Elementary Axioms and Pictures for Infinitesimal Calculus

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Our contemporary idea of the number line (Fig. 1)

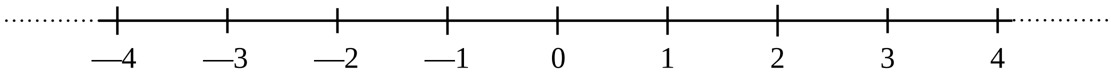

Figure 1

is to conceive it as a system of real numbers given by decimal expansions or, more formally, as a complete ordered field. But it was not always so; prior to the formalisation of the real number concept in the late nineteenth century the number line was often considered to include infinitesimal quantities and their infinite reciprocals. It is this ambivalence which is at the heart of the new theory of non-standard analysis and which can be exploited to give a satisfactory theory of infinitesimal calculus. Instead of *one* number line we must imagine *two*, a number system K of "constants" and a larger system K^* of "quantities." To the naked eye a pictorial representation of these would look the same, the number line of the picture above, but the number line K^* contains infinitesimal detail and infinite structure not present in K.

The purpose of this paper is to present an elementary list of axioms to describe these two number systems and to explain how the additional subtleties present in K^* may be represented pictorially in a satisfactory manner. The axioms are based on the work of Keisler (1976a, 1976b) and the pictures are an extension of the microscopes and telescopes of Stroyan (1972). There are essentially three parts to the axioms: first, to distinguish between K and K^*, second, to describe how to pass from the larger system K^* to the smaller system K, and third, to describe how to pass in the opposite direction from K to K^*. The first is easy. As every ordered field may be considered to contain the rational numbers we can give a formal definition of a positive infinitesimal x as an element satisfying

$$0 < x < r, \text{ for every positive rational number } r.$$

The distinction between K and K^* is that K^* contains infinitesimals but K does not. In other words there are infinitesimal quantities (in K^*) but no infinitesimal constants (in K). Because the reciprocal of a (positive) infinitesimal may be shown to be a (positive) infinite element (greater than any rational number), there are also infinite quantities in K^*; on the other hand, all constants are finite. The passage from K^* to K is negotiated by insisting that every finite quantity $x \in K^*$ differs from a constant by an infinitesimal:

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$x = c+\varepsilon$, ($c \in K$, ε infinitesimal).

This means that every quantity in K^* is either infinite (and too far off to the left or right to figure in a finite picture) or finite (and an infinitesimal quantity from a constant, too close to distinguish in a normal scale picture). In Fig. 2, ε represents a positive infinitesimal quantity.

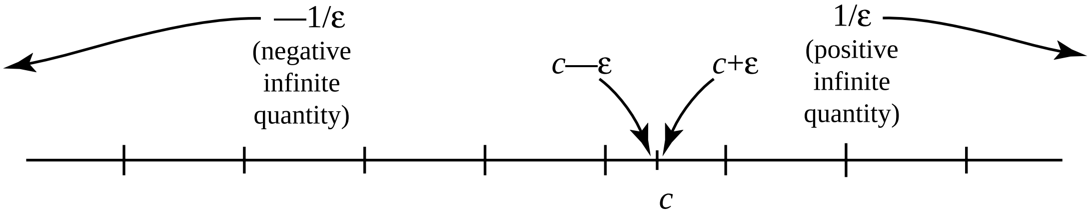

Figure 2

When $x = c+\varepsilon$ as above, the constant c will be called the *constant part* (or *standard part*) of x and denoted by $c=st(x)$. This gives us a map *st* from the finite elements of K^* to K. It does not allow us, as it stands, to see any infinite structure and it loses all the infinitesimal detail. However, if we first alter the scale on K^* using a map $\mu: K^* \to K^*$ of the form

$$\mu(x) = (x-\alpha)/\delta, \quad (\alpha,\delta \in K^*, \delta \neq 0),$$

we can move a portion of K^* into the finite part (using μ) and then apply *st* to reveal an image in K. In this way we get a perfectly ordinary picture of the subtle structure of K^* that would otherwise be invisible to the naked eye. By taking δ to be infinitesimal we see certain infinitesimal detail near α; by taking α to be infinite we see part of the infinite structure.

The final part of the axiomatic system, the passage from K to K^*, is more elusive. We have to begin with a picture in K^n and somehow enrich it in an appropriate way to give a picture in K^{*n}, with extra infinitesimal detail or infinite structure, as appropriate. Any subset $D \subseteq K^*$ needs a corresponding subset $D^* \subseteq K^{*n}$ where $D \subseteq D^*$. If D is given by simple formulae, such as the semicircle:

$$D = \{(x,y) \in K^2 \mid x^2+y^2<1, y \geq 0\},$$

including the straight edge but excluding all points on the circumference, then a natural candidate for D^* is

$$D^* = \{(x,y) \in K^{*2} \mid x^2+y^2<1, y \geq 0\},$$

To the naked eye, D, D^* look alike, but D^* has enriched structure, including extra quantities of the form $(a+\varepsilon, b+\delta)$ where a, b are constants, ε, δ arc infinitesimals and

$$(a+\varepsilon)^2+(b+\delta)^2<1, \quad b+\delta \geq 0.$$

A similar process can be applied to any subset D given as the solutions of a list of formulae which are equalities or inequalities between polynomials in the coordinates. More generally we suppose that every $D \subseteq K^n$ extends to D^* where $D \subseteq D^* \subseteq K^{*n}$ and every function $f: D \to K$ extends to $f: D^* \to K^*$ (agreeing with f on the subset D). An equality or inequality between functions, say $f \leq g$ (where $f: D \rightarrow K$, $g: E \rightarrow K$) can be interpreted as having solutions of the form

$$S(f \leq g) = \{x \in K^n \mid x \in D \cap E,\ f(x) \leq g(x)\}.$$

and the broader set of solutions

$$S^*(f \leq g) = \{x \in K^{*n} \mid x \in D^* \cap E^*,\ f(x) \leq g(x)\}$$

If a subset $X \subseteq K^n$ were of the form $X = S(f \leq g)$, then we could take X^* to be the extended solution set $X^* = S^*(f \leq g)$ This is the key that unlocks the whole theory—and it took 300 years to formalise, from the first insightful steps in the calculus by Newton in 1666 to the arrival of non-standard analysis (Robinson, 1966).

A *list* L is just a finite set of equalities or inequalities between functions from subsets of K^n to K. Its *solution set* $S(L)$ is just the set of common solutions in K^n of all the equalities or inequalities and its extended solution set $S^*(L)$ is just the corresponding set of solutions in K^{*n}. If $X = S(L)$ then the extended set X^* is taken to be $X^* = S^*(L)$.

Example

If $X = \{(x,y) \in K^2 \mid x = y^2, x \geq 0, y \geq 0\}$, then

$$X^* = \{(x,y) \in K^{*2} \mid x = y^2, x \geq 0, y \geq 0\}.$$

In this case X is the graph of the function $y = \sqrt{x}$ and X^* can be taken as the graph of the extended function, $y = \sqrt{x}$ for $x \in K^*$, $x \geq 0$.

In general we must make sure that if we were to describe a set X in two different ways as a solution of a list, $X = S(L) = S(M)$, then the two extended solution sets $S^*(L)$, $S^*(M)$ are the same. We shall assume this property in an equivalent form as the fourth, and final, part of the axiomatic system.

The Subset Axiom. If $S(L) \subseteq S(M)$ for lists of formulae L, M, then $S^*(L) \subseteq S^*(M)$.

Interchanging the roles of L, M, we immediately obtain:

The Extension Property. If $S(L) = S(M)$ then $S^*(L) = S^*(M)$.

1. The axioms

All the axioms have been introduced and now it is just a matter of assembling them in a single list.

I. Constants and quantities

1(a). K is an ordered subfield of an ordered field K^*.

1(b). K^* contains a non-zero infinitesimal but K does not.

II. Restriction

2. Every finite element $x \in K^*$ is of the form

$$x = c + \varepsilon \quad (c \in K,\ \varepsilon \text{ infinitesimal}).$$

III. Extension

3(*a*). Every $D \subseteq K^n$ has an extension $D \subseteq D^* \subseteq K^{*n}$ and every function $f{:}D \to K$ has an extension $f{:}D^* \to K^*$ (agreeing with f on D).

3(*b*). If $f{:}K^n \to K$ is a polynomial, then its extension is the same polynomial extended to K^{*n}, $f{:}\ K^{*n} \to K^*$.

IV. Uniqueness

Using the notation introduced earlier,

4. (The Subset Axiom). For lists L, M,

$$S(L) \subseteq S(M) \Rightarrow S^*(L) \subseteq S^*(M).$$

These axioms are not minimal assumptions, for instance the statement $D \subseteq D^*$ and the fact that the extension function agrees with f on D may both be deduced from the other assumptions (in the manner of Keisler, 1976b, page 12). The phraseology given here is one that has proved suitable for mathematics students beginning to study the theory; at this stage the clarity of "extension functions and sets" proves to be more valuable than the mathematician's urge to give minimal axiomatic requirements.

Notice that the completeness axiom (every non-empty subset of K which is bounded above has a least upper bound) is not one of the given axioms. This is one of the most difficult principles in analysis to grasp and its ramifications prove to be a stumbling block for many studying analysis.

In essence the proof is as follows. The completeness axiom may be taken in the form

> "every increasing sequence of real numbers which is bounded above tends to a real limit."

To establish the completeness property for the field of constants K, it must be established that a sequence s(1), s(2),… in K, bounded above by $c \in K$, has a limit in K. This requires some routine computations (omitted here) that the function $s{:}\boldsymbol{N} \to K$ (where $\boldsymbol{N}$ is the set of natural numbers 1, 2, ...) extends to a function $s{:}\boldsymbol{N}^* \to K^*$ where the set $\boldsymbol{N}^*$ contains infinite elements. (In fact every unbounded set X extends to a set X^* containing infinite elements.) It is then a straightforward matter to deduce that, because

$$s(1) \le s(n) \le c, \text{ for every } n \in \boldsymbol{N},$$

then

$$s(1) \le s(n) \le c, \text{ for every } n \in \boldsymbol{N}^*.$$

Thus $s(n)$ is finite for infinite n and a routine calculation shows that the limit l of the sequence is

$$l = st(f(n)) \text{ for any infinite } n \in \boldsymbol{N}^*.$$

In the knowledge that this can be established from the axioms, when the dust settles we see that the field of constants is a complete ordered field and so it can only be the field of real numbers. With this in mind, those who are meeting this system for the first time can take heart in the fact that the axioms have a perfectly satisfactory informal description as follows.

> 1(a). The ordinary number line of constants (which can be thought of as limits of decimal expansions) are considered as part of a larger number system K^* of quantities which can also be thought of as a number line and satisfy the usual rules of arithmetic (addition, subtraction, multiplication, division and order).
>
> 1(b). All the ordinary constants are finite (in the sense that they can be represented pictorially on a finite number line produced sufficiently far in one direction or the other) but some quantities (in K^*) are infinite and too far off to be represented on such a finite line, no matter how far produced. The reciprocals of such infinite quantities are infinitesimal (smaller in size than any positive rational) and are indistinguishable from zero to any normal scale.
>
> 2. If a quantity x is finite, then it is of the form $x=c+\varepsilon$ where c is an ordinary constant and ε is an infinitesimal (positive, negative or zero).
>
> 3(a). Every subset D in ordinary constant space K^n can be enhanced to a subset D^* which looks the same as D to the naked eye, but includes additional quantities. (It may be shown that the extra elements of D^* arise in one of two ways: if $x \in K^n$ is a limit point of D, then D^* includes points infinitesimally close to x and if D is unbounded, then D^* includes infinite points.) Every function defined on D extends to apply to D^*.
>
> 3(b). In particular a polynomial function with constant coefficients applies to quantities by using the same polynomial formula.
>
> 4. If L and M are finite lists of equalities or inequalities between functions then if the constant solutions of all the formulae in L are a subset of the constant solutions of all the formulae in M, then the same is true of the quantity solutions of L and M.

A good example of axiom 4 in action is the extension of a “piecewise defined function.” Consider $D = \{x \in K \mid x \geq -1\}$ and $f{:}D \to K$ given by

$$f(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0 \\ 2x^2 & 0 \leq x \leq 1 \\ x + 1/x & x > 1 \end{cases}$$

Then

$$\mathrm{S}(-1 \leq x,\ x \leq 0) \subseteq \mathrm{S}(f(x)=0)$$

implies

$$\mathrm{S}^*(-1 \leq x,\ x \leq 0) \subseteq \mathrm{S}^*(f(x)=0)$$

and this tells us that all quantities $x \in K^*$ such that $-1 \leq x \leq 0$ satisfy $f(x)=0$. The same kind of thing works for the other two parts of the domain and so the extended function $f{:}D^* \to K^*$ is just given by the same formulae as the original function, now applying to $x \in D^*$ where

$$D^* = \{x \in K^* \mid x \geq -1\}.$$

Both functions have graphs which look like this to a finite scale (Fig. 3).

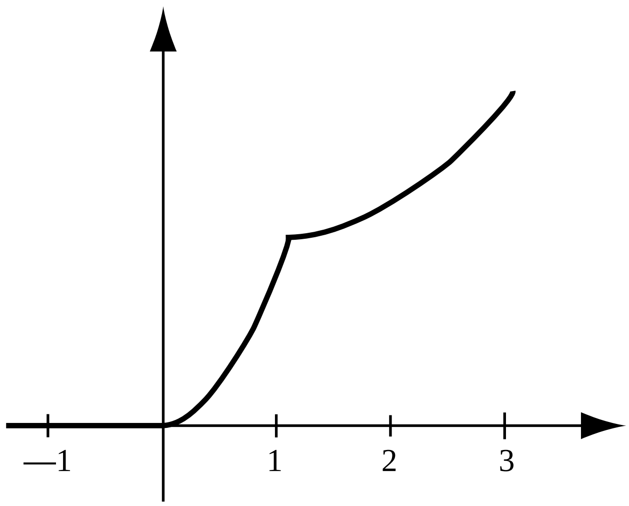

Figure 3

The extra structure of the graph of the extended function can be revealed in ordinary pictures by using optical lenses.

2. The pictures

To represent details in K^* in an ordinary picture, we follow the idea of Stroyan and first move elements of K^* using the map $\mu{:}K^* \to K^*$ given by

$$\mu(x) = (x-\alpha)/\delta, \quad (\alpha, \delta \in K^*, \delta>0).$$

(We take $\delta>0$ because this maintains the order of points on the line, multiplying by a negative number would reverse them.)

For instance, if we take $\alpha=c$, $\delta=\varepsilon$, then

$$\mu(c-\varepsilon) = -1, \mu(c) = 0, \mu(c+\varepsilon)=+1.$$

Hence the three points $c-\varepsilon$, c and $c+\varepsilon$ (for constant c and positive infinitesimal ε) which we mentioned as being indistinguishable to the naked eye in the introduction are now spread out and mapped onto clearly distinct points –1, 0, 1 (Fig. 4).

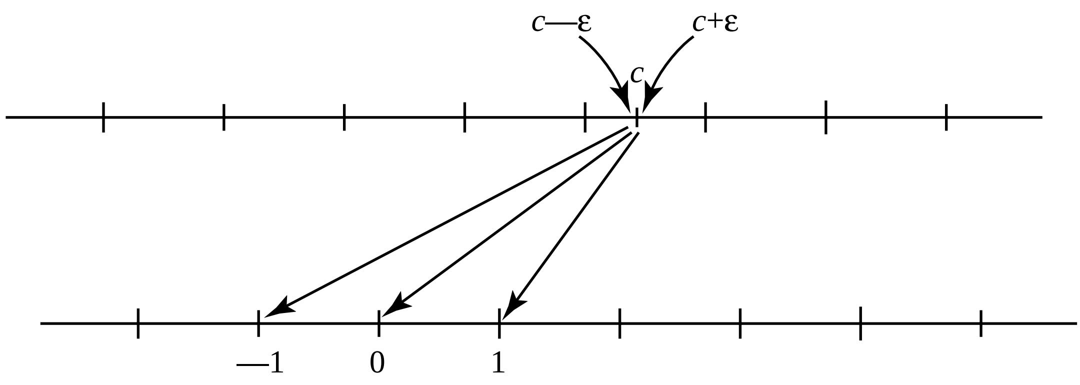

Figure 4

When using this map μ, however, a point $x \in K^*$ such that $(x-c)/\varepsilon$ is not finite cannot be seen in the magnified picture. In general, the map μ where $\mu(x) = (x-\alpha)/\delta$ will be called the *δ-lens pointed at a.* The *field of view* of μ is the set of $x \in K^*$ such that $\mu(x)$ is finite. The map μ translates the field of view and moves it to fill the whole set of finite elements. Following μ by taking the constant part gives a map from the field of view to the set *K* of constants which is called the *optical δ-lens pointed at* α. Optical δ-lenses are what give the required ordinary pictures of phenomena involving infinitesimal and infinite quantities. They work just as well in two dimensions and more generally in *n* dimensions by applying a lens to each coordinate. For instance a map

$$\mu : K^{*2} \to K^{*2}$$

given by

$$\mu(x, y) = ((x-\alpha)/\delta, (y-\beta)/\rho)$$

is called the (δ, ρ)-lens pointed at (α, β). If $\delta \neq \rho$, we say that the lens is *astigmatic.* Usually we shall take $\delta = \rho$, in which case we simplify the notation and refer to μ as a δ–lens in two dimensions. Following μ by taking the constant part of each quantity then gives an optical δ–lens in two dimensions defined on the field of view and taking constant values in K^2.

As an example, let us look at the graph $f: K^* \to K^*$ given by $f(x) = x^2$ considered as the subset

$$\{(x, x^2) \in K^{*2} \mid x \in K^*\}$$

Part of the finite portion of this graph is shown in Fig. 5.

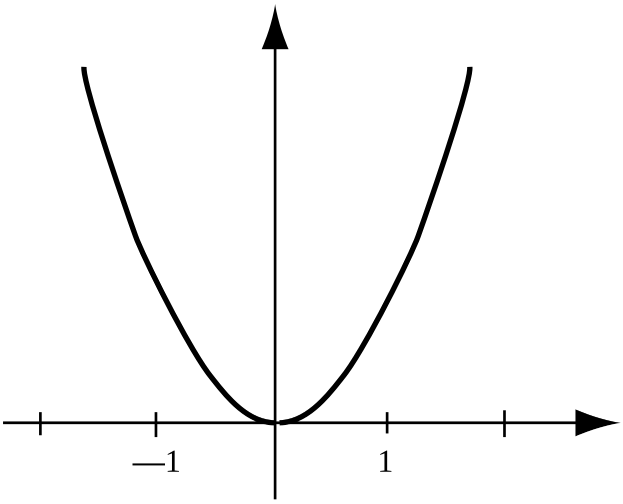

Figure 5

To the naked eye it looks just like the ordinary graph of a parabola because the infinitesimal detail is too small to see. It can be revealed by magnifying it through an appropriate lens, say a δ-lens pointed at (a,a^2). We have

$$\mu(x, y) = ((x–a)/\delta, (y–a^2)/\delta).$$

A nearby point $(a+\lambda, (a+\lambda)^2)$ on the graph, when viewed through μ reveals

$$\mu((a+\lambda), (a+\lambda)^2) = (\lambda/\delta, (2a\lambda+\lambda^2)/\delta)$$

Suppose that $\kappa=\lambda/\delta$ is finite, then for infinitesimal δ we must have λ infinitesimal and so $\lambda^2/\delta=\lambda\kappa$ is also infinitesimal. Taking constant parts, we have

$$st(\mu(a+\lambda, (a + \lambda)^2)) = (st(\lambda/\delta), st(2a\lambda/\delta+\lambda^2/\delta))$$

$$= (st(\lambda/\delta), 2ast(\lambda/\delta))$$

for constant a. Putting $st(\lambda/\delta)=l$, we see that the points on the graph in the field of view are mapped on to a straight line $(l, 2al)$ as l varies.

As in Tall (1980) it is a useful device to use the “map-making technique” of denoting the pictorial image by the same symbol as the original point. This is analogous to calling the place on a map “London” rather than “the image of London on this map.” In the given example this means denoting the image $(l, 2al)$ by the original symbol $(a+\lambda, (a+\lambda)^2)$ and the image (0, 0) (where $\lambda=0$) by (a, a^2), giving the following picture (Fig. 6).

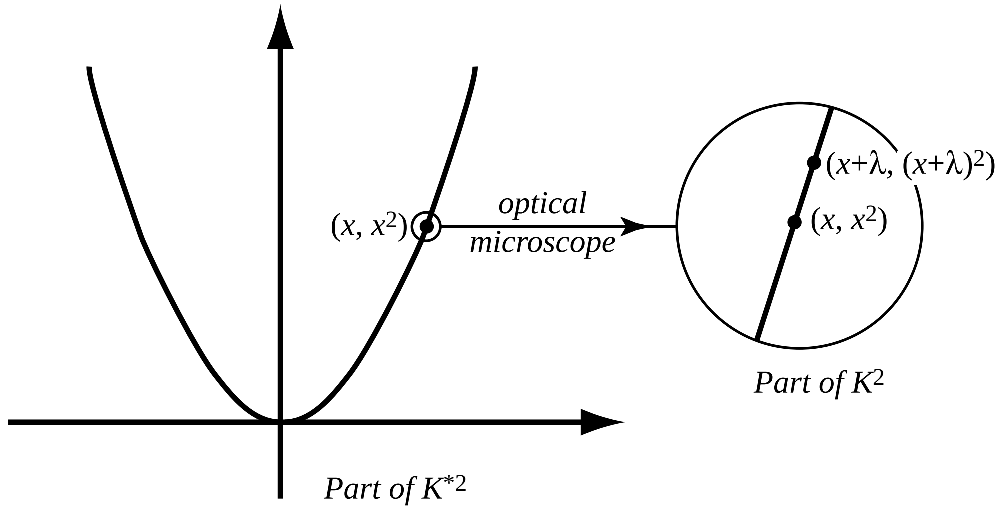

Figure 6

Lenses of different types are named as follows: if δ is infinitesimal then the lens is called a *microscope*, if δ is finite and not infinitesimal, then the lens is a *window*, if δ is infinite the lens is a *macroscope.* A window pointed at a point with at least one infinite coordinate is called a *telescope.* (These definitions modify those I gave in Tall, 1980, and bring them into line with the definitions of “microscope” and “telescope” used by Stroyan, 1972.) Microscopes reveal infinitesimal details (as in the above example), windows retain essentially the same scale, but used as telescopes they pull in structure at infinity, macroscopes allow one to “stand back” and see an infinite range. Microscopes and telescopes prove to be the most useful; through an optical microscope a differentiable function looks like a straight line and through an optical telescope two asymptotic curves look identical.

Of course a microscope does not reveal *all* the infinitesimal detail, just as in real life microscopes of different magnifications are needed to reveal different levels of accuracy. Given any $x, \delta \in K^*$, we say that x is of *lower order* than δ if x/δ is infinitesimal, it is the *same order* if x/δ is finite but not infinitesimal and *higher order* if x/δ is infinite. A δ-lens pointed at α reveals details which differ from α by the same order as δ. Higher order detail is too small to see and lower order detail is too far away to be in the field of view. Two points in the field of view which differ by a quantity of higher order than δ look the same through an optical δ-lens.

3. Applications

Infinitesimal notions prove a natural way to express the basic ideas of calculus and analysis, for instance, continuity, differentiability, the fundamental theorem of calculus, asymptotes and convergence. Some of these ideas will now be illustrated by examples.

Example 1: Continuity. A function $f:D\to K$ is continuous at $a\in D$ in infinitesimal terms if

.$x\in D^*$, $st(x)=a$ implies $st(f(x))=f(a)$.

In other words, if x and a are infinitely close together, so are $f(x)$, $f(a)$. The function $f:D\to K$ given by $D = \{x\in K \mid x\geq -1\}$

$$f(x)=\begin{cases} 0 & \text{for } -1\leq x\leq 0 \\ 2x^2 & 0\leq x\leq 1 \\ x+1/x & x>1 \end{cases}$$

is continuous everywhere in D. For instance, it is continuous at a=0 because if $st(x)$=0, then x is infinitesimal and

$f(x)$=0 for x negative,

$f(x) = 2x^2$ for x positive;

then

$st(f(x)) = 0 = f(0)$

in both cases.

(Note: Two common misconceptions about continuity are that a continuous function must be given by a single formula or that the graph must be in a single piece. Example 1 shows that a function given by different formulae on different subintervals can be continuous. The function $f(x)=1/x$ on the domain $\{x\in \boldsymbol{R} \mid x\neq 0\}$ gives an example of a continuous function whose graph is *not* "in one piece"; it is, however, in one piece over each connected piece of the domain, namely when x<0 or when x>0. A further discussion of this idea occurs in Tall, 1982.)

Example 2: Differentiability. If a function $f:D\to K$ *is* differentiable at $a\in D$, then an optical microscope pointed at a reveals the graph as being a straight line. In infinitesimal terms we say that f is differentiable at $a\in D$ with derivative $f'(a)\in K$ if

$$st\left(\frac{f(a+\varepsilon)-f(a)}{\varepsilon}\right)=f'(a)$$

for all (non-zero) infinitesimal ε with $a+\varepsilon\in D^*$.

For instance, the function f of example 1 is differentiable at x=0 with derivative $f'(0)$ = 0 because $(f(0+\varepsilon) - f(0))/\varepsilon$ equals

0, for ε<0

and

2ε, for ε>0.

In each case the constant part is 0.

Viewing the graph of f through a microscope pointed at (0, 0) reveals an infinitesimal portion of the graph as an (optical) straight line.

The same function is not differentiable at x=1, however, for a simple calculation gives

$$st\left(\frac{f(1+\varepsilon)-f(1)}{\varepsilon}\right)=\begin{cases} 4 \text{ for } \varepsilon<0 \\ 0 \text{ for } \varepsilon>0 \end{cases}.$$

It is a simple extension of the theory to see that the left derivative and right derivative both exist here, with and so values 4, 0, respectively.

In general, if left and right derivatives both exist, viewed through an optical microscope the graph looks like the meeting of two straight half lines (Fig. 7).

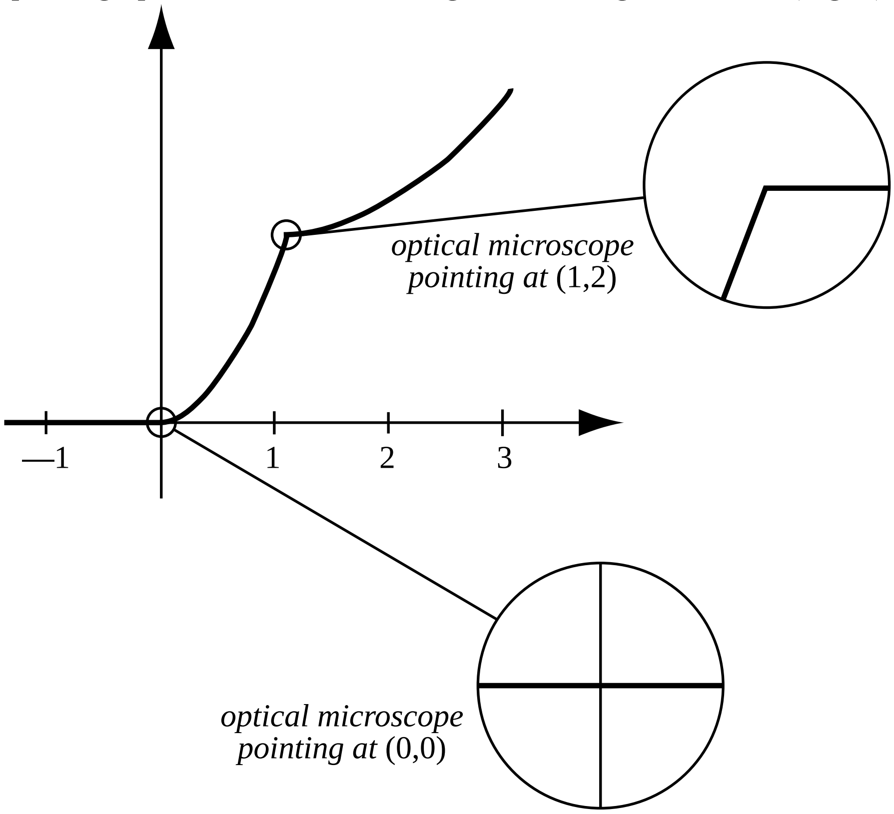

Figure 7

Example 3: Asymptotes. Asymptotic curves look identical when viewed through an optical telescope (Fig. 8). For instance, the curves $f(x)=x+1/x$ and $g(x)=x$ are asymptotic. If a δ-lens is pointed at (ω, ω) for δ=1 and infinite ω, then for finite h and $x=\omega+h$ we have the difference between y-coordinates is

$$st((f(\omega+h)-g(\omega+h))/1)$$

$$= st(\omega+1/(\omega+h)-\omega) = 0.$$

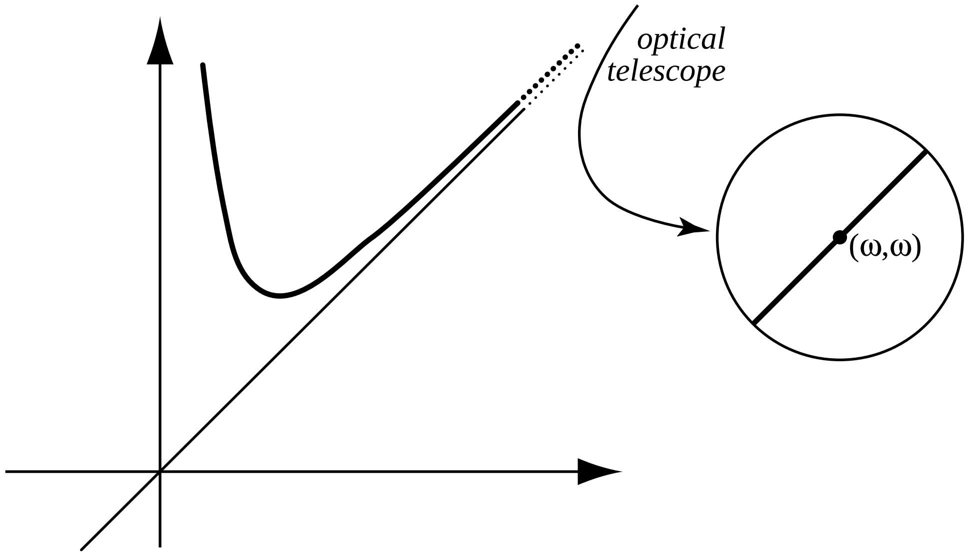

Figure 8

Example 4: Integration. Integrating a continuous function is straightforward in infinitesimal terms, thinking of the integral geometrically as the area under the graph. Let $f{:}D \to K$ be a continuous function and suppose that D contains the interval $[a, b]$, then if $A(t)$ is the area under the graph from $x = a$ to $x = t$, we have $A(t + \varepsilon) - A(t)$ is the area from $x = t$ to $x = t+\varepsilon$. Taking ε to be infinitesimal, then the continuity of f gives $f(t+\delta)$ to be infinitesimally close to $f(t)$ for all $t+\delta$ in the interval from t to $t+\varepsilon$. Viewing the picture through an *astigmatic* lens (Fig. 9), which leaves the vertical scale unchanged, but expands the horizontal scale by a factor $1/\delta$, and taking constant parts to lose the infinitesimal detail, the infinitesimal element of area $A(t+\varepsilon)-A(t)$ is revealed as a *rectangle*, height $f(t)$, width ε. Thus

$$A(t+\varepsilon)-A(\mathrm{t}) = f(t) \times \varepsilon + \text{higher order terms}$$

and so

$$\frac{A(t+\varepsilon)-A(t)}{\varepsilon} = f(t) + \text{infinitesimal terms.}$$

Taking the constant part, we get

$$A'(t) = f(t).$$

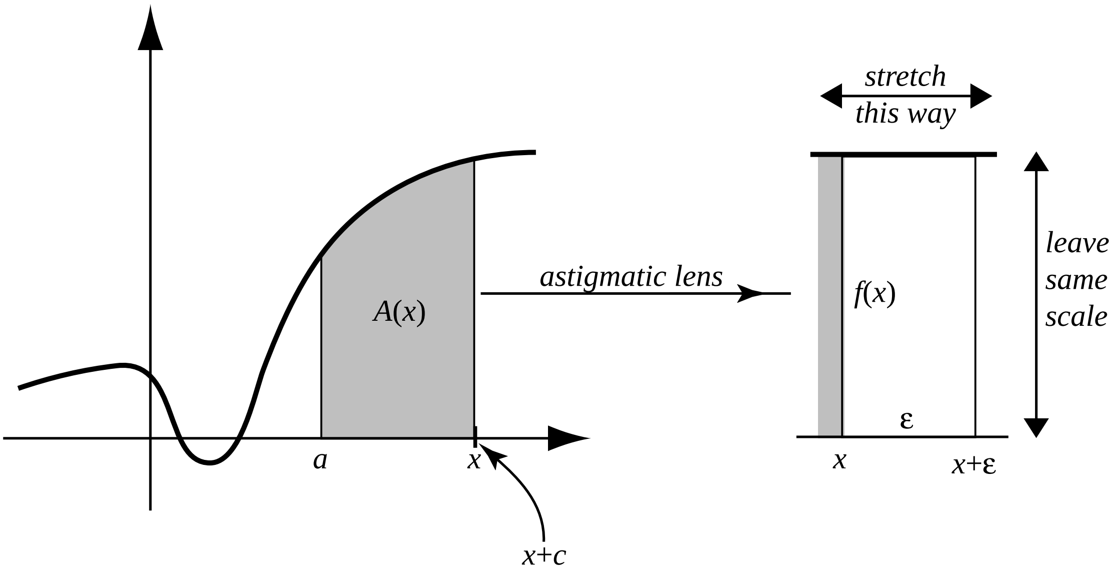

Figure 9

Example 5: Convergence. All the usual convergence notions can be dealt with in simple infinitesimal terms. For instance, a sequence (a_n) converges to a if $st(a_N) = a$ for all infinite N. (Think of the sequence as a function $a{:}\boldsymbol{N}\to K$ where $\boldsymbol{N}$ is the set of natural numbers, extend to $a{:}\boldsymbol{N}^*\to K^*$ and take infinite N in $\boldsymbol{N}$) Likewise. a sequence (a_n) is a Cauchy sequence if $a_M - a_N$ is infinitesimal for all infinite $M, N \in \boldsymbol{N}^*$.

The most enlightening cases occur with convergence of *functions*. For instance, the Fourier series with

$$F_n(x) = \sin x + \tfrac{1}{3}\sin 3x + \ldots + \frac{\sin(2n-1)x}{2n-1}$$

has the limit

$$F(x) = \begin{cases} -\pi/4 & \text{for } -\pi < x < 0 \\ 0 & \text{for } x = 0, \pi \\ \pi/4 & \text{for } 0 < x < \pi \end{cases}$$

with values repeating with period 2π.

The sequence of *continuous* functions F_1, F_2, … has a *discontinuous* limit F. Furthermore, the function F_n has a maximum at $x = \pi/(2n)$ which persists as n grows large, giving spikes approximately 1.179 times higher than the horizontal part of the limit function. This is called Gibb’s phenomenon. In ordinary analysis the behaviour seems somewhat anomalous. But in infinitesimal terms it has a simple explanation (Cleave, 1971). For infinite n the graph is similar in nature to the picture for large n. The Gibb’s spike still occurs at $\pi/(2n)$ but now this x-value is infinitesimal and so the graph is almost a vertical line segment. Viewed through an optical window the picture is a vertical line segment through the origin, rising up to the Gibb’s spike then (with infinitesimal wobbles) it settles down to the horizontal segment of the limit curve. Thus infinitesimal analysis reveals a behaviour familiar to any engineer trying to generate a square wave which does not have such a natural explanation in the limit theory of ordinary analysis.

Rather than perform the necessary calculations, an analogous phenomenon can be demonstrated using graphs made out of straight lines. The function

$$f_n(x) = \begin{cases} -1 & \text{for } x < -1/n \\ nx & \text{for } -1/n \le x \le 1/n \\ 1 & \text{for } x > 1/n \end{cases}$$

(Fig. 10).

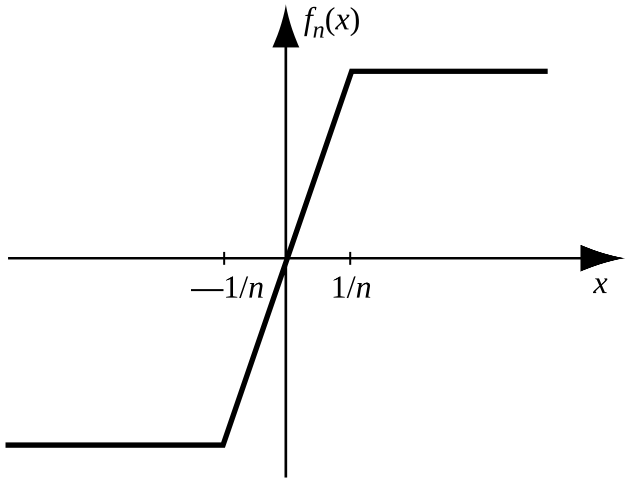

Figure 10

tends to the limit function

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

(Fig.11).

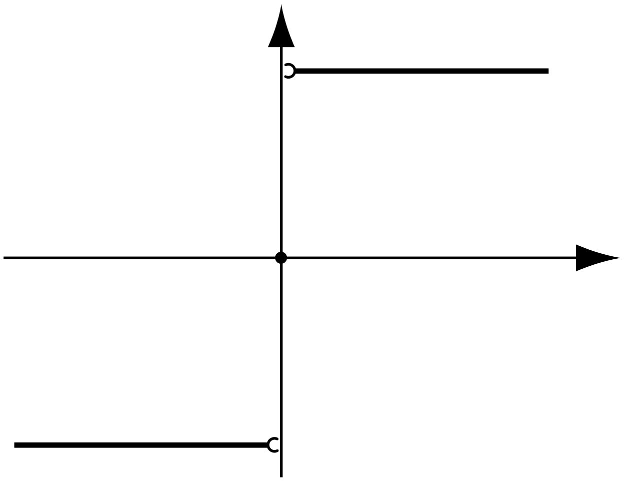

Figure 11

But for infinite N, the graph of f_N is given by the same formula as for a finite integer n, namely

$$f_N(x) = \begin{cases} -1 & \text{for } x < -1/N \\ Nx & \text{for } -1/N \le x < 1/N \, . \\ 1 & \text{for } x > 1/N \end{cases}$$

Because N *is* infinite, the line segment between $x=-1/N$ and $x=1/N$ has infinite gradient. Looking at the graph of f_N through an optical window given a picture which is the natural limit picture of the graphs of f_N, though it is not the graph of a function, namely a horizontal half-line $x=-1$ for $x<0$, a vertical line segment from (0,–1) to (0,1) and a horizontal half-line $x=1$ for $x>0$ (Fig. 12).

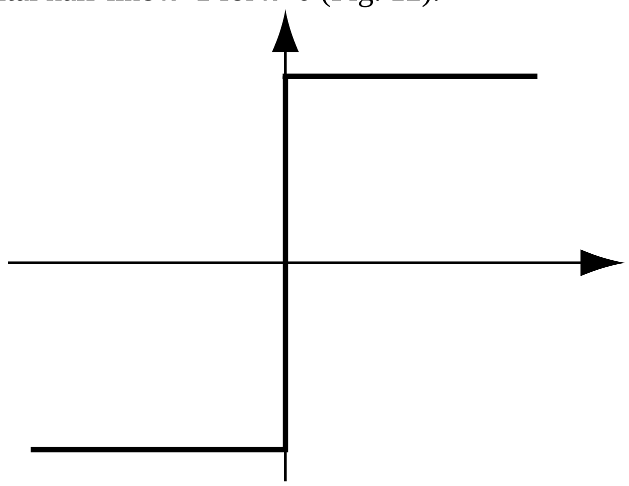

Figure 12

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