# The Dynamics of Understanding Mathematics[1]

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## Introduction

Recent issues of *Mathematics Teaching* have begun what looks like being a very fruitful discussion on the nature of mathematical understanding, including Richard Skemp's article on "relational and instrumental understanding" (MT 77), Byers & Herscovics' addition of two more possible categories ("intuitive" and "formal") (MT 81), then John Backhouse's reply (MT 82). There must be nearly as many views of what constitutes "understanding" as there are mathematical educators, but I would like to inject in the discussion the suggestion that we consider the *dynamics* of mathematical understanding. By the latter I mean the constantly changing mental patterns that give rise to mathematical understanding and which characterise mathematical thinking. The brain is a hive of mathematical activity—it takes in information, processes it, distinguishes between things, sees similarities, makes deductions, forgets things, them remembers them later, it has mental blocks and leaps of insight. So many of theses activities are bewilderingly quick or subconscious that we cannot say what is happening in our own minds. A verbal commentary at the time is inadequate and subsequent explanation is often only a partial rationalisation of what we think has happened (or even what we think we think has happened).

As an example, consider the case of a four year old child who can count up into the hundreds and knows a few number bonds. He is asked "What's seven and seven?" and replies (after a brief pause) "fourteen". When asked "Why?" he replies "because it's two off sixteen". "Oh, I see, … why is that?" … "Because eight and eight are sixteen and it is two less." Now we can postulate the child remembered 8+8=16 (though not in that notation) and had a partial relational understanding of number to use this to deduce that 7+7=14.

A week later he was asked again, "What's seven and seven?" and he couldn't remember, then after a long pause he said, "fourteen". "How do you know?" "It just is." "OK, what's eight and eight?" "Eighteen." His elder brother (aged seven) was given his own sum to think about—"eighty plus eighty" (a real "toughie") whilst his younger brother considered his own problem. The elder brother replied "a hundred and sixty," and his younger brother immediately responded, "it's sixteen". "How do you know?" "Well he …", (points to his older brother), "said a hundred and sixty, then 1 just knowed[2]."

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[1]This is the submitted version of a paper published in *Mathematics Teaching*, (1978), **81**, 50–52. At the request of the editor, the theoretical mathematical model later in the article was replaced by practical examples.

[2] Notice his use of a "regularisation" of the verb ending as "knowed", rather than the irregular "knew". The way in which he generates language by regular patterns is possibly similar to his use of patterns in arithmetic.

How did he know? Did the “six” in “a hundred and sixty” jog his memory, or did he visualise the more complex problem as ten times his own in some sense? He has some experience with “tens” sums, but not as big as “eighty plus eighty”, and at this stage he didn't want to play any more.

Any description of mathematical understanding must allow for this kind of dynamic process—understanding and deducing one week, forgetting and remembering the next.

The example given above was one of success. It is far more difficult to describe the process of *not* understanding. We have all experienced the blank look of a mental blockage, and here the person concerned is often unable to explain the cause of the difficulty. It is on this count that the “tetrahedron of understanding” in Byers & Herscovics article may be found wanting. Briefly they suggest that it is helpful to consider their four kinds of understanding (instrumental, relational, intuitive and formal) as vertices of a tetrahedron, then any blend of the four may be represented as a point inside the tetrahedron. Such a representation only exhibits the *ratio* between the four kinds, as may be seen by considering the centroid, where all kinds are present in equal proportions. But how does one distinguish in this representation between all four being fully present, or all four being totally absent? (If one wishes to represent four quantities $x_1$, $x_2$, $x_3$, $x_4$, each lying between 0 and 1, this requires a point in 4-space, $(x_1, x_2, x_3, x_4)$, not a point in a tetrahedron in three.) Furthermore, using the classification of understanding into four (or more) kinds, we must be prepared to acknowledge the type(s) of understanding being used as a function of time. In the example given, perhaps the memory of 8+8 = 16 is an *instrumental* act, then the realisation that 7+7 is “two off” a *relational* one and, for all we know, the final deduction (“he said a hundred and sixty, then I just knowed”) an *intuitive* one?

This minor quibble apart, the Byers & Herscovics paper gives much valuable food for thought, building on Richard Skemp’s ideas. I’d now like to suggest a dynamic interpretation which sees these ideas in a different light, and I hope answers John Backhouse’s fears about the proliferating categories of understanding by seeing them all within a single development.

## The dynamics of understanding

By a schema I shall mean a coherent pattern of mental activity in the mind of an individual. This exists in time and changes with time. As a child gains experience of life in general, and mathematics in particular, the constraints on his mental activity change. They can develop and get more versatile, they can decay, they can change by conscious and subconscious reformulations of ideas as the child attempts to make a coherent pattern out of the universe he lives in. Understanding which comes about through this search for coherence I would term “relational understanding”. The example “seven and seven is fourteen … because it’s two off sixteen,” exhibits this kind of understanding, though it includes the remembering of "8+8=16" to produce the final result. "Instrumental understanding", on the other hand can simply be an exercise of the memory, or worse, be characterised on occasions by compartmentalisation of ideas, not wishing to make an overall pattern and preferring the comfort of a limited closed system. This closure manifests itself as "rules without reason" in Skemp's description. I feel personally that the distinction between these two types of understanding is often one of attitude—the desire to make a coherent pattern out of different pieces of information distinguishing the one from the other. Note that relational understanding, according to this viewpoint, can occur in very rudimentary situations, as with the child's addition example, where he not only had very few number bonds available to him, he also could not even write the numerals, yet he understood certain relationships between numbers.

What of "intuitive understanding"? I consider that this occurs with a developing schema that is insufficient for the purpose at hand, yet there are facets of the problem to be understood which seem to link with the current available schema. In fact, apparent linkages with the current schema may exercise such a pull that the jump in mental state to the final state may come as a blinding revelation, strongly imprinting it on the memory. The important distinction in intuitive understanding is that the person concerned has not reflected on his schema and has not rationalised the way that he thinks about it.

John Backhouse, in his article, does not find the concept of intuitive understanding necessary, though he is more sympathetic to the notion of "intuitive thinking". However, earlier in his article he mentions the experience of ideas fitting into place "ah, now I get it", "it's clicked", and so on. Such a feeling can eminently occur when a problem is solved "without prior analysis" which characterises the Byers and Herscovics' definition of intuitive understanding. In fact the 'aha' experience of such a solution can have such a strong imprint on the individual that it seems even more true than a deduction made with cool unblemished logic.

In considering the formal category, John Backhouse suggests that certain examples of "formal understanding" given by Byers & Herscovics are no more than "understanding of form". Looking at the matter in terms of the schematic development of the individual, it may he helpful to distinguish two clearly distinct interpretations. On the one hand, formal understanding may be the type of understanding in which the individual has reflected on his schema and rationalised his thinking as to how it fits together coherently. This is an individual thing, and is akin to the use of the term by Piaget. On the other hand, there is the ability to put the mathematics in a formal context, using the correct notation and so on. This is a corporate thing where the individual has learnt to share the schemas of mature mathematicians in the topic under consideration. Byers and Herscovics give an example which they claim shows formal understanding is absent in which the student writes

$$f(x) = x^2 = f'(x) = 2x.$$

This is clearly a case of lack of formal understanding in the corporate sense because the student uses the wrong notation, but if the second "=" is read as "implies" then the student may have an individual formal understanding of what going on without manifesting it in the correct notation.

**Towards a mathematical model of understanding and non-understanding**

At this stage I do not wish to refine the above explanations any further, or add any additional categories of "understanding" (e.g. we have "intuitive" and "formal", so what has happened to Piaget's "concrete"), simply to stress the role of the dynamic development in the individual in the understanding process.

Having suggested that understanding be seen within this wider context, what of *non-understanding*? How does this fit into the picture? In a recent article in "Mathematics Teaching", written with Rolph Schwarzenberger, we discussed some of the factors which could cause non-understanding in a particular development of a mathematical topic [5]. That article was written with a particular model of psychological development in mind which was not explicitly mentioned at the time, though it has been explored in [4]. Perhaps it is pertinent to mention it here, because it sets non-understanding in an appropriate light which is complementary to understanding. It also links up with John Backhouse's emphasis on what has been termed (though not by him in his article), the "aha" experience. Whilst the "aha" experience can occur suddenly with understanding, it never seems to occur with non-understanding.

When our knowledge of brain activity is more advanced, it may prove possible to give a precise mathematical model of its state at any given point in time. For example, following Zeeman [6], one could postulate that the level of activity is measured in each brain cell and the measurements recorded as the coordinates of a point in a very high dimensional space $\mathbb{R}^N$ where $N$ is about $10^{10}$ because there are approximately $10^{10}$ brain cells. The point is restricted in its movement with time by a changing dynamical flow influenced by external factors to the brain and the nature of brain activity itself. Such a model is not available to us yet, but the simplification of imagining brain activity represented by a single point in a complicated space can sometimes be illuminating.

If this is a little mind-boggling, consider the analogy of a solid body in space, whose position is fixed when we choose three points in the body, $P_1$, $P_2$, $P_3$ and then give coordinates $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$ to these three points in space. The position of the body is given by the nine coordinates which he can regard as a point $(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)$ in $\mathbb{R}^9$. These coordinates are restricted by the fact that we must have

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2= d^2$$

where $d$ is the distance $P_1P_2$, with two other similar equations for $P_2P_3$, $P_3P_1$, all three occurring because the body is solid and these distances are fixed. Thus the position of the body is represented by a point in $\mathbb{R}^9$ under certain restrictions.

The brain model uses the same idea, measuring the activity in each brain cell and recording the full information as a point in $\mathbb{R}^N$. The moving point representing the state of the brain at any given moment in time is subject to a dynamical flow, rather like a complicated magnetic field on a point particle. In the passage of time, the field changes and the point moves subject to the constraints of the field. Remember that the field is caused by external constraints and internal ones, so this does not mean that the model adheres to a particular psychological theory. (In particular, the presence of the internal activity of the brain means that this model is not meant just to represent some stimulus-response theory interpretation.)

There may be attractors which draw in the point, repellers which force it away, or more complex figurations, whorls, loops, and so on. We postulate that the brain attempts to find a position of equilibrium in the changing dynamical field. A position which is tenable at one time may cease to be an equilibrium position and then the point must move elsewhere towards another equilibrium. This jump to another position is the essence of the modern theory of catastrophes, the leap being the catastrophe. A graphic illustration occurs when “the penny drops” and suddenly one can see the light. This is the “aha” experience. Such a major catastrophe as this has external manifestations the sudden change from furrowed brow to look of delight, and so on. When the brain receives external stimulus, the configuration of the dynamical system changes, and as the brain reacts to the stimulus, the configuration continues to change. The presence of attractors in the system will draw in the point representing the state of the brain. They may be suitable, in the sense that they are appropriate to the problem at hand, or unsuitable, in the sense that they are not.

The presence of unsuitable attractors or repellers in the system may make certain points of the space inaccessible. This gives a qualitatively different behaviour. When pulled into an attractor, the location of the attractor is known, corresponding to a cognitive sense of equilibrium, “knowing where one is”. When repelled by a repeller, the actual location of the repeller is not so clearly sensed, corresponding to the cognitive confusion and a feeling of not knowing what is happening.

Possible practical examples of this phenomenon are discussed in “Conflicts in the Learning of Real .Numbers and Limits” (Mathematics Teaching 82), though not using catastrophe theory language. The examples show how conflicts in the mind can lead to confusion and lack of understanding.

The model which I have just outlined is by no means finalised, but as mathematicians we are used to making postulates and deducing from them. If we argue the existence of such a model, we must then put flesh on it to be of any use. That is where the articles by Richard Skemp [3], Byers & Herscovics [2] and John Backhouse [1] point the way towards a useful discussion. It is to be hoped that the comments made in this article guide such a discussion into the dynamic development of thinking and see the various types of understanding as possible classifications of parts of the process.

A classification may be useful, but it may also blind us to other possible factors. Any useful classification must exhibit the realities of mathematical understanding, the leaps of insight, the existence of mental blockages and the various other phenomena which characterise mathematical thinking.

**References**


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6. E. C. Zeeman: *Catastrophe Theory* (Addison Wesley) p. 287.