Intuitive infinitesimals in the calculus

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Intuitive infinitesimals

Intuitive approaches to the notion of the limit of a function

“$\lim_{x \to a} f(x) = l$” or “$f(x) \to l$ as $x \to a$”

are usually interpreted in a dynamic sense

“$f(x)$ gets close to l as x gets close to a.”

The variable quantity x moves closer to a and causes the variable quantity $f(x)$ to get closer to l. As an implied corollary resulting from the examples they do, students often believe (see [5]) that $f(x)$ can never actually reach l. For $l = 0$ this gives a pre-conceptual notion of “very small” or “infinitely small.” A limit is often interpreted as a never-ending *process* of getting close to the value of l rather than the value of l itself.

For instance, using the Leibnizian notation, students with a background of intuitive limiting processes might interpret dx as

$$\lim_{\delta x \to 0} \delta x.$$

The latter is not thought of as zero, but the process of getting arbitrarily close to zero.

In the same way

$$\lim_{n \to \infty} a_n$$

is usually regarded as a never-ending process, so that

$$0 \cdot 999\ldots9\ldots$$

is regarded as *less* than 1 because the process never gets there.

The extent of these phenomena will depend on the experiences of the students. In a questionnaire for mathematics students arriving at Warwick University, they were asked whether they had met the notation

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

and, if so, they were asked the meaning of the constituent parts δx, δy, dx, dy. Of course some would have been told that dx and dy have no meaning in themselves, only in the composite symbol $\frac{dy}{dx}$, a few others may have been told that dx is any real number and $dy = f'(x)dx$. Out of 60 students completing the questionnaire I classified their interpretation of dy as follows:

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dy has no meaning by itself....................................15
dy means "with respect to *y*" (as in integration)................ $7\frac{1}{2}$
dy is the differential of *y* .... $7\frac{1}{2}$
dy is an infinitesimal increase in *y*................................$11\frac{1}{2}$
dy is a "*very* small" or "smallest possible" increase in *y* .... $4\frac{1}{2}$
dy is the limit of δy as it gets small................................. 6
No response 8

The first three categories total 30 (including $\frac{1}{2}$'s which represent students giving more than one answer) which might be considered "orthodox" the next three categories total 22 which contain pre-conceptual infinitesimal notions.

The case of $0\cdot999\ldots9\ldots$ has been reported on several occasions (including [5], [10]). Out of 36 first year university students, 14 thought $0\cdot\dot{9} = 1$, 20 thought $0\cdot\dot{9} < 1$ and 2 hedged their bets.

Many reasons given incorporated infinitesimal arguments:

> $0\cdot\dot{9}$ is just less than 1 but the difference between it and 1 is infinitely small.

> I think $0\cdot\dot{9} = 1$ because we could say "$0\cdot\dot{9}$ reaches one at infinity" although infinity does not actually exist, we use this way of thinking in calculus, limits, etc.

Such intuitive notions of infinitesimals are reinforced in the English sixth form by occasional use of notations like

$$\lim_{x\to1^-}\frac{1}{x-1}=-\infty$$

where 1^- may then be interpreted as being infinitesimally smaller than 1 and $-\infty$ is "negative infinity".

A further corollary of this type of thinking is that the pictorial number line may be considered to include infinitesimals and infinite elements.

Of course, such infinitesimal notions as these are not at this stage part of a coherent theory and may lead to conflicting conclusions. If one first notes that $\frac{1}{3}=0\cdot333\ldots$, then asks the meaning of $0\cdot999\ldots$, a student who earlier had stated that $0\cdot\dot{9} < 1$ might now assert that $0\cdot\dot{9} = 1$. (Fourteen of the twenty students in the previously mentioned investigation reacted in this way.)

The notion of relative rates of tending to infinity (for instance 2^n tends to infinity faster than n^2) can lead to the notion of relative *sizes* of infinity, and deductions such as

$$\lim_{n\to\infty}\frac{n^2}{2^n}=\frac{\text{"infinity"}}{\text{"a bigger infinity"}}=0\ .$$

Though such notions are normally expunged in formal analysis courses they often remain as intuitive techniques.

In many ways, therefore, limiting processes in analysis lead to a cognitive feeling for intuitive infinitesimals.

Historical considerations

The notion of infinitesimal as a variable quantity which approaches zero has a very respectable antecedent in the work of Cauchy in the first half of the nineteenth century. This was essentially banished from formal mathematics by the $\varepsilon-\delta$ techniques of Weierstrass in the second half of the century. Infinitesimals did not die. Hilbert used then as late as the 1920s and applied mathematicians continued to find them useful. In the 1960s Robinson [4] legitimised their full use in analysis through logical constructions, but the mathematical community at large failed to warm to them.

Keisler's experiment teaching calculus with infinitesimals

An approach to teaching introductory calculus with infinitesimals was developed by Keisler [2] based on a teaching experiment during 1973–4. Infinitesimals were handled in a gentle axiomatic way and a supplementary text for supervisors [3] demonstrated a viable approach to Robinson's non-standard analysis using set theory instead of first-order logic. Comparing students following Keisler's approach with a control group, Sullivan [6] demonstrated that those using infinitesimal techniques subsequently had a better appreciation of $\varepsilon-\delta$ techniques as well.

Meanwhile, Bishop [1] fiercely criticised Keisler's text for adopting an axiomatic approach when it is not clear to the reader that a system exists which satisfies the given axioms. Although Bishop's review adopts an extreme viewpoint, the benefits reported by Sullivan have failed to convince the vast majority of mathematicians to switch to infinitesimal techniques.

Existence of infinitesimals

When we speak of "existence", we may do so from several different points of view. Bishop's criticism is directed at the fact that the axioms of non-standard analysis assert that certain concepts exist which cannot be constructed in any genuine sense. For instance we take (a_n) to be the sequence of decimal places in the expansion of π then $a_1=3$, $a_2=1$, $a_3=4$, and so on. Non-standard analysis not only asserts that infinite hyper-integers exist, but that a_H is defined for each positive infinite hyperinteger H. One may even prove that a_H is an ordinary integer between 0 and 9, yet no-one can give a precise value of a_H (which is only reasonable because H is not precisely specified).

The students in Keisler's experimental programme were not concerned with existence in this sense, but with what I shall term *cognitive existence*, that the concepts become part of an acceptable coherent structure in their mind. As we shall see, belief in cognitive existence is improved by coherent use of the concepts.

A smaller system with infinitesimals suitable for the calculus of Leibniz

My sympathies very much lie with Robinson and Keisler. Not only have infinitesimals played a large part in the historical development of the calculus, they still occur intuitively in standard analysis. I therefore developed a simpler system of calculus with infinitesimals published in [8], which proved to be sufficient to handle the calculus of Leibniz [7]. Briefly one adjoins a single infinitesimal ε to the real numbers (together with the necessary algebraic expressions in ε), giving the system of superreal numbers. The algebraic expressions concerned include power series in ε and allow the use of infinitesimal techniques for the calculus of analytic functions. The system is nowhere near as powerful as Robinson's, which can handle *all* functions, but it does not require any logical techniques, only straightforward algebra. There is also an allied geometrical theory which allows one to view graphs using infinite magnification factors which give real pictures of infinitesimal phenomena. More important, there is a correspondence between functions f and their values $f(\varepsilon)$ at $x=\varepsilon$, giving a correspondence between functions which tend to zero and infinitesimals, recalling the historical precedent of Cauchy, [8].

In the Spring of 1980 this superreal theory was used as an auxiliary introduction to non-standard analysis (as in [3]) for third year honours university mathematics students with two years of standard analysis. The reasons behind the superreal preface were threefold:

1) *pedagogical*: an initial approach to infinitesimals given them cognitive existence in complementary algebraic and geometric terms,
2) *logical*: no first order logic is necessary,
3) *historical*: the system has certain properties in common with the calculus of Leibniz.

In the course the superreals were not explicitly constructed (though they easily could have been). The students were just introduced to them as power series which could be manipulated algebraically and visualised geometrically. The hyperreals of the non-standard part of the course, on the other hand, were first described axiomatically, then constructed using

Zorn's Lemma as in [3]. Immediately following the construction, the students were asked to respond to the following question:

> Do you consider the following exist as coherent mathematical ideas? Respond as follows:
>
> 1. definitely yes 2. fairly sure 3. Neutral/no opinion
> 4. confused 5. fairly sure not 6. definitely not.

The responses were as follows:

N=42	1	2	3	4	5	6
natural numbers	40	2	0	0	0	0
real numbers	39	3	0	0	0	0
complex numbers	32	8	1	0	1	0
infinitesimals	23	9	8	0	2	0
supperreal numbers	18	12	8	2	2	0
hyperreal numbers	15	7	11	6	3	0

Notice that more believe in infinitesimals *per se* than in either infinitesimal system and more believe in superreals than hyperreals, though the difference was not very significant. The students lacked the sophistication to notice that the existence of the superreals had not been shown by an explicit construction. They had little or no previous experience of Zorn's Lemma and the subtleties raised by Bishop escaped the majority.

Five weeks later at the end of the course, in response to the same question, 46 students responded:

N=46	1	2	3	4	5	6
natural numbers	43	3	0	0	0	0
real numbers	39	5	1	0	0	0
supperreal numbers	27	13	4	2	0	0
hyperreal numbers	15	20	5	4	2	0

Familiarity had warmed their feelings towards the systems with infinitesimals. Although 11 out of 46 had studied the construction of the hyperreals at length, this warming cannot be ascribed to logical deduction alone. A small number remained cognitively uncertain of the infinitesimal systems (categories 3–6) but the majority were at least fairly sure (categories 1, 2).

Another questionnaire showed that only 7 out of 46 considered it a flaw that the superreals were not formally constructed, only 10 were positively unhappy with the use of Zorn's Lemma in the construction of hyperreals, though 22 felt that a construction was essential. With the lowering of cognitive belief in existence, a construction becomes more necessary.

An essential difference between these students and those in Keisler's experiment was that these have extensive experience in ε–δ analysis. When several students, representing a cross-section of all abilities, were interviewed in depth after the course, it became clear that their heavy investment in ε–δ analysis made them have a high regard for it, even though it still presented them with technical difficulties.

The vast majority of university teachers have a similar investment, so a cultural resistance to non-standard analysis is only natural. In addition to Keisler's move to replace logical notation by a formulation more acceptable to mathematicians, a factor which may help the reconciliation of ε–δ analysis and infinitesimal techniques is likely to be the Cauchy-like notion of the correspondence between functions which tend to zero and infinitesimals. As we have seen, the limits of such functions in standard analysis already contain the seeds of intuitive infinitesimal concepts.

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