Visualizing Higher Level Mathematical Concepts Using Computer Graphics

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ABSTRACT The computer is going to revolutionize mathematical education, not least with its ability to calculate quickly and display moving graphics. These facilities have been utilized in interactive programs to demonstrate the ideas in differentiation and integration, evolving new dynamic concept images.

Theoretical background

The work described in this paper is the result of a happy accident of history. Over a number of years mathematics educators have studied the concept imagery generated by students when learning the calculus and now microcomputers have become available which can draw moving pictures to provide powerful cognitive support for this imagery. Though by no means a total solution, it is hoped that interactive work on the computer can give fruitful insight into the calculus that is potentially more meaningful.

The research of Orton (1979) confirmed that a group of students taught by current methods in the U. K. had great difficulty with a number of ideas in the calculus requiring relational understanding. These included the idea of rate of change between two points on a graph with all the possible signs involved, the notion of the derivative as a limit, the idea of the area as the limit of a sum and the meaning of positive and negative areas.

Other authors have noted interference in mathematical meaning through the use of words that have different colloquial connotations. For instance, the idea of a “limit” being unreachable (Cornu, 1981) or the term “gets close to” carrying the implication “not coincident with” (Schwarzenberger & Tall, 1978). Ervynck (1981) has also documented problems with limits and suggests the value of pictures to visualize the processes involved. Standard pictures found in text-books have two major problems: they are static, and so fail to fully convey the dynamic nature of many of the concepts, and they also tend to be limited in variety, leading to a restricted concept image being developed from too few exemplars. For instance, the classical differential triangle is usually drawn as in figure 1, with the increments δx, δy both positive and the

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graph sitting neatly in the first quadrant. As Orton has observed, a significant proportion of his students interpreted the symbolism

$$\delta y/\delta x \to dy/dx$$

to mean "$\delta y/\delta x$ gets smaller until it becomes dy/dx."

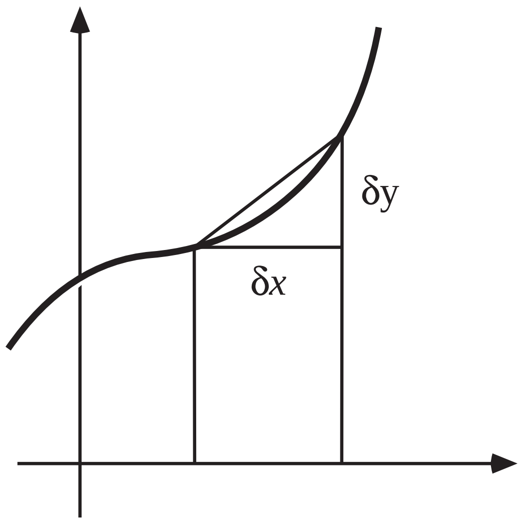

Figure 1

In an example such as figure 1, the gradient $\delta y/\delta x$ *does* get smaller until, to all intents and purposes, it is indistinguishable from dy/dx. By simplifying the examples presented to students in this way, hoping to help them in the initial stages, the net result may be a restricted concept image in the student's mind which later conflicts with the formal theory (Tall & Vinner, 1981).

The computer programs

To combat the conceptual limitations of the kind just described, a suite of three computer programs were written by the first-named author to help generate more appropriate concept images: *Gradient*, *Area* and *Blancmange*. *Gradient* draws moving pictures of the gradient of a graph, leading to ideas of differentiation, *Blancmange* draws an everywhere continuous, nowhere differentiable function (Tall, 1982) to prevent too limited a set of exemplars being encountered, and *Area* computes and displays the area under a graph in various ways, leading to ideas in integration.

Gradient and *Area* both allow the input *of* a function in normal analytic notation (e.g. $\mathbf{f(x)=sin2x}$ or $\mathbf{f(x)=(x^2-1)^{1/2}}$) and draw the graph over a chosen range, indicated places where the function becomes undefined or has an asymptote. *Gradient* offers two main routines, the first simulating the limiting process at a point in which the chord is drawn between two chosen points $(a, f(a))$, $(b, f(b))$ and then b moves in steps to a as the gradient is displayed numerically on the screen. On one computer (the 380Z) arithmetic accuracy is such that the gradient can only be obtained to about three figures, seriously prejudicing the concept image of the limiting process, but on another (the BBC) five digit accuracy allows a much more successful simulation. Both computers are markedly better in the second routine, displaying the gradient as a *function* $g(x)=(f(x+c)-f(x))/c$, for fixed (non-zero) c and variable x. To simulate this dynamically the program draws a sequence of chords from x to $x+c$ as x increases by steps, simultaneously plotting the gradient of the chord as a point $(x, g(x))$. The static picture in figure 2 (a computer printout) fails to convey the impact of this idea, but the moving picture on the screen, with the function and the step c specified by the user, leaves an unforgettable impression so that the graph of $g(x)$ is visibly seen to be the gradient of the graph $f(x)$.

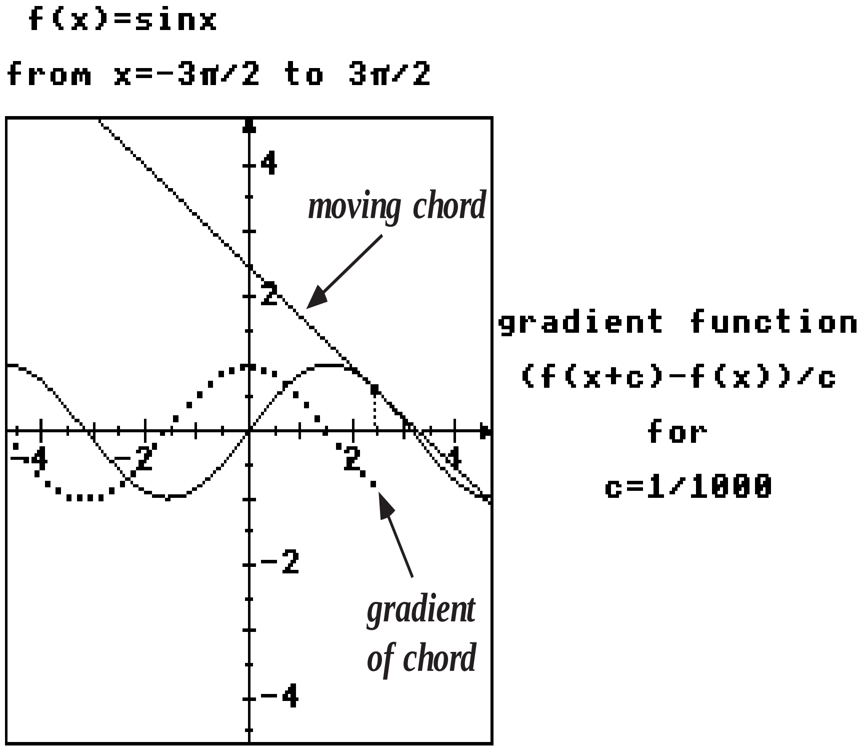

Figure 2

As one student put it. “I never understood what it meant to say that the derivative of sin$x$ was cos$x$ until I saw it grow on the computer.”

The program *Area* also has two major routines, one to display the approximate area under the graph as computed by various methods, the other to draw the “area-so-far function” from a to x as a function of x. Positive and negative areas are displayed in different colours and the reasons for the signs become more obvious when it is realized that the area is calculated as the product of two signed lengths drawn on the screen. For instance, calculating the area from right to left has *negative* step times the signed ordinate, giving a negative area above the axis and positive area below.

The programs are designed both for demonstration purposes and for student investigations, allowing students the freedom to explore and enrich their concept images in a more personal way. It is interesting to see the students regarding the computer as an authority which does not present the same threat to them as the teacher. Indeed, they seem far more willing to discuss conceptual difficulties thrown up by the computer than they would difficulties in understanding a teacher's explanation.

Research

The second-named author has initiated two research studies using the programs. A cross-sectional study is being conducted on a group of about 30 A-level (= senior high school) students. The group is being divided so that some use the *Gradient* program, some the *Area* program and some both. A questionnaire is being administered to all the students to assess their understanding on about 30 different concepts, including a number pinpointed by the Orton study. The data is being analysed in an attempt to identify those concepts that appear affected by the activities.

A longitudinal study is also being conducted on a group of twelve adults attending a one year, two-evenings-a-week class designed to take them to degree standard in mathematics. The continuous assessment and teaching style of the small group discussion make this especially suitable for a study that relies on interpreting written work and contributions to the class. They have recently begun work on the calculus and used the programs in groups of about six students each. Though the full analysis must await the end of the course, preliminary impressions show some interesting reactions.

In the first session the students used the computer to study the gradient of the graph of $\sin x$ at a number of individual points. Initially they were invited to choose points a,b quite far apart and to see that as the step between the points decreased the gradients of the chords formed no obvious pattern. They readily appreciated that the step had to be relatively small before the sequence of values for the gradient converged. There was interest and scepticism when for very small steps the gradient began to wander again after having appeared to converge. (This was the 380Z computer with limited accuracy.) They investigated positive and negative steps and one group became interested in the number of stages it took before the gradient stabilized, concluding that it depended on the curvature of the graph at the point.

In the next session the students were introduced to the gradient function, drawing $\sin x$, $\cos x$, then powers of x, and guessing the formula for the derived function which was drawn and compared with the gradient graph. They were very familiar with the graphs of standard functions and correctly conjectured the derivatives in every case. Exponential functions provided the first instance of student generated work. The graph of 2^x was drawn on the screen and the gradient function plotted. The two curves were clearly related and it was discovered by trial and error that the derived function was about $\frac{2}{3}2^x$, a quite reasonable approximation. The graph of 3^x was similarly examined and then e^x. It became apparent that e was the number which, when raised to the power x and differentiated, had a derivative which equalled the original function. Further student investigations initiated by the group led to various other conjectures, the *pièce de résistance* being the conjecture that the derivative of arctan(x) was $1/(1+x^2)$!

The first session devoted to integration was noticeable for the large amount of discussion around the idea of negative area. With many other groups this has created no problems, but this time, principally among students who were primary teachers, there was some resistance to accepting the idea of negative area at all. The program was invaluable in that it could focus the discussion on a picture where the students could see why the area of a strip came out negative in a variety of cases, and that integrating from a higher limit to a lower one gave the same answer as the other way round, but with a change in sign. One group using the program decided to find the "paintable area" between the curve and the axis by dividing the calculations into segments above and below the axis and taking the absolute value before adding. It was then realized that the whole calculation could be done in one go by taking the original function, squaring, then square-rooting before calculating the area. The second group came to similar conclusions but used the abs function instead.

The drawing of the "area-so-far function" from various arbitrary points has naturally given a meaning to the constant of integration and highlighted once more the importance of the change in sign when integrating from high to low rather than vice versa.

It is too early to say what effect the computer has had on the students' responses to assessment questions, however, it is already noticeable that graph-oriented questions submitted so far have been very well done.

Most student reaction has been positive. One student remarked "It was helpful, fantastic, just being able to draw the graphs … it would have been such a hassle any other way." Another said after the very first session, "It's interesting that it's only looking at the graph that it's made any sense. You know I said I didn't understand what the derived function was all about – I could do all the odd things before but until now I didn't have a clue what it meant."

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