Calculations and Canonical Elements

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Part 1

1. Introduction

Equivalence relations are the basis of modern approaches to many topics in school mathematics, from the first ideas of cardinal number (through matching activities and correspondence between sets) through definitions of negative numbers (using ordered pairs of natural numbers), to equivalence of fractions, modular arithmetic, vectors, and many more advanced topics. We contend that these approaches to the subject have been based on an inadequate theoretical framework, causing an unnecessary schism between traditional mathematics and "modern" approaches. The missing link is the concept of a *canonical element.* Reintroducing this idea gives a much more coherent relationship between the structural elegance of equivalence relations in modern mathematics and the traditional aspect of computation. We tackle this in Part 1 of this paper, which follows. This in turn gives a clearer insight, as we shall see in Part 2, into certain technical and educational problems.

2. Equivalence Relations

We shall assume that the reader is familiar with the notion of an equivalence relation $\sim$ on a set S. Being a relation means that for each ordered pair of elements $a, b \in S,$ we either have $a \sim b$ (a, b are related), or $a \nsim b$ (a, b are not related). An equivalence relation satisfies the further properties:

(E1) $a \sim a$ for every $a \in S$

(E2) if $a \sim b,$ then $b \sim a$

(E3) if $a \sim b, b \sim c,$ then $a \sim c.$

This partitions the set S into *equivalence classes,* where for any $x \in S$ we denote the equivalence class containing x by

$$E_x = \{y \in S \mid x \sim y\}.$$

We find that $E_x = E_y$ if and only if $x \sim y$, and if $x \nsim y$ then $E_x \cap E_y$ is empty, so E_x, E_y are disjoint. An alternative notation for E_x which we shall often use is $[x]$, so

$$[x] = \{y \in S \mid x \sim y\}.$$

Further discussion of equivalence relations may be found in [9] or [11, p. 74 ff.]. Here we content ourselves with some typical examples met in teaching, which we shall explore further as this article develops.

Example 1: Cardinal Number

Two sets are equivalent if there is a bijection (or one-to-one correspondence) between them. Some modern syllabuses in primary schools, inspired by Piaget (for instance [1], [5], [7]), begin by laying emphasis on matching activities as a preliminary basis for cardinal number. In other quarters (for example [6, p. 192]), this approach is vigorously rejected. We shall consider this conflict later on.

Example 2: Egyptian Number Symbols

(An interesting historical example which could be used with advantage in primary schools.) The ancient Egyptians used stroke | to denote a unit, a hoop ∩ to denote 10 and a scroll 𓍢 to denote 100, with other symbols for higher powers of I0. Two collections of such symbols are equivalent if they represent the same number, for instance 12 strokes | | | | | | | | | | | | are equivalent to one hoop and two strokes ∩ | |.

Example 3: Integers (Positive and Negative) using Ordered Pairs of Natural Numbers

Let N denote the set of natural numbers 0, 1, 2, 3, . . . If wish to "construct" the integers Z (including negative integers) then one method is to consider ordered pairs (m, n) which $m, n \in N$ and to define an equivalence relation.

$$(m_1, n_1) \sim (m_2, n_2) \text{ if and only if } m_1 + n_2 = m_2 + n_1$$

Thus, for example, $(1, 0) \sim (2, 1) \sim (3, 2) \sim \ldots \sim (n+1, n) \sim \ldots$, $(2, 0) \sim (3, 1) \sim (4, 2) \sim \ldots \sim (n + 2, n) \sim \ldots$.

This cunning device yields a "definition" of the integers (positive and negative) as the equivalence classes, namely

$$^{+}1 = \{(1, 0), (2, 1), \ldots (n+1, n), \ldots\},$$

$$^{+}2 = \{(2, 0), (3, 1), \ldots (n+2, n), \ldots\},$$

$$\ldots$$

also

$$0 = \{(0, 0), (1, 1), \ldots, (n, n), \ldots\}$$

$$^{-}1 = \{(0, 1), (1, 2), \ldots, (n, n+1), \ldots\}\}$$

$$^{-}2 = \{(0, 2), (1, 3), \ldots, (n, n+2), \ldots\}$$

In essence the integer ^{+}k or ^{-}k denotes the amount by which the first element in an ordered pair exceeds the second, as can be seen by looking at the above instances.

This method (which we do not advocate for teaching purposes) has been called (rather affectedly) the "Doncaster method" in [1], though it is based on a well-known mathematical construct and has been used before by other mathematical educationists in other forms (such as [3] or the Nuffield Primary Mathematics Project).

Example 4: Fractions

Two fractions p/q, m/n (where p q, m, n are non-zero natural numbers) are equivalent (and we write simply $p/q = m/n$), if $pn = mq$. (This can also be treated using ordered pairs in a manner analogous to example 3 by writing

$$(p, q) \sim (m, n) \text{ if } pn = mq,$$

but we won't dwell on that here. For details, see [11, Chapter 10], or [1].)

Example 5: Modular (or "Clock") Arithmetic

We shall work modulo 3 for simplicity (the same phenomenon, of course, works with 3 replaced by any other positive integer). Two integers are "equivalent modulo 3" if their difference is divisible by 3. For instance, 1 is equivalent to 4, 2 is equivalent to 368, –1 is equivalent to 2. The equivalent classes are:

$$E_0 = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\} = E_3 = E_6 = \ldots ,$$

$$E_1 = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\} = E_4 = E_7 = \ldots ,$$

$$E_2 = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\} = E_5 = E_8 = \ldots .$$

of course we also have $E_0=E_{-3} = E_{-6} =\ldots$, and so on.

These examples will be sufficient for our purposes, but many more occur in mathematics, see, for instance [9]. We shall consider other examples in the second article.

3. Canonical Elements

Wherever there is an equivalence relation, we can select canonical elements. Quite simply, we do this by choosing a single element from each class. Then that element is called the *canonical element* (or *representative)* for that class. As an example we can take the set of children in a classroom and use the equivalence relation "is the same sex as", which divides the children into two equivalence classes: the boys and the girls. (For this example we shall assume the children are mixed to prevent the discussion being trivial.) A canonical choice of elements is then one boy (a particular one, say, Joe), and one girl (say, Ann). It doesn't matter which boy or girl we select as long as we have in mind precisely one of each. We don't have to pick the best, or the biggest, or "the most typical", as long as we have just one from each equivalence class. There is no coloration in the meaning of a "canonical element" beyond that (arbitrary) choice.

In the mathematical examples we considered in the last section, however, there is in each case a "natural" choice of canonical element which is usually made by mathematicians. In fact these canonical elements were the stuff of mathematics before modern set theory was a twinkle in Cantor's eye. We shall see that it is the manipulation of these canonical elements that constitutes the traditional art of calculation in mathematics.

Example 1: Cardinal Number

In school we are concerned only with finite sets. Counting a finite set of elements is performed by pointing to each element in turn and saying "one", "two", "three", … until we have pointed at each elements precisely once. The last number that we recite is the number of elements in the set. In practice this means that a set with (say) four elements is put into one-one correspondence with the set (one, two, three, four). All sets equivalent to the given set can also be put into one-one correspondence with (one, two, three, four), so the latter is a natural choice of canonical element. Of course, we don't have to wrap it up in set theoretic rigmarole. We shall return to this point in Section 6.

Example 2: Egyptian Number Symbols

The canonical choice (for numbers less than 1,000) is the representation which has less than 10 of each symbol. This corresponds to our decimal notation in the obvious way, for instance in the equivalence class containing I I I I I I I I I I I I (12 strokes), the canonical element is one hoop and two strokes, $\cap$ I I.

Example 3: Integers using ordered pairs of natural numbers.

If $m \geq n$, a sensible choice of canonical element in $E_{(m,n)}$ (the equivalence class containing (m, n)) *is* $(m-n, 0)$. For $m < n$ we choose $(0, n-m)$. In this way every ordered pair is equivalent to a canonical element of the form $(k, 0)$ for $k \geq 0$, or $(0, k)$ for $k<0$. Of course this makes $(k, 0)$ the canonical choice in the class ${}^{+}k$ and $(0, k)$ the choice for ${}^{-}k$. As we shall see later, there isn't much in a name.

Example 4: Fractions

The natural choice of canonical element for equivalent fractions is "the one in lowest terms".

Example 5: Modular Arithmetic

Working modulo 3, the natural choice of canonical elements in the classes E_0, E_1, E_2 are 0, 1, 2 respectively.

4. Calculations

All the examples we have mentioned above were specifically selected because we do calculations with them. In all cases we have an operation of addition and also multiplication. For the moment, let us concentrate on the former.

The primitive addition of cardinal numbers (say three plus five) may be achieved by choosing a set of three objects (in one-one correspondence with lone, two, three)) then a *disjoint* set of five objects, putting the two sets together and counting them: "one, two, …, eight" to find "three plus five is eight". There is a subtlety here in the choice of disjoint sets which is not found in adding two Egyptian numbers, where one simply takes the two collections of symbols all together. We can symbolise this using a (we hope) forgivable mixture of ancient and modem notation by writing 26+15 as:

$$\cap\cap||||||+\cap||||| = \cap\cap||||||\cap|||||$$

This combination of symbols is not in canonical form, but we can complete the calculation by replacing the answer by the canonical element in its class (effected by replacing ten strokes by an equivalent hoop), thus:

$$\cap\cap||||||\cap||||| = \cap\cap\cap||||||||||\ \ |$$
$$= \cap\cap\cap\cap|.$$

This, in essence, is the procedure that first school children use when they perform addition on an abacus (see, for example, [2, book 1, p. 9]). Here "twenty-six plus fifteen" is represented as:

tens	units
oo	oooooo
o	ooooo

which becomes, on exchanging 10 units for one ten:

tens	units
oo	o
oo	

which is 41.

In example 3, addition of ordered pairs of natural numbers may be defined by adding the components:

$(m, n)+(p, q)=(m+p, n+q)$.

This also allows us to add equivalence classes, for instance, to add $^{+}2$ and $^{-}3$, we select any elements we wish from the appropriate equivalence classes, say (2, 0) and (0, 3) to get

$$(2, 0) + (0, 3) = (2, 3).$$

Now (2, 3) is equivalent to (0, 1) and both are in the equivalence class $^{-}1$, hence

$$^{+}2 + {}^{-}3 = {}^{-}1.$$

In example 4, when adding fractions m/n and p/q, we put them over a "common denominator", say nq, to get:

$$m/n + p/q = (mq + np)/nq.$$

We then proceed to reduce the answer to lowest terms. In practice we may develop more refined techniques; we do not need to use the common denominator nq because the lowest common multiple of n and q will do, thus:

$$\begin{aligned} 1/2 + 1/6 = 3/6 + 1/6 &= 4/6 \\ &= 2/3 \text{ (in lowest terms).} \end{aligned}$$

Finally, example 5, modular arithmetic, involves similar principles. For instance, to calculate 2 + 2 (modulo 3) we first compute 2 + 2 = 4, then find the canonical element equivalent to 4, namely 1, so

$$2 + 2 = 1 \text{ (modulo 3)}$$

Some mathematicians prefer to write this as

$$2 + 2 \sim 1 \text{ (modulo 3)},$$

yet others prefer to use equivalence class notation:

$$E_2 + E_2 = E_{2+2} = E_4,$$

and, of course, $E_4 = E_1$ so we get

$$E_2 + E_2 = E_1.$$

As in this last case, all the examples can be interpreted using either equivalence classes or canonical elements. The modern approach favours equivalence classes, the traditional computational one favours canonical elements.

In each case we have a set S on which an operation of addition is defined. (It could be any other binary operation, such as multiplication, but for simplicity of explanation we shall use addition.) Also S has on it an equivalence relation.

The case of addition of integers

$$^{+}2 + {}^{-}3 = {}^{-}1$$

is typical, in that we can extend the addition of elements (in this case addition of ordered pairs (m, n)) to the equivalence classes themselves.

The method is this:

To add equivalence classes E_x, E_y form the equivalence class E_{x+y} containing $x+y$, and this is the answer. Thus

$$^{+}2 + {}^{-}3 = E_{(2,0)} + E_{(0,3)} = E_{(2,3)} = {}^{-}1$$

This is the modern formulation using only equivalence classes.

A traditional formulation, using only elements, follows the pattern of adding fractions:

$$1/2 + 1/6 = 3/6 + 1/6 = 4/6 = 2/3.$$

The link between these two types of formulation is the notion of a canonical element, as we see in Figure 1.

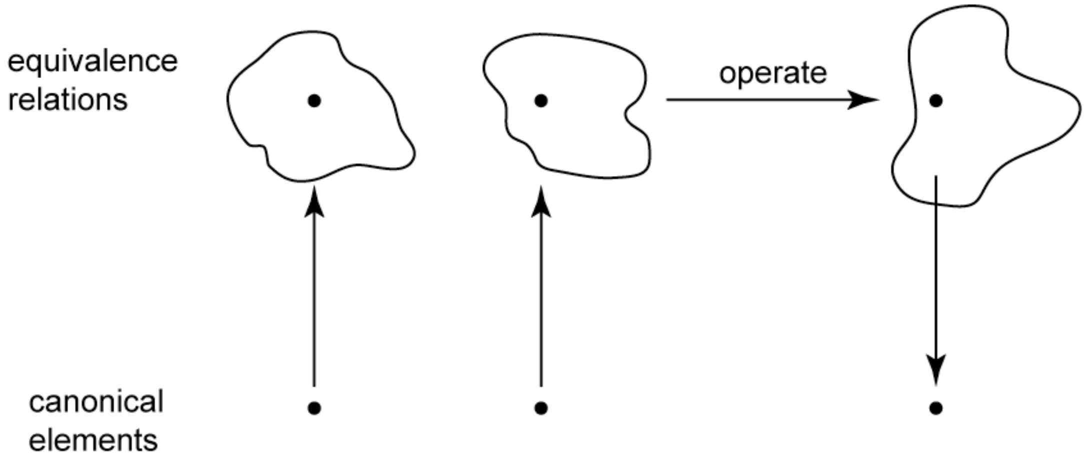

Figure 1

The upward vertical arrows correspond to finding the equivalence class to which an element belongs (usually easy), the horizontal arrow is the operation of addition on classes, the downward arrow is picking out the canonical element in a given class (usually harder and more algorithmically contrived).

The modern formulation usually involves only the top of this diagram (see Fig. 2).

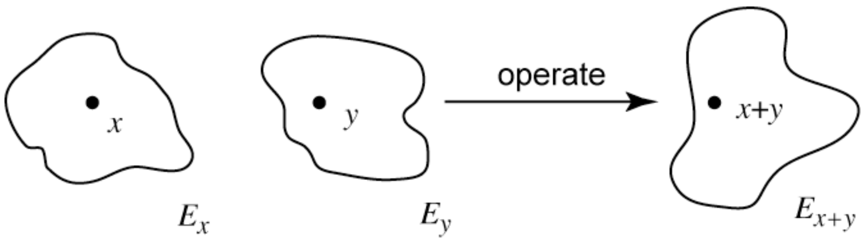

Figure 2

It is just the definition

$$E_x + E_y = E_{x+y} \, .$$

The traditional formulation usually avoids all reference to equivalence classes and works on the element level, as in Figure 3.

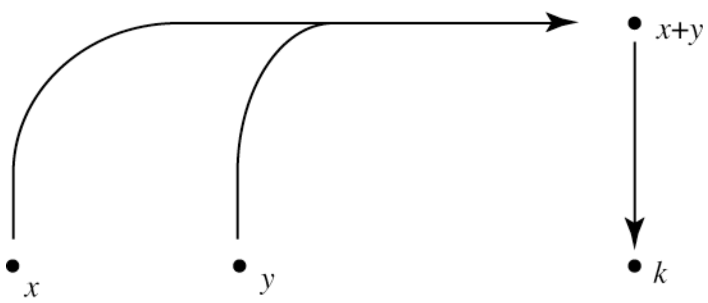

Figure 3

In this diagram we start with elements x, y, then operate to get $x + y$, then find the equivalent canonical element k.

The difference between Figures 2 and 3 represents the schism between modern and traditional approaches. The unification in a single framework is given by Figure 1. It is in analysing the total structure that we shall obtain a blend of the old and new. But before we do this we must digress for a while to consider an important technicality.

5. A Technical Problem

In carrying over a binary operation from elements to equivalence classes there is a technical problem which proves difficult for beginners to grasp yet which forms an essential ingredient in the whole picture. In defining the sum of E_x and E_y as E_{x+y}, we have skated over the problem of the name of the equivalence class E_{x+y}. For instance if x' is another element equivalent to x, then $E_{x'}$ and E_x are one and the same. The symbols $E_{x'}$ and E_x, are just different names for the same equivalence class. Suppose that we calculate $E_{x'} + E_y$ instead of $E_x + E_y$, then we would get the answer:

$E_{x'} + E_y = E_{x'+y}$.

The central problem is: can we be sure that the equivalence class $E_{x'+y}$ is the same as E_{x+y}? Worse still, suppose we compound the problem by considering another y' such that $y \sim y'$ (in which case $E_y = E_{y'}$). Would we find that

$E_{x'} + E_{y'} = E_{x'+y'}$.

is the same equivalence class as E_{x+y}? If we can't, then the whole process of defining the sum of E_x and E_y breaks down because we may get different answers E_{x+y} and $E_{x'+y'}$ depending on which elements we select from the equivalence classes to perform the computations. For the sum to carry over from elements to equivalence classes we require the condition

if $x \sim x'$ and $y \sim y'$, then we must also have $x+y \sim x'+y'$

As an example, consider addition modulo 3. Here, by computation, we have:

$E_1 + E_2 = E_3$,

but

$E_1 = E_4$, $E_2 = E_8$,

and

$$E_4 + E_8 = E_{12}.$$

Fortunately, we have $E_{12} = E_3$, (because 12 – 3 is divisible by 3), so we encounter no problem in this case. In fact we encounter no problem in any of the examples given, because in all cases, $x \sim x'$, $y \sim y'$, implies $x+y \sim x'+y'$ as the reader may verify.

But what of a general operation ∘ on a set S with equivalence relation ~ ? We might wish to carry over the operation o to equivalence classes by defining

$$E_x \circ E_y = E_{x\circ y}.$$

To do this requires the general property

$$x \sim x',\ y \sim y' \text{ implies } x\circ y \sim x'\circ y'.$$

If this does not hold, the whole thing breaks down. As an example (taken from [11, p. 77]), consider the operation on integers

$$x \circ y = x^y.$$

Denoting the equivalence class of n modulo 3 by $[n]$, we might attempt to define the taking of powers for numbers modulo 3 by:

$$[x]\circ[y] = [x\circ y]$$

or, in other words,

$$[x]^{[y]} = [x^{\,y}].$$

This does not work. For instance, if $x = 2$, $y = 2$, we get

$$[2]^{[2]} = [2^{\,2}] = [4] = [1].$$

But [2] = [5]

and

$$[2]^{[5]} = [2^{\,5}] = [32] = [2].$$

The classes [1] and [2] are *different*.

This is because

$$x \sim x',\ y \sim y' \text{ does } \textit{not} \text{ imply } x^y = (x')^{y'}$$

in general. As a counter example we have 2~2, 2~5, but $2^2 \nsim 2^5$. This warns us that, in general, just blindly pressing on without checking that

$$x \sim x',\ y \sim y' \text{ implies } x\circ y \sim x'\circ y'. \qquad (*)$$

can sometimes lead to nonsense in handling computations with equivalence relations.

The astute reader may notice that we do not get any such problems in defining operations on the corresponding *canonical elements,* at least as far as the definition itself is concerned. We can always define an operation on canonical elements alone by starting with canonical elements x, y, then forming the composite $x \circ y$; this of course need not be a canonical element, but it is equivalent to a unique canonical element k. We therefore define a new operation on canonical elements associated with $\circ$ by defining x, y to be the unique canonical element k which is equivalent to $x \circ y$.

For instance, if $x \circ y = x^y$, we can define a new operation on the canonical elements 0, 1, 2 (modulo 3) in this fashion. To compute 2^2 (where 2 is now thought of as a canonical element), we first compute 2^2 (as an integer) to get 4, then take the canonical element equivalent to 4 modulo 3, namely 1. In this way we are able to compute all powers x^y where x, y run through the values 0, 1, 2 (considered as canonical elements modulo 3).

Although we have been able to make such a definition, we haven't really gained anything, because the simple rules of powers break down. For example we can compute

$2^1 = 2$, $2^2 = 1$ (as canonical elements),

but

$(2^2)^2 = 1^2 = 1$

$2^{2\times2} = 2^1 = 2$

$(2^2)^2 \neq 2^{2\times2}$.

Hence in general we may have

$(x^m)^n \neq x^{mn}$ when computing with canonical elements modulo 3.

It is all a matter of swings and roundabouts: the difficulty simply pops up somewhere else.

Returning to the case where the fundamental property (*) *does* hold, we find genuine differences between the equivalence class approach and that using only canonical elements. If the operation $\circ$ on the elements has basic algebraic properties, for instance, associativity or commutativity, then these can be seen to carry over easily to the induced operation on equivalence classes. For instance, if we suppose that

$x \circ y = y \circ x$

then we find

$E_x \circ E_y = E_{x \circ y}$ (by definition)

$= \mathrm{E}_{y \circ x}$ (since $x \circ y = y \circ x$)

$= E_y \circ E_x$, (by definition again).

Thus a commutative operation on elements induces a commutative operation on equivalence classes; the analogous statement for associativity follows just as easily.

However, at the canonical element level there is more difficulty with the associativity proof. We compute $(x \circ y) \circ z$, first by finding $x \circ y$ equivalent to a canonical element k, then computing $k \circ z$ and finding the canonical element equivalent to this. On the other hand, in finding $x \circ (y \circ z)$, we first find the canonical element c which is equivalent to $y \circ z$, then, when we compute the canonical element equivalent to $x \circ c$, it is by no means clear we end up with the same element as before. In fact to show this is so we have to appeal to the full force of condition (*). We know

$x \circ y \sim k,\ y \circ z \sim c$

hence, using (*), we get

$k \circ z \sim (x \circ y) \circ z = x \circ (y \circ z) \sim x \circ c.$

Because $\sim$ is an equivalence relation, we deduce $k \circ z \sim x \circ c$, and so the two elements $k \circ z$, $x \circ c$ are equivalent to the same canonical element.

Thus the proof of associativity for the product of canonical elements has to use (*) for elements such as $x \circ y$, $y \circ z$, which may not be canonical, and we are forced to see the canonical elements within a broader framework.

This demonstrates that traditional computations with canonical elements are better viewed within the context of equivalence relations.

On the other hand, mathematicians must be able to compute, so they need in the end to be able to handle the manipulation of canonical elements. Hence the best solution is to gain a global view of the whole picture, placing canonical elements within the framework of the theory of equivalence classes. This we shall do in Part 2 of this article, with special reference to mathematical concepts taught in school.

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