effect on the relationship with the sixth-formers. They were much more willing to discuss ideas illustrated on the computer as they typed in expressions themselves than they might have been if the concepts were introduced with the authority of a teacher’s “talk and chalk”. They were able to conjecture what might happen, suggest possible formulae for derivatives. test them with the computer and investigate ideas experimentally before they were proved formally. Their insight into the geometrical ideas proved far greater than comparable groups of students who had not used the computer, without

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their ability to do the algebraic manipulations being impaired. In a word they did mathematics *actively* rather than simply learn passively at the teacher s instigation. They learned not only mathematics, they learned how to learn.

Elementary Graph-Sketching

Some syllabuses postpone graph-sketching until after differentiation to take advantage of the derivative in determining maxima and minima. As our approach to differentiation depended on an experience of graphs. we studied polynomials and trigonometric functions in four stages:

(i) sketching graphs of polynomials and rational functions,

(ii) derivatives of polynomials and powers (including negative and fractional powers),

(iii) trigonometric functions in radians and trigonometric relationships,

(iv) derivatives of trigonometric functions.

Supergraph is such a flexible computer program that it can be used by any level of pupil or teacher. One version of the program allows superimposition of any number of cartesian, parametric or polar graphs and another sacrifices the parametric and polar options for a wide variety of other facilities, including tangents, normals, lines, zoom options etc. Both versions allow normal algebraic input (with cursor movement for inserting powers) and letters other than x,y are taken as constants. For example a straight line could be typed as

$$y=mx+c$$

or a quadratic as

$$y=ax^2+bx+c$$

and the constants may be varied to superimpose variations on the graph including families and envelopes. For instance, figure 1 began as the line

$$y=mx+c \ (m=1,\ c=0)$$

with a succession of graphs fixing m and increasing c by 1 each time, we see that the graphs are all parallel. Alternatively, fixing c and varying m shows that the graph always passes through the point given by $x=0,\ y=c$. By taking m negative it is possible to see a straight line graph with negative gradient, or $m=0$ gives a horizontal graph.

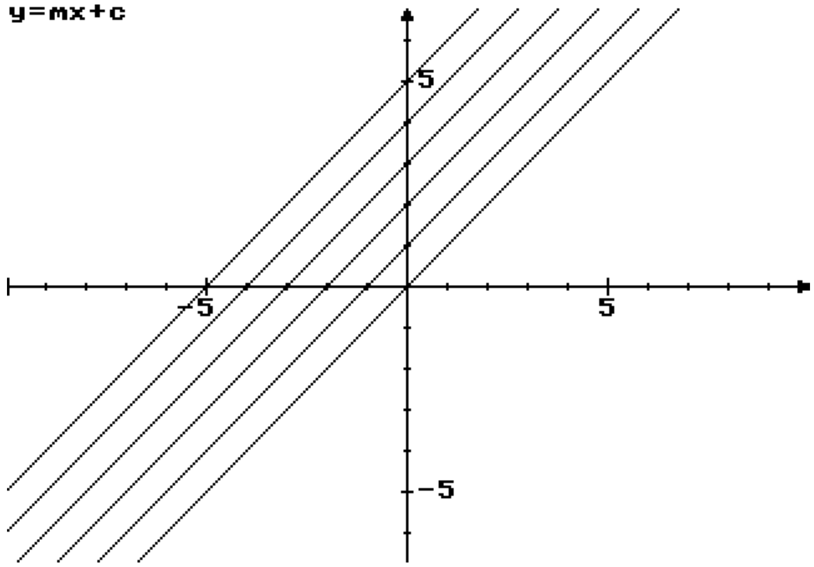

Figure 1 : a family of straight line graphs

We didn't need to have any special book of directions to use Supergraph in graph-sketching, we simply worked from our usual (SMP) text-book and illustrated graphs as they turned up.

One nice discussion we had concerned the graph of

$$y=1/(x-a)(x-b)$$

(note that Supergraph gives division precedence over implied multiplication).

We first drew the graph $y=(x-a)(x-b)$

for $a=1$, $b=2$ and then superimposed the graph of the reciprocal (figure 2). We could see that the two graphs had the same sign but where $(x-a)(x-b)$ was zero the reciprocal had asymptotes.

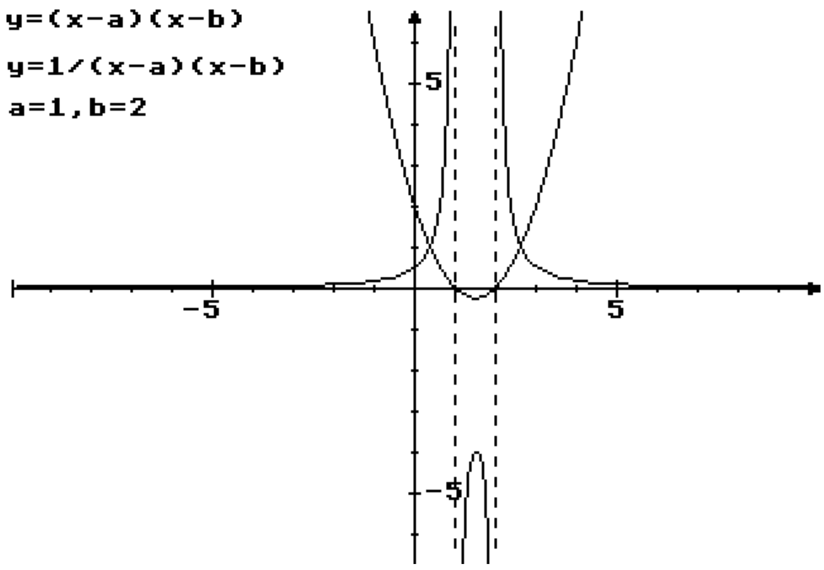

Figure 2 : sketching related graphs

By considering the sign of the product of the terms for $x<a$ and $a<x<b$ we could see that the algebraic expression $1/(x–a)(x–b)$ is very large and negative before a and very large and negative after it. We reinforced this message by substituting $x=a+h$ to get the expression

$$y=1/h(a–b+h)$$

and considering the sign for small values of h. In this way we linked the algebraic ideas with a pictorial representation.

The interesting investigation was to take $a=b$ and see what happened.

before drawing the graph the algebraic substitution $x=a+h$ gave

$$y=1/h^2$$

and it could easily be seen that the graph should be large and positive both before and after a. This was confirmed by drawing the picture.

From here it was a simple matter to conjecture what would happen to

$$y=1/(a–x)^3,\ y=1/(a–x)^4,\ \ldots$$

and test the result on the computer. Likewise we could look at a combination such as

$$y=1/(a–x)(b–x)^2.$$

In some cases the students were not sure of the scale of the picture, so they used automatic scaling to get a sighting before drawing the graph using equal scales. This helped them to think about the kind of ranges it is appropriate to draw the graphs. They soon started checking the graph sketches given in the text-books and were amazed to see how inaccurate they were. It threw the whole question of graph-sketching by old methods somewhat into disrepute!

The gradient of a graph through magnification

Not all students work at the same pace and one day Allan finished his exercises well before the others. He was given the program "Magnify" from the Graphic Calculus Pack and told to draw some graphs, magnify them, and report what happened. Two or three minutes later he said "They look less curved". He was joined by other students who tested their conclusion by magnifying other graphs. They were asked to explain their ideas to the rest of the class.

Was this property of graphs always true, or did it sometimes fail? Discussion followed. Most of the class were convinced that it was always true. But was it?

They tried all sorts of expressions to cause the idea to break down and failed. The graph y=abs(x) (the "absolute value" or "modulus" of x) was suggested by the teacher. It clearly had a "corner" at x=0. Other graphs, such as y=abs(x^2–1) also had corners.

Thus it was that the students began to appreciate that some graphs looked straight under high magnification and some didn't. Another program in the pack was drawn to show the "blancmange function", which is so wrinkled. no matter how highly it is magnified, it never looks straight. An amusing and fascinating discussion followed. Clearly there were many curves in nature that failed to "look straight" when highly magnified. Even a ruler is wrinkled viewed under a microscope.

The next lesson we looked at the gradient of a graph through this approach. If the graph had the special property of approximating to a straight line under a magnifying glass, we could talk about its gradient in a small segment as being the gradient of the magnified (almost) straight portion.

This proved easy to see (literally!) with the computer. Although we had a routine to display the limit of the chord from a to b as b tended to a, we found it much more profitable to attack the gradient of the graph as a dynamic picture. The idea is simple: draw the chord from x to x+c (where c is small, and then let it click along the curve as x increases and, at each click, plot the gradient of the chord as a point, leaving a trace of gradient points behind (figure 3).

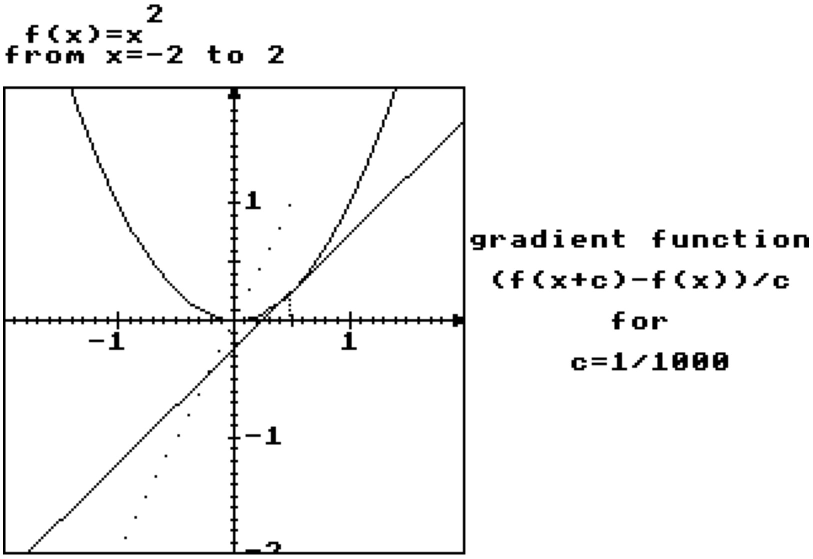

Figure 3: The gradient function of f(x) = x^2

The gradient of y=x^2 proved to be a revelation, it was virtually a straight line when c was small!

By checking the algebra, we found the gradient from (x,x^2) to (x+c, (x+c)^2) to be

$$\frac{(x+c)^2-x^2}{c}$$

$$=2x+c \text{ (for } c\neq 0)$$

and it was evident that for small c the gradient would approximate to y=2x.

The software included a routine to type in y=2x and make a comparison with the gradient. Give or take a pixel, the graphs proved to be identical.

Interestingly enough, the odd pixel or so difference provoked an interesting discussion in which it was realized that the graph-drawing routines only calculate a few points and join them up by lines. As the picture is actually made up of dots of light (less than 200 by 200 in the graph-square) it is self-evident that there are likely to be inaccuracies in drawing anyway.

Putting on a show

That evening was the Open Evening, giving parents the opportunity to see the kind of work going on in the sixth form centre. As usual the Science Departments had all their experiments on display and mathematics was keen to compete. The obvious weapon was the computer. On the spur of the moment we organised small groups of students from the calculus set to come and investigate the gradients of graphs in front of the visitors. Each group was given a sheet of challenges. They knew the gradient of x^2 was $2x$, could they guess the gradient of x^3, and when they had done that, could they guess the formula for x^n? This was easy. The first group drew the gradient function (figure 4) and saw that it was a U-shape but clearly not x^2, so they guessed from their earlier experience that it might be x^4. By superimposing this graph and comparing with the gradient they saw the error of their ways and immediately moved on to try $2x^2$, then refined their guess to $3x^2$. It worked!

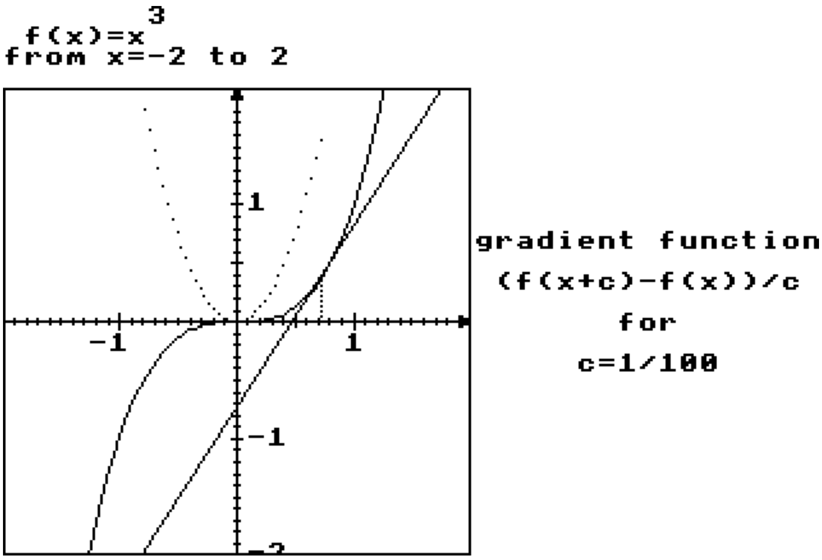

Figure 4: The gradient of f(x) = x^3

A quick conference and they guessed that the gradient of x^n must be nx^{n-1}. Any mathematician might think they would try out this expression systematically with n=3. Not a bit of it: their first check was n=33. The computer creaked a bit and dutifully drew the graphs and confirmed their suspicions.

The next task was to try the formula for n=–1, –2 and others such as n=–1/2? and it went according to plan.

Following this the challenges leapt ahead with a brief description of angles in radians and a challenge to find the gradient function for sinx, drawing the graph from –π to 2π. (The programs allow π to be typed in as pi which translates to Greek before the operator's very eyes.) They cracked the problem immediately on seeing the gradient drawn and guessed the gradient of cos$x$ too. They were pressing on eagerly now.

The next bastion to fall was the discovery that 2^x and 3^x had similar shaped gradient functions and that somewhere in between was a number k such that the derivative of k^x is again k^x. Then they conjectured the derivative of the natural logarithm of x, and even managed that of ln(abs(x))...

This was expected to take them some time, but it was all over in a few minutes with them asking for more. They were challenged to find the gradient function of tan$x$ (figure 5).

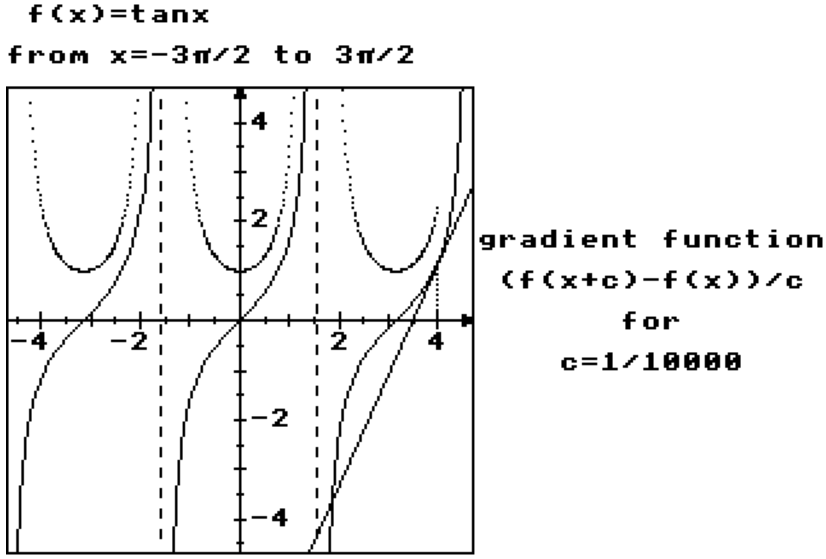

Figure 5 : The gradient of tan$x$

The response was that the gradient of tan$x$ looked like the square of the tanx graph, except it was 1 where $(\tan)^2 x$ was 0, so the gradient was probably $1+\tan^2 x$. (They had to be shown that the version of the program being used required the input as $1+(\tan x)^2$. It wasn't possible to fit every refinement into the tiny BBC memory.)

They followed this up with other curves of their interest and eventually admitted defeat when they couldn't guess the formula for the derivative of the semicircle $(1—x^2)^{1/2}$.

Other groups had similar successes. The difficulty was keeping the earlier experimenters away to give the later ones a chance.

New Methods of Approach

We followed up with more student use of the programs in the regular mathematics classes. Ww quickly realized that with one micro for sixteen students we needed a more systematic approach for its use than at the Open Evening. An extra computer at this stage would have been invaluable. With only one computer we divided the class into groups of three or so. As they went through a set of regular differentiation exercises, they took it in turns to go to the computer, type in their functions and check that their result actually worked. The rules were simples they were assigned exercise numbers, the first group to the first question. the second to the second, and so on, until all the groups were exhausted, then the first group had the next question and the process repeated. They could go up at any time when they had completed a given exercise and the system of staggering the questions meant that they could get on with the exercises until there was a free space to try the computer. They were told to bring any interesting features of the graph to the attention of the whole group, so occasionally work stopped to see what they had found.

They soon got used to the idea of visualizing the gradient function as a dynamic process, looking along the graph, seeing the gradient change, and visualizing the gradient as another graph.

Fitting in with the Physics Department

One day several of the students who had missed the Open Evening experience came to the class saying that the Physics department had started on Simple Harmonic Motion and they didn't understand the derivatives of sine and cosine. We switched on the computer and went through the exercise for them. Even though we had yet to cover the trigonometric functions in radians, this proved most helpful and they professed satisfactory insight into what was going on. Except Brian. He didn't see what all the fuss was about. What he wanted was to be told the formula so that he could learn it and pass the exam. All this computer stuff was a waste of time.

More Graph Sketching

We returned to graph-sketching to study the trigonometric functions. The students were encouraged to sketch the graphs of sine and cosine – the time-honoured method of drawing a circle radius 1 and transferring the values of $y=\sin x$ to an x-y graph as the angle x turned (several times) through full circle (figure 6).

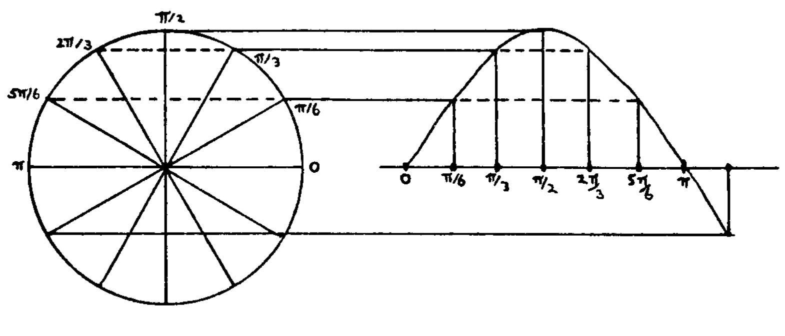

Figure 6

We could have programmed a computer to do the drawing for us. but there is something essential in the act of reading the values round the circle and physically transferring them to the graph. In the same lesson they drew $\cos x$, then moved on to the functions $\sin 2x$, $2\sin x$, and other similar expressions.

Once again the physical act of drawing proved indispensable. For instance y=2sin$x$ requires the y-value for y=sin$x$ to be read, multiplied by 2 and the new graph plotted whilst y=sin2x requires the y-value to be read from y=sin$x$, then the x-value halved to give the new graph. None of this action would have come out on the computer.

But when the initial graphs had been drawn and the physical effort made, it was a luxury to sit back and draw sin$x$, 2sin$x$, 3sin$x$, to see the growing y-values, then sin$x$, sin2x, sin3x to see the reducing x size.

After studying the angle formulae such as

$$\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b)$$

which we "proved" by the "modern" matrix method, we found that it was met with some resistance. But superimposing the graphs of

$$\sin(a+x)$$

and

$$\sin(a)\cos x+\cos(a)\sin x$$

at least gave some feeling of confidence.

We had an interesting experience with drawing $y=a\sin x+b\cos x$.

What would the graph look like, say for a=3, b=4 ? The students had no idea. Even the physical sketching proved hard. So we compared this with the formula for

$$k\sin(p+x)=k\cos(p)\sin x+k\sin(p)\cos x.$$

We required $a=k\cos(p)$, $b=k\sin(p)$ which gave $k=(a^2+b^2)$, $\tan(p)=b/a$.

So we drew $y=a\sin x+b\cos x$ (for a=3, b=4) and superimposed $y=k\sin(p+x)$ where

$$k=\mathrm{sqr}(a^2+b^2),\ p=\mathrm{atn}(b/a).$$

Lo and behold, the superimposed graph was, pixel for pixel, identical with the original.

Local Behaviour

The students had grave difficulties sketching the graph of $y=\sin^2 x$.

Imagining the graph of y=sin$x$ and squaring, most of them ended up with a sketch with cusps where the graph met the x-axis (figure 7).

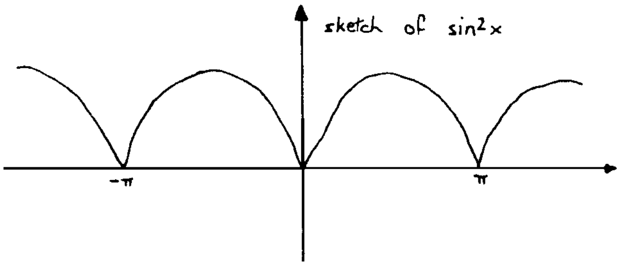

Figure 7

When the true graph was drawn, they expressed some mild surprise. But it was easy to focus their attention on the fact that the graph of $y=\sin x$ near the origin was much the same as $y=x$, so the graph of $y=(\sin x)^2$ must be similar to $y=x^2$. Hence the rounded shape at the origin, and also at every multiple of π.

Following up this idea, what would the graph of $y=\sin^3 x$ look like? They sketched it and compared it with the computer picture which proved to look much like x^3 at every multiple of π (figure 8). A picture in a book only conveys part of the feeling of the graph growing in front of your eyes on a computer screen, especially when that graph is drawn to your bidding.

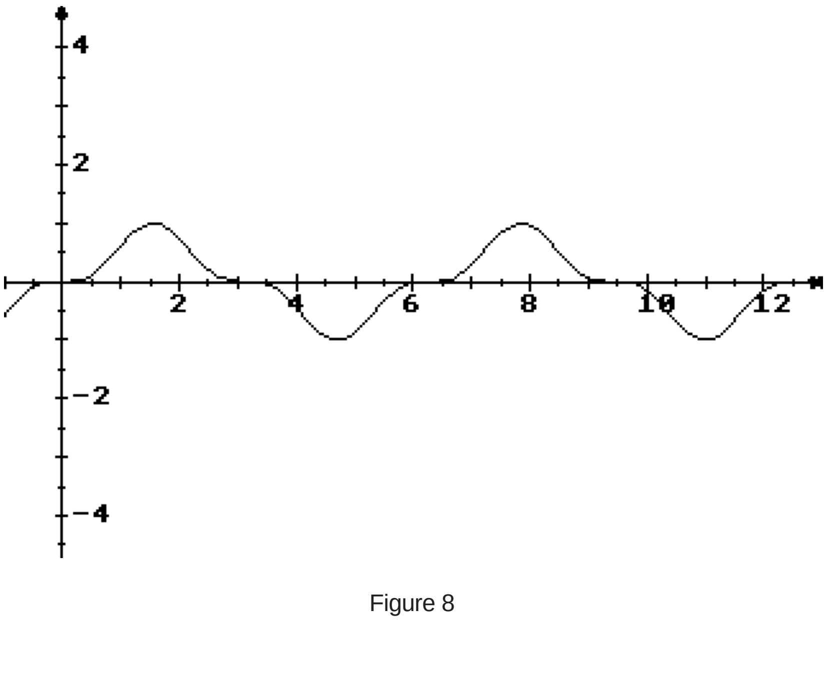

Figure 8

Proof of the formulae

When we came to derive the formulae for the derivatives of sine and cosine we were in the fortunate position of knowing the answers before we went through the trigonometric manipulation. The students experience of the gradient program stood them in good stead and they could see that the gradient from x to $x+h$ would be much the same as the gradient from $x–h$ to $x+h$. So the approximate gradient was

$$\frac{\sin(x+h)–\sin(x–h)}{2h}$$

The fabled formulae transformed this to

$$\frac{\sin x\cos h+\cos x\sin h–(\sin x\cos h—\cos x\sin h)}{2h}$$

or

$$\cos x\frac{\sin h}{h}\ .$$

Their experience with the local behaviour of the graph of sin$x$ showed them that, near the origin,

$$\frac{\sin x}{x}$$

was approximately 1 and this was reinforced by further discussions of well-known ideas. For small h therefore the gradient of the sine curve approximated to cos$x$, which was as expected. The cosine curve followed similarly, including the magic minus sign in the derivative –sin$x$. But this minus sign now had a physical interpretation: when the gradient function for cos$x$ was drawn it simply turned out to be the graph of sin$x$ upside down (figure 9).

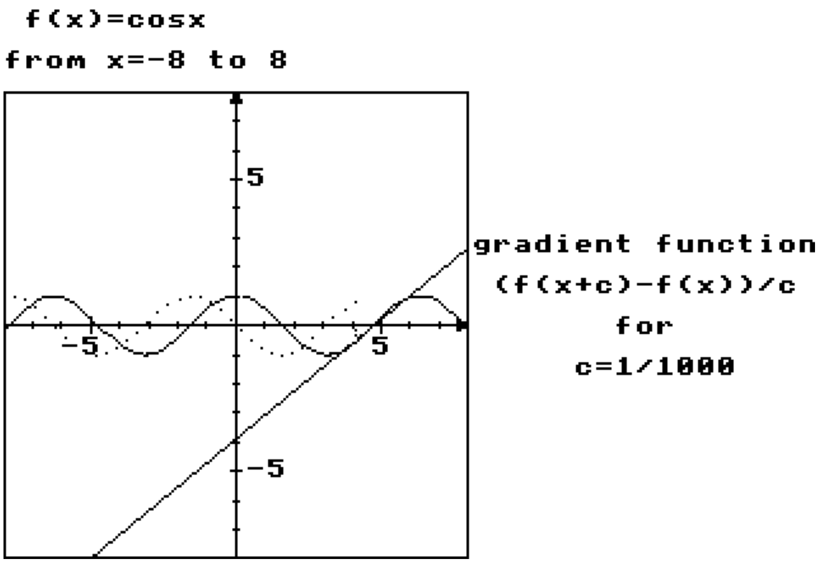

figure 9 : The graph of the gradient of sin$x$

Gradient Sketching

By this time, without being taught explicitly, the students were very good at sketching gradients. Those following a standard A-level course faced with the graph in figure 10 might consider drawing its gradient in a number of stages.

> "The graph looks like x^3+1, so its derivative will be $3x^2$ and a sketch of this can now be drawn."

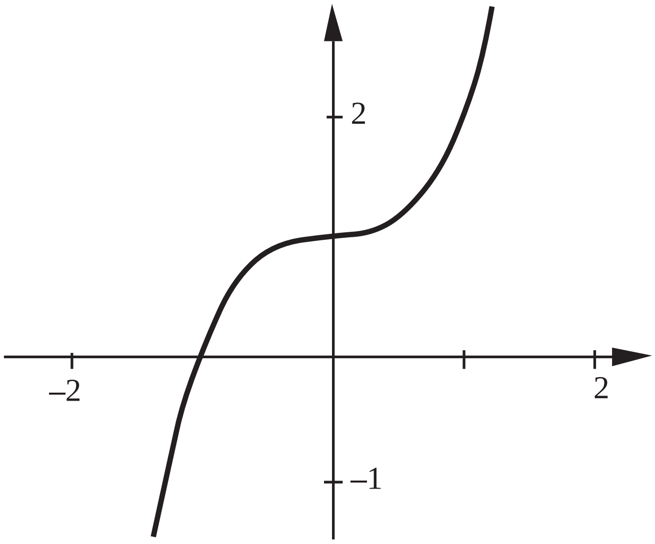

figure 10 : what is the gradient of this graph?

But a student with a dynamic view of the gradient would say

> "The gradient starts off large and positive, it diminishes rapidly until the gradient is zero at the origin, then increases just as rapidly again beyond this point."

Thus a sketch of the derivative could be drawn in a single procedure. Slightly more complicated graphs which would confuse the standard student (because (s)he couldn't guess the formula) would be equally well handled by the student with a dynamic image of the gradient. In tests between control and experimental groups the improvement was statistically highly significant.

Hand-waving

Using the programs for differentiation had an unforseen effect. They outlasted their usefulness very quickly. This is not to say that they weren't used on occasions in the later stages because they were always there to draw a difficult derivative. But the students could now interpret a static picture on the blackboard or a vague hand-wave in the air as a dynamic representation of a gradient function. When we came to discuss the properties of maxima and minima they responded immediately to questions about the gradient of the graph at, before, or after, the turning points.

Anti-differentiation and the arbitrary constant

The idea of antiderivative is usually blessed with the name "indefinite integral" in our A-level. It is a misnomer if ever there were one. It means that we know f(x) and we want to find a function A(x) such that A'(x)=f(x). We can look at this two ways: the standard way (I want to look for a formula which differentiates to give f(x)) and the pictorial way (I know the gradient f(x) of the graph y=A(x), can I draw a suitable graph for A(x)?). It doesn't take long to consider the pictorial approach and it has an unforseen benefit.

Through an array of points in the plane we draw short line segments of gradient f(x). Then y=A(x) is obtained by following along the directions given (figure 11). Because the gradient depends only on x, not on y, the possible graphs through a vertical line all have the same direction. Thus the solutions differ by a constant.

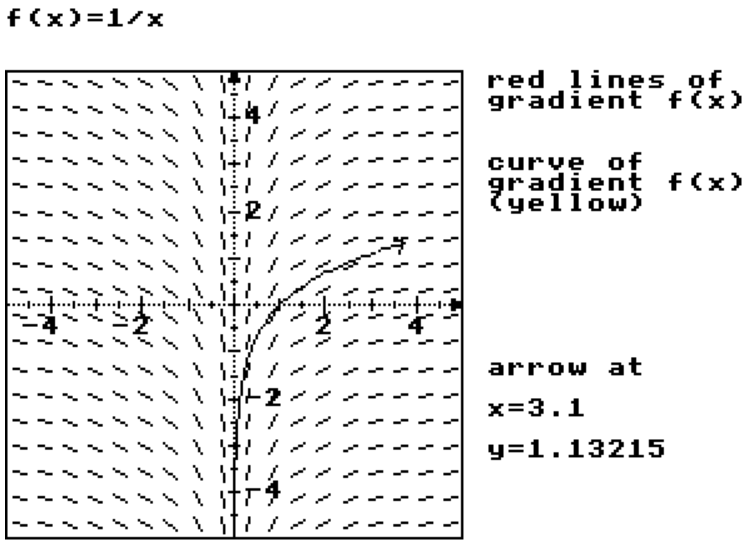

Figure 11 : A curve of gradient 1/x

However, if the function f(x) is undefined somewhere, for instance, f(x)=1/x is undefined for x=0, then we can't trace the solutions through this value of x. A solution of A'(x)=1/x by tracing the directions will lie on one side of the origin only and not cross it. Thus the "added constant" only applies on one side of the origin. It is perfectly possible to shift the parts of the solution on either side of the origin up or down by different constants. Thus we get the truth about the "arbitrary constant": it is only valid in a *connected* part of the domain of f(x). Too difficult for students to understand? Try it and see!

Effects on Algebra

The algebraic techniques necessary to answer standard sixth form questions were developed in conjunction with the appreciation of visual concepts. Interestingly, it became obvious that skills in one of these areas did not necessarily imply skill in the other. Colin, who could soon talk fluently and intelligently about the behaviour of functions near asymptotes, sketch derived functions, appreciate differentiability and the geometry of translation and reflection of graphs, found great difficulty in successfully developing three or more lines of algebra. By contrast David, who found little difficulty with algebraic manipulation, often being amongst the first in finishing an exercise, needed several prompts to appreciate the picture of some of the functions he was dealing with. However, the results shown graphically certainly provided motivation for proceeding to a more formal algebraic approach, with the sequence

illustration → conjecture → proof

being the most successful method of interesting the class in the nature of algebraic proof such as the derivative of x^n.

The computer graphics also provided a good way of checking results. The sense of triumph when $y=(l–x^2)^{1/2}$ was differentiated by the chain rule. the result drawn on the computer and found to be identical to the superimposed computer-drawn derivative far exceeded that usually experienced by looking up the answer in the answer book!

In a similar way, innocuous looking algebraic errors in using the formulae for differentiation were shown to produce markedly different graphs from the actual derivative, thus underlining the importance of careful algebra.

Algebraic difficulties are such a recurring problem that they need a total rethink in how they are introduced earlier in the curriculum. (See Tall & Thomas 19??.)

Insight into area

The next stage in the course was to investigate the area under a graph by summing rectangles. The computer picture of the process quickly showed the effect of increasing the number of rectangles by reducing the width of each rectangle to gain a better approximation to the area under the curve. The program shades the rectangles differently according to the sign of the area (figure 12). Before any calculation was performed by the class they were able to appreciate how the signs of the ordinate and the step combine to give the sign for the area of each rectangle.

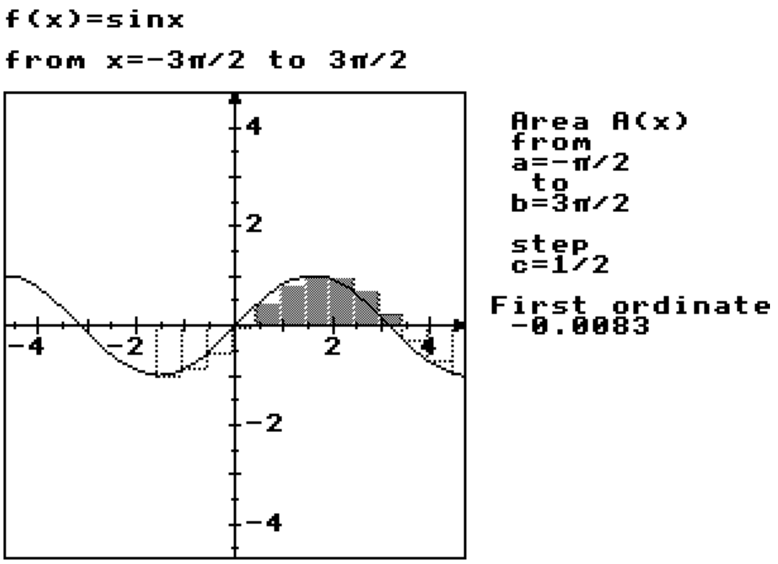

Figure 12 : The area, calculated taking sign into account

When it came to performing the algebra necessary to sum the area of rectangles under $y=x^2$ there was the usual trouble with Σn^2 , but they did make more sense of the the limit of the sum as the width of the strip tended to zero after seeing the calculation performed on the computer. After looking at the results for the area under $y=x$ and $y=x^2$ from $x=0$ to $x=b$ they were willing to conjecture the result for the area under $y=x^3$ for $x=0$ to $x=b$. The algebra required to show this result would have certainly lost many of the group, but the computer was able to support their conjecture.

At this point we looked briefly at the other numerical methods for estimating area given in the package, such as the trapezium rule, Simpson's rule and the mid-ordinate rule, to compare results. The class were soon able to see the relative efficiency of these methods and make good guesses as to whether they would overestimate or underestimate the area under a given curve.

We progressed to the standard notation for integration. After seeing the rectangles being drawn, it was a relatively simple matter for the class to appreciate that the area from $x=a$ to $x=c$ is the sum of the areas from $x=a$ to $x=b$ and $x=b$ to $x=c$ and, more importantly, that the area from $x=b$ to $x=a$ is *minus* the area from $x=a$ to $x=b$.

They had conjectured the result of the Fundamental Theorem of Calculus for some easy polynomials, had it confirmed for more difficult examples, and were now ready for some form of proof of the general result.

The idea behind the fundamental theorem was illustrated on the computer by stretching the x-range of a graph over a small interval. The y-range was left unchanged and the increase in area from x to x+h was examined. We chose $f(x)=\sin x$ as a typical illustration, using ranges from $x=0.99$ to 1.01 and $y=-2$ to 2 (figure 13). It can be seen that a horizontal stretch flattens out the graph to give the area calculation in the form

$$A(x+h)-A(x) \approx h\mathrm{f}(x).$$

The more the *x*-range is stretched, the flatter the graph gets and the better the approximation becomes. By rewriting it in the form

$$\frac{A(x+h)-A(x)}{h} \approx \mathrm{f}(x)$$

and allowing h to tend to zero gives the fundamental theorem:

$$A'(x) = \mathrm{f}(x).$$

The students found it quite natural to examine the graph stretched in this way and we felt that they had a better understanding of the fundamental theorem than other students following a more standard text-book approach.

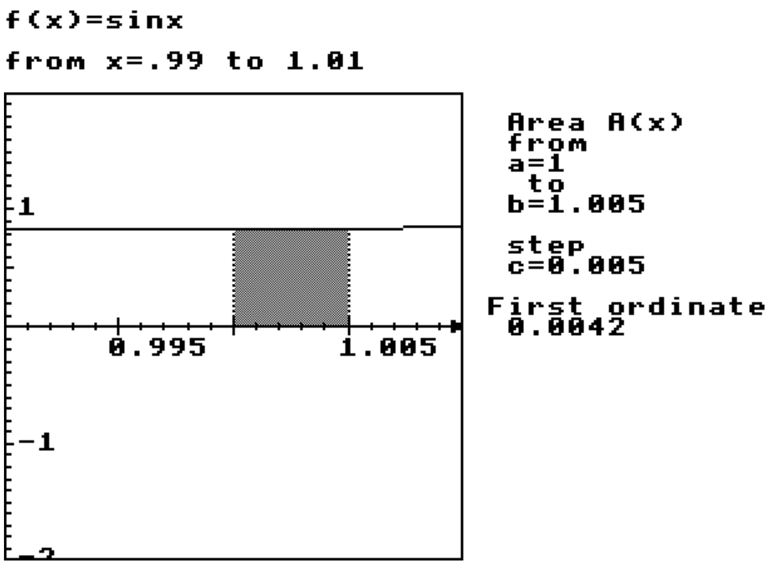

Figure 13

What we didn't do

Although students worked interactively with prepared software throughout the year we did not ask them to write their own programs or give detailed explanations of the construction of the programs in the package. This was a conscious decision, bearing in mind the scope of the present A-level syllabus and the restricted time available. But this is not to say that such developments would not have been desirable: the great majority of the students were interested in programming in BASIC, most having their own micros, and they often expressed interest in how the programs worked. It is easy to see that programs on the lines of “132 Short Programs for the Mathematics Classroom” published by the Mathematical Association could be employed in an A-level course, with the algorithms employed enhancing the understanding of many concepts. At present lack of time restricts such developments in the current A-level syllabus.

The Upper Sixth

Most of the foundations of the A-level syllabus have been laid in the first year and the computer played an integral part throughout. In the upper sixth the computer will continue to be available in every lesson. Our experience using the computer at this level shows that it is most valuable when we meet such topics as the exponential function, logarithmic function and Taylor's series. However, it is likely to be used less than in the lower sixth as we become involved with answering standard examination questions. We still use the programs on occasion to check results graphically but, perhaps as important, the graphical images which students can now visualize in their mind's eye will be appealed to even in quick blackboard and chalk explanations.

The future

Our experience shows that the computer can be used as a useful adjunct to the current A-level. Even more mileage can be achieved through designing a new curriculum that takes the new possibilities into account. A fully integrated approach, with the students programming their own algorithms as well as using prepared software would require a provision for microcomputers currently beyond the financial resources of the average school. In this respect schools lag behind the provisions students have in their own homes. (Our class proved to have 18 computers between sixteen students, only one not having his own machine.) One may look forward optimistically to a time when computers can realize a fuller potential in a much broader mathematics curriculum.

In the meantime it is clear that a single computer in a classroom can be used to advantage in the current syllabus. The same techniques should also prove helpful at an earlier age, in particular the exploratory investigations of a dynamical graphic approach on the computer can be used to give a meaning to the calculus concepts taught for the 16+ examination.